Problems for the Practice Problems on the Mean Value Theorem for Exam 2

Problem 1. Verify that \( g(x) = x^3 - 2x^2 - 4x + 2 \) satisfies the hypotheses of Rolle’s Theorem on \([-2, 2]\) and find all points \( c \in [-2, 2] \) for which \( f'(c) = 0 \).

Solution. Since \( g(x) \) is a polynomial, it is continuous and differentiable everywhere. Also, \( g(2) = g(-2) = -6 \), so \( g \) satisfies the conditions for Rolle’s Theorem. To find the roots of \( g'(x) = 3x^2 - 4x - 4 \):

\[
3x^3 - 4x - 4 = 0 \\
(3x + 2)(x - 2) = 0 \\
x = 2, -2/3
\]

Rolle’s Theorem promises a solution in the open interval \((-2, 2)\), so the only point satisfying Rolle’s Theorem is \( c = -2/3 \).

Problem 2. Let \( f(x) \) be a function which is continuous on \([a, b]\), differentiable on \((a, b)\) and such that \( f(a) \) and \( f(b) \) have different signs (i.e. one is strictly positive and the other is strictly negative). Prove that if \( f'(x) \) is either strictly positive or strictly negative on \((a, b)\), then the equation \( f(x) = 0 \) has exactly one solution in the interval \([a, b]\).

Solution. By the Intermediate Value Theorem, \( f \) has at least one root in the interval \((a, b)\). Suppose for a contradiction that \( f \) had two roots in \((a, b)\), say \( f(c_1) = f(c_2) = 0 \) with \( c_1 \neq c_2 \) and \( c_1, c_2 \in (a, b) \). We can assume \( c_1 \) and \( c_2 \) are indexed so that \( c_1 < c_2 \). (If not, switch which point is called \( c_1 \) and which is called \( c_2 \).

Because \( c_1, c_2 \in (a, b) \), our hypotheses tell us that \( f \) is continuous on \([c_1, c_2]\) and differentiable on \((c_1, c_2)\). By Rolle’s Theorem (since \( f(c_1) = f(c_2) \)), there must be a point \( d \in (c_1, c_2) \) such that \( f'(d) = 0 \). But, \( d \in (a, b) \) and \( f'(d) = 0 \) contradicts the hypothesis that \( f'(x) \) is either strictly positive or strictly negative on \((a, b)\).

Problem 3. Let \( 0 < a < b \) be real numbers and let \( n \geq 2 \) be an integer. Use the Mean Value Theorem to explain why

\[
na^{n-1}(b - a) \leq b^n - a^n \leq nb^{n-1}(b - a)
\]

Solution. Following the hint, apply the Mean Value Theorem to the function \( f(x) = x^n \) on \([a, b]\). There is a point \( c \in (a, b) \) such that \( f'(c) = (f(b) - f(a))/(b - a) \). In other words,

\[
nc^{n-1} = \frac{b^n - a^n}{b - a}
\]

Since \( a < c < b \) and the function \( f'(x) = nx^{n-1} \) is strictly increasing on \([a, b]\) (because \( a \) and \( b \) are strictly positive), we have \( na^{n-1} < ac^{n-1} < nb^{n-1} \). Substituting for \( nc^{n-1} \) using the equation above gives

\[
na^{n-1} < \frac{b^n - a^n}{b - a} < nb^{n-1}
\]
or in other words (because $b - a$ is positive), $na^{n-1}(b - a) < b^n - a^n < nb^{n-1}(b - a)$.

**Problem 4.** Let $f(x)$ be a function which is continuous on $[a, b]$ and such that $f''(x)$ exists at every point in $(a, b)$. Suppose that the line segment joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of $f(x)$ at a point $(c, f(c))$ where $a < c < b$. Prove that there is at least one point $d \in (a, b)$ at which $f''(d) = 0$.

**Solution.** Since $f''(x)$ exists on $(a, b)$, we know $f'(x)$ exists on $(a, b)$. Since $f$ is also continuous on $[a, b]$, we can apply the Mean Value Theorem to $f$ on $[a, b]$. Because the line from $(a, f(a))$ to $(b, f(b))$ goes through $(c, f(c))$, we know that the slopes of the line segments from $(a, f(a))$ to $(c, f(c))$ and from $(c, f(c))$ to $(b, f(b))$ are equal. (Both of these line segments are part of the line from $(a, f(a))$ to $(b, f(b))$.) Draw a picture if this is not clear.) In other words, 

$$(f(c) - f(a))/(c - a) = (f(b) - f(c))/(b - c).$$

Applying the Mean Value Theorem to $f$ on $[a, c]$, there is a point $c_1 \in (a, c)$ such that

$$f'(c_1) = \frac{f(c) - f(a)}{c - a}.$$

Applying the Mean Value Theorem to $f$ on $[c, b]$, there is a point $c_2 \in (c, b)$ such that

$$f'(c_2) = \frac{f(b) - f(c)}{b - c}.$$

The intervals $(a, c)$ and $(c, b)$ are disjoint, so $c_1 \neq c_2$. Because $(f(c) - f(a))/(c - a) = (f(b) - f(c))/(b - c)$, $f'(c_1) = f'(c_2)$.

We are given that $f''(x)$ exists on $(a, b)$ and therefore $f'(x)$ is continuous on $[c_1, c_2]$ and differentiable in $(c_1, c_2)$. Since $f'(c_1) = f'(c_2)$, we can apply Rolle’s Theorem to $f'(x)$ on $[c_1, c_2]$. Therefore, there is a point $d \in (c_1, c_2)$ such that $f''(d) = 0$. But, $a < c_1 < c_2 < b$, so $d \in (a, b)$ as required.

**Problem 5.** Let $I = (a, b)$ be an open interval and let $f$ be a function which is differentiable on $I$. Use the Mean Value Theorem to prove the following statements. (For this problem, I won’t give a solution. I will let you work on modifying what we did in class.)

9(a). If $f'(x) = 0$ for all $x \in I$, then there is a constant $r$ such that $f(x) = r$ for all $x \in I$.

9(b). If $f'(x) > 0$ for all $x \in I$, then $f(x)$ is strictly increasing on $I$.

9(c). If $f'(x) < 0$ for all $x \in I$, then $f(x)$ is strictly decreasing on $I$. 