Math 2141: Practice Problems on Mean Value Theorem for Exam 2

These problems are to give you some practice on using Rolle's Theorem and the Mean Value Theorem for Exam 2. You do not need to hand them in.

Problem 1. Verify that $g(x) = x^3 - 2x^2 - 4x + 2$ satisfies the hypotheses of Rolle's Theorem on $[-2, 2]$ and find all points $c \in [-2, 2]$ for which $f'(c) = 0$.

Problem 2. Let $f(x)$ be a function which is continuous on $[a, b]$, differentiable on (a, b) and such that $f(a)$ and $f(b)$ have different signs (i.e. one is strictly positive and the other is strictly negative). Prove that if $f'(x)$ is either strictly positive or strictly negative on (a, b), then the equation $f(x) = 0$ has exactly one solution in the interval $[a, b]$.

Problem 3. Let $0 < a < b$ be real numbers and let $n \geq 2$ be an integer. Use the Mean Value Theorem to explain why

$$na^{n-1}(b - a) \leq b^n - a^n \leq nb^{n-1}(b - a)$$

Hint. Let c be the point in the conclusion of the Mean Value Theorem applied to the function $f(x) = x^n$ on the interval $[a, b]$. Since $f'(x) = nx^{n-1}$ is increasing, what is biggest value of $f'(x)$ on $[a, b]$? What is the smallest value of $f'(x)$ on $[a, b]$?

Problem 4. Let $f(x)$ be a function which is continuous on $[a, b]$ and such that $f''(x)$ exists at every point in (a, b). Suppose that the line segment joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of $f(x)$ at a point $(c, f(c))$ where $a < c < b$. Prove that there is at least one point $d \in (a, b)$ at which $f''(d) = 0$.

Hint. Use the Mean Value Theorem to show that there are distinct points $c_0, c_1 \in (a, b)$ such that $f'(c_0) = f'(c_1)$. Now use Rolle's Theorem to get a point d such that $f''(d) = 0$.

We proved one part of the last problem in class. As this is one of the most important applications of the Mean Value Theorem in calculus, it is well worth reviewing the proof of this part and proving the other two parts.

Problem 5. Let $I = (a, b)$ be an open interval and let f be a function which is differentiable on I. Use the Mean Value Theorem to prove the following statements.

9(a). If $f'(x) = 0$ for all $x \in I$, then there is a constant r such that $f(x) = r$ for all $x \in I$.

9(b). If $f'(x) > 0$ for all $x \in I$, then $f(x)$ is strictly increasing on I.

9(c). If $f'(x) < 0$ for all $x \in I$, then $f(x)$ is strictly decreasing on I.