Math 2141 Final Review Problems

Problem 1. Evaluate the following integrals using integration by parts.

\[ \int_0^1 (x^2 + 1)e^x \, dx \]
\[ \int (\ln x)^2 \, dx \]
\[ \int e^{-x} \sin x \, dx \]

Problem 2. Evaluate the following integrals by using a substitution and then doing integration by parts.

\[ \int_1^4 e^{\sqrt{x}} \, dx \]
\[ \int x^3 e^{x^2} \, dx \]

Problem 3. Evaluate the following trig integrals.

\[ \int_0^\pi \cos^5 x \, dx \]
\[ \int \tan^3 x \sin^2 x \, dx \]

Problem 4. Use trig substitutions to evaluate the following integrals.

\[ \int_0^2 x^3 \sqrt{x^2 + 4} \, dx \]
\[ \int \frac{1}{x^2 \sqrt{16x^2 - 9}} \, dx \]
\[ \int \sqrt{5 + 4x - x^2} \, dx \]

For the last integral, you may want to use the trig identities $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin 2\theta = 2 \sin \theta \cos \theta$. 
Problem 5. Evaluate the following integrals.

\[
\int_0^1 \frac{x - 1}{x^2 + 3x + 2} \, dx \\
\int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6} \, dx \\
\int \frac{x^3}{(x + 1)^3} \, dx \\
\int \frac{x^2 - x + 6}{x^3 + 3x} \, dx \\
\int_0^1 \frac{x}{x^2 + 4x + 13} \, dx
\]

Problem 6(a). Let \( f(x) = x - 1 - \ln(x) \) for \( x > 0 \). Prove that \( f(x) \geq 0 \) and that \( f(x) > 0 \) for \( x \neq 1 \).

6(b). Let \( g(x) = \ln(x) - 1 + 1/x \) for \( x > 0 \). Prove that \( g(x) \geq 0 \) and that \( g(x) > 0 \) for \( x \neq 1 \).

6(c). Use the results in 6(a) and 6(b) to explain why the inequalities

\[
1 - \frac{1}{x} < \ln(x) < x - 1
\]

hold for \( x > 0 \) and \( x \neq 1 \).

6(d). Use the inequalities in 6(c) to prove that

\[
\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1
\]

Hint. Write the inequality in 6(c) with \( 1 + x \) in place of \( x \). Note that this new inequality holds when \( 1 + x \neq 1 \), which is the same as \( x \neq 0 \).

Problem 7. Give an alternate proof that

\[
\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1
\]

by letting \( f(x) = \ln(x) \), using the fact that \( f'(1) = 1 \) and using the limit definition of \( f'(1) \).

Problem 8. Let \( k > 1 \) be a natural number. Prove that

\[
\sum_{i=2}^{k} \frac{1}{i} < \ln(k) < \sum_{i=1}^{k-1} \frac{1}{i}
\]

Hint. You know \( \ln(k) = \int_1^k 1/x \, dx \), so \( \ln(k) \) is the area under the curve \( y = 1/x \) on the interval \([1, k]\). Consider the partition \( P = \{1, 2, \ldots, k\} \) of \([1, k]\). Use the left endpoints and
the right endpoints of this partition to approximate the area under the curve from below and above.

**Problem 9.** Use the Mean Value Theorem to prove that if \( f'(x) = 0 \) for all \( x \) in an open interval \( I \), then there is a constant \( c \) such that \( f(x) = c \).

**Problem 10.** Prove that if \( f'(x) = g'(x) \) for all \( x \) in an open interval \( I \), then there is a constant \( c \) such that \( f(x) = g(x) + c \).

*Hint.* Consider the function \( h(x) = f(x) - g(x) \).

**Problem 11.** Prove that if \( f'(x) = rf(x) \) for some fixed real number \( r \), then \( f(x) = f(0)e^{rx} \).

*Hint.* Consider the function \( h(x) = f(x)e^{-rx} \). What is \( h'(x) \)? How does that help you?