Reverse Mathematics and the Coloring Number of Graphs

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We use methods of Reverse Mathematics to analyze the proof theoretic strength of certain graph theoretic theorems involving the notion of coloring number. Classically, the coloring number of a graph $G = (V, E)$ is the least cardinal κ such that there is a well ordering of V such that below any vertex in V, there are fewer than κ many vertices connected to it by E. A theorem which we will study in depth, due to Komjáth and Milner, states that if a graph is the union of n forests, then the coloring number of the graph is at most $2n$. In particular, we look at the case when $n = 1$. In doing the above, it is necessary for us to formulate various different Reverse Mathematics definitions of coloring number; we also analyze the relationships between these definitions.

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Chapter 1

Introduction

1.1 Computability Theory

We assume the reader has a basic knowledge of Computability Theory.

Fix an effective enumeration of all partial computable functions $\varphi_0, \varphi_1, \varphi_2, \ldots$ and fix an effective enumeration of all Turing functionals $\Phi_0, \Phi_1, \Phi_2, \ldots$. We think of the functions as computer programs (that are allowed an arbitrary finite amount of time to run) and the functionals similarly as computer programs, with the only difference being that functionals are allowed to have access to a set as an oracle. To denote that a function $\varphi_e : \mathbb{N} \to \mathbb{N}$ halts on input x and outputs y, we write $\varphi_e(x) \downarrow= y$. To denote that φ_e halts on input x (and has some output), we write $\varphi_e(x) \downarrow$. We write $\varphi_{e,s}(x) \downarrow= y$ if $e, x, y < s$ and y is the output of the program φ_e on input x for $< s$ many steps. Therefore

$$\varphi_e(x) \downarrow= y \leftrightarrow (\exists s)[\varphi_{e,s}(x) \downarrow= y].$$

It is clear that for a given stage s, $\varphi_{e,s}(x)$ should be a computable predicate, since it is computable to run a program for finitely many steps.

If $\varphi_e(x)$ does not halt, then it is undefined, and we write $\varphi_e(x)\uparrow$, and so

$$\varphi_e(x)\uparrow\leftrightarrow(\forall s)[\varphi_{e,s}(x)\uparrow].$$

To say that a Turing functional Φ with oracle A halts on input x and outputs y, we write $\Phi^A(x)\downarrow= y$. We can define Φ running for s many steps similarly to the way we defined it for functions. Define *the halting set*

$$K := \{e \in \mathbb{N} : \varphi_e(e)\downarrow\}.$$

We define the *jump of* A, written A' (and spoken "A jump" or "A prime") as

$$A' := \{e \in \mathbb{N} : \Phi_e^A(e)\downarrow\}.$$

Therefore $K = \emptyset'$. A set is computable if its characteristic function is (total) computable. It is well-known that K is not computable. It is one of the most basic examples of a set which is not computable. (In fact, for any set A, $A' >_T A$.) For more information on the subject, I would direct the reader to Soare [9].

1.2 Reverse Mathematics

Reverse mathematics deals with the analysis of the proof theoretic strength of theorems. It works best in the context of countable or essentially countable mathematics—for example, we can analyze theorems in number theory, countable algebra and countable combinatorics. We can also study theorems in real and complex analysis, or more generally about complete separable metric spaces, since they can be understood in terms of a countable dense subset. Reverse mathematics is less useful for studying heavily set-theoretic subjects such as abstract functional analysis, general topology or set theory itself. The reason is that we restrict our axiomatic focus from set theory (ZFC) to second order arithmetic (Z_2).

All of the theorems we analyze in reverse mathematics are in the language of Z_2. The first order part of Z_2 has constants and variables, which are intended to range over elements of $\mathbb{N}$, and the usual addition and multiplication. The second order part has set variables which are intended to range over subsets of $\mathbb{N}$, and the $\in$ relation. Formulas of Z_2 consist of formulas that are put together from the above with the usual logical symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$, and the quantifer symbols $\forall$ and $\exists$, which are intended to quantify over both number and set variables. We classify the proof theoretic strength of a theorem by finding the weakest subsystem of Z_2 in which the theorem is still provable.

What are the subsystems of Z_2? A subsystem is distinguished by the level of comprehension (set existence) that it allows. The comprehension scheme in Z_2 is given by

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(x)$ is any formula in which the set variable X does not occur. Essentially, if we restrict what kind of formula φ is allowed to be, then we get a new subsystem. There are infinitely many subsystems of Z_2, but it turns out that there are five that occur most often in reverse mathematics. They are RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0. We think of the system RCA_0 as a system which is just strong enough to prove the existence of computable subsets of $\mathbb{N}$. WKL_0 (Weak König's Lemma) is a type of compactness that asserts the existence of paths through infinite binary branching trees. ACA_0 (Arithmetic Comprehension Axiom—a stronger form of compactness) asserts the existence of sets definable by formulas that only quantify over number variables. ATR_0 (Arithmetic Transfinite Recursion) is equivalent to any two countable well orders being comparable, while Π^1_1-CA_0 asserts the existence of Π^1_1 sets. A Π^1_1 set is one that is definable by a Π^1_1 formula (one that has a universal set quantifier and unrestricted number quantifiers after it).

The subscript 0 means that we have restricted what kind of induction scheme we are allowed to use in our proofs. Z_2 has full second order induction, given by the set induction principle

$$(0 \in X \wedge \forall n[n \in X \to n+1 \in X]) \to \forall n(n \in X)$$

but we weaken the level of induction using the schema

$$(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))) \to \forall n \varphi(n)$$

and restricting what kind of formula φ can be.

Definition 1.2.1. [8] The scheme of Δ^0_1 *comprehension* consists of all axioms of the form

$$(\forall n)[\varphi(n) \leftrightarrow \psi(n)] \leftrightarrow (\exists X)(\forall n)[n \in X \leftrightarrow \varphi(n)]$$

where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and X is not free in $\varphi(n)$.

Definition 1.2.2. For each $k \in \omega$, the scheme of Σ^0_k *induction* consists of all axioms of the form of the induction schema given above, where $\varphi(n)$ is any Σ^0_k formula of the language of second-order arithmetic. Similarly, Π^0_k induction consists of all axioms of the same form, except that $\varphi(n)$ is any Π^0_k formula.

Definition 1.2.3. [8] RCA_0 is the formal system in the language L_2 of second-order arithmetic whose axioms consist of the basic axioms, in addition to the schemes of Δ^0_1 comprehension and Σ^0_1 induction.

We should note that RCA_0 does not say that noncomputable sets do not exist; but it is not strong enough to prove they do exist. We can talk about noncomputable sets in RCA_0 using the formulas that define those sets, as we will see later.

Definition 1.2.4. [8] The axioms of ACA_0 are the basic axioms and the induction axiom together with comprehension axioms

$$(\exists X)(\forall n)[n \in X \leftrightarrow \varphi(n)]$$

where φ is any arithmetical formula in which X does not occur freely.

Note that an arithmetical formula is one that does not quantify over sets, only over numbers. We do allow set parameters in an arithmetical formula. As an example, the set K exists in ACA_0, as it is definable by the arithmetical (Σ^0_1) formula $(\exists s)[\varphi_{e,s}(e)\downarrow]$.

We will use the following lemma [8] extensively.

Lemma 1.2.5. The following are pairwise equivalent over RCA_0.

1. ACA_0.
2. Σ^0_1 comprehension, i.e., the comprehension axioms $(\exists X)(\forall n)[n \in X \leftrightarrow \varphi(n)]$ where φ is any Σ^0_1 formula in which X does not occur freely.
3. For all one-to-one functions $f : \mathbb{N} \to \mathbb{N}$ there exists a set $X \subseteq \mathbb{N}$ such that $(\forall n)[n \in X \leftrightarrow \exists m(f(m) = n)]$, i.e., X is the range of f.

Definition 1.2.6. [8] (RCA_0) We define a *finite set* to be a set X such that $(\exists k)(\forall i)[i \in X \to i < k]$.

Theorem 1.2.7. (Theorem II.2.5 from [8]) (RCA_0) For any finite set $X \subseteq \mathbb{N}$ there exist k, m and $n \in \mathbb{N}$ such that

$$\forall i[i \in X \leftrightarrow (i < k \wedge m(i+1) + 1 \text{ divides } n)]$$

Every finite set X can be encoded as a unique natural number. The *code* of the finite set of natural numbers X is the least number of the form $\langle k, \langle m, n \rangle\rangle$ such that the above formula holds. Note that $\langle i, j \rangle = (i+j)^2 + i$ is the standard pairing map, which is a one-to-one map of $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$.

Definition 1.2.8. (RCA_0) Let A be a set. We define the set of codes for finite subsets of A,

$$\mathrm{Fin}_A := \{c \in \mathbb{N} : c \text{ is a code for a finite subset of } A\}.$$

Definition 1.2.9. (Definition II.2.6 from [8]) (RCA_0) A *finite sequence of natural numbers* is a finite set X such that $(\forall n)[n \in X \to \exists i \exists j(n = \langle i, j \rangle)]$ and

$$\forall i \forall j \forall k[(\langle i, j \rangle \in X \wedge \langle i, k \rangle \in X) \to j = k]$$

and $(\exists l)(\forall i)[i < l \leftrightarrow \exists j(\langle i, j \rangle \in X)]$. The number l is uniquely determined and is called the *length* of X. The code of a finite sequence X is just the code of X as a finite set.

Definition 1.2.10. (RCA_0) Let A be a set. We define the set of codes for finite sequences of elements from A (identified with partial functions $\sigma : \mathbb{N} \to A$),

$$\mathrm{Seq}_A := \{c \in \mathbb{N} : c \text{ is a code for a finite sequence of elements from } A\}.$$

Sometimes we write $A^{<\mathbb{N}}$ in the place of Seq_A.

For $\sigma \in \mathrm{Seq}_A$, we let $|\sigma|$ denote the length of σ. For $\sigma, \tau \in \mathrm{Seq}_A$, we write $\tau \subseteq \sigma$ to say that τ *is an initial segment of* σ, i.e.,

$$|\tau| \leq |\sigma| \wedge (\forall i < |\tau|)[\sigma(i) = \tau(i)],$$

which we could also write as $\tau = \sigma \upharpoonright dom(\tau)$.

Definition 1.2.11. We say that $T \subseteq 2^{<\mathbb{N}}$ is a *tree* if

$$(\forall \sigma \in T)(\forall \tau \in T)[\tau \subseteq \sigma \to \tau \in T].$$

In words, the above is equivalent to saying that T *is closed under initial segments.* A *path* in a tree T is a function $f : \mathbb{N} \to 2$ such that $(\forall n)[f \upharpoonright n \in T]$.

Definition 1.2.12. *Weak König's lemma* is the statement that every infinite tree $T \subseteq 2^{<\mathbb{N}}$ has a path.

Definition 1.2.13. [8] The axioms of WKL_0 are those of RCA_0 plus weak König's lemma.

Now we turn to graph theory and formulate a few definitions.

Definition 1.2.14. (RCA_0) A *graph* G is a pair (V, E), where V is the set of vertices and E is an irreflexive, symmetric relation on V. (Note that our graphs are undirected with no edges from any vertex to itself.)

Definition 1.2.15. (RCA_0) Let $G = (V, E)$ be a graph, and $u, v \in V$, $u \neq v$. A *path* in the graph G is a nonempty sequence $\sigma \in \mathrm{Seq}_V$ such that

$$(\forall i \neq j < |\sigma|)[\sigma(i) \neq \sigma(j)] \wedge (\forall i < |\sigma| - 1)[\sigma(i) E \sigma(i+1)].$$

The *collection of all paths in* G is given by

$$\mathrm{Path}_G := \{\sigma \in \mathrm{Seq}_V : \sigma \text{ is a path in } G\}.$$

The *collection of all paths from* u *to* v *in* G is given by

$$\mathrm{Path}_G^{u,v} := \{\sigma \in \mathrm{Path}_G : \sigma(0) = u \wedge \sigma(|\sigma| - 1) = v\}.$$

Definition 1.2.16. (RCA_0) An *acyclic graph* is a graph $F = (V, E)$ such that

$$(\forall u, v \in V)[|\mathrm{Path}_F^{u,v}| < 2].$$

A *forest* is an acyclic graph. A *tree* is a forest $T = (V, E)$ such that

$$(\forall u, v \in V)(\mathrm{Path}_T^{u,v} \neq \emptyset).$$

Notice that this definition of a tree is different than what we have already described. While this difference could become problematic, it should be clear from the context which definition we intend to use when we say "tree."

Definition 1.2.17. Let $G = (V, E)$ be a graph. The *component of* G *with representative vertex* v is the subgraph of G that is induced by the set of vertices given by $\{u \in V : \mathrm{Path}_T^{u,v} \neq \emptyset\} \cup \{v\}$.

When we say "component of G" we mean a component of G with representative vertex v for some $v \in V$. Note that, in general, ACA_0 is required to know that for an arbitrary vertex $v \in V$, the component of G with representative vertex v exists. Of course, if G only has finitely many components, it would make sense (and is in fact true, as we will see later) that RCA_0 is enough to know that, for an arbitrary vertex $v \in V$, the component of G with representative vertex v exists.

Definition 1.2.18. We say a graph $G = (V, E)$ has finitely many components if there is a finite set $X \in \mathrm{Fin}_V$ such that X contains exactly one vertex from each component.

For the following, we reason within RCA_0. Let $T = (V, E)$ be a tree. For all $X \in \mathrm{Fin}_V$ and all $y \in V \setminus X$ we can form the set of all paths from the induced subgraph on X to the vertex y

$$\mathrm{Path}_T^{X,y} := \{\sigma \in \mathrm{Path}_T : \sigma(0) \in X \wedge \sigma(|\sigma| - 1) = y\}.$$

Because T is a tree (and hence acyclic), for each $x \in X$ there is a unique path from x to y. It follows that $\mathrm{Path}_T^{X,y}$ is a finite set because X is a finite set.

Let $n = \min\{|\sigma| : \sigma \in \mathrm{Path}_T^{X,y}\}$. For any $\sigma \in \mathrm{Path}_T^{X,y}$ with $|\sigma| = n$, we have $\forall i[1 \leq i < |\sigma| \rightarrow \sigma(i) \notin X]$.

We call such a σ with $|\sigma| = n$ a *path from X to y*. Since the induced subgraph on X need not be connected, there may be more than one such path, so choose the one with least code to define the function

$$P : \mathrm{Fin}_V \times V \rightarrow \mathrm{Path}_T$$

such that

$$P(X, y) = \begin{cases} \emptyset \text{ if } y \in X \\ \\ \sigma \text{ if } y \in V \setminus X, \text{ where } \sigma \text{ is a path from } X \text{ to } y \text{ with least code.} \end{cases}$$

Notice that if the induced subgraph on X is connected, then there is a unique path from X to y for any $y \in V \setminus X$. The existence of the function P in RCA_0 will be useful to us later.

1.3 Coloring Number

Definition 1.3.1. (RCA_0) We say that a binary relation $\leq$ is a *linear order* on the set X if the following axioms are satisfied:

1. $(\forall x, y \in X)[(x \leq y \wedge y \leq x) \rightarrow x = y]$

2. $(\forall x, y, z \in X)[(x \leq y \wedge y \leq z) \rightarrow x \leq z]$

3. $(\forall x, y \in X)[x \leq y \vee y \leq x]$

Definition 1.3.2. (Classical) The *coloring number* of a graph G, written $\mathrm{Col}(G)$, is the least cardinal κ for which there is a well-ordering of the vertex set in which every vertex is joined by an edge to fewer than κ smaller vertices.

If $\mathrm{Col}(G) = \kappa$, then we may assume that the ordering which witnesses this has order type $|V|$. This is a well known result of Erdös and Hajnal [2], which we give here as the following lemma.

Lemma 1.3.3. (Erdös, Hajnal) Let $G = (V, E)$ be a graph. If $|V| = \lambda$ and $\mathrm{Col}(G) = \kappa$, then there exists a well ordering of V with the order type λ witnessing $\mathrm{Col}(G) = \kappa$.

In Reverse Mathematics we restrict ourselves to work with only countable graphs. (So from now on, when we say "infinite graph," we really mean "countably infinite graph.") Considering the above lemma, we are particularly interested in well orderings of the vertex set V that have order type ω. Of course, to get such a well ordering of type ω may require nontrivial axioms in the sense of Reverse Mathematics, and since the proof provided by Erdös and Hajnal of the lemma uses transfinite induction, it is not immediately clear which subsystem is actually necessary. It would be a reasonable Reverse Mathematics question to explore this lemma to determine its proof-theoretic strength.

Definition 1.3.4. (RCA_0) Let $G = (V, E)$ be a graph and let $k \in \mathbb{N}$, $k \geq 1$. A *k-order* of V is a linear order $\leq_V$ of V such that for every $x \in V$ there are at most $k - 1$ many $y \in V$ such that $y \leq_V x$ and $E(x, y)$ holds.

Notice that if $G = (V, E)$ is a graph, then the existence of a k-order which is a well-order on V classically implies that $\mathrm{Col}(G) \leq k$. We now restate the classical definition of coloring number for countably infinite graphs as

Definition 1.3.5. (Classical) For $k \geq 1$, $\mathrm{Col}(G) \leq k$ if there is a k-order of V of type ω.

In many ways the classical definitions of coloring number given above are unsatisfactory in terms of Reverse Mathematics. For instance, how do we define (in RCA_0) what it means for a linear order of V to be of type ω? This leads us to formulate a few new definitions.

The following definition gives a strong way of saying that a linear ordering $\leq_V$ on a set V has order type ω by specifying, for each element $v \in V$, exactly how many elements are below v in the ordering $\leq_V$.

Definition 1.3.6. (RCA_0) We say that an ordering $\leq_V$ of a set $V = \{v_0, v_1, v_2, \dots\}$ has *strong ω-type* if there is a bijection $f : \mathbb{N} \to V$ such that

$$i \leq_{\mathbb{N}} j \iff f(i) \leq_V f(j).$$

In other words, f explicitly gives the ordering $\leq_V$, by specifying $f(0) =$ the first element of V in the ordering $\leq_V, \ldots, f(n) =$ the element of V in the $n+1$ position in the ordering $\leq_V$.

The following definition gives a weaker way of saying that a linear ordering $\leq_V$ on a set V has order type ω. Under this definition, we cannot tell exactly how many elements are below a given vertex v in the ordering $\leq_V$, only that there is some finite bound on the number of elements below v in the ordering $\leq_V$.

Definition 1.3.7. (RCA_0) We say that an ordering $\leq_V$ of a set $V = \{v_0, v_1, v_2, \ldots\}$ has *weak ω-type* if

$$(\forall i)(\exists j)(\forall m \geq_{\mathbb{N}} j)[v_i \leq_V v_m].$$

Here are some variations on the Reverse Mathematics definition of coloring number. For the following, let $G = (V, E)$ be a graph, and $k \in \mathbb{N}$ with $k \geq 2$.

Definition 1.3.8. (Linear order coloring number) (RCA_0) We say that $\mathrm{Col}_{LO}(G) \leq k$ if there is a k-order of V.

Definition 1.3.9. (Strong ω coloring number) (RCA_0) For an infinite graph G we say that $\mathrm{Col}^S_\omega(G) \leq k$ if there is a k-order of V of strong ω-type.

Definition 1.3.10. (Weak ω coloring number) (RCA_0) For an infinite graph G we say that $\mathrm{Col}^W_\omega(G) \leq k$ if there is a k-order of V of weak ω-type.

It is not hard to see we have the following string of classical implications:

$$\mathrm{Col}^S_\omega(G) \leq k \iff \mathrm{Col}^W_\omega(G) \leq k \implies \mathrm{Col}_{LO}(G) \leq k$$

The converse of the last implication above is false in general. Classically, $\mathrm{Col}_{LO}(G)$ and $\mathrm{Col}(G)$ are not the same. Consider the following to see this fact.

Lemma 1.3.11. $\mathrm{Col}_{LO}(G) \leq k$ if and only if $\mathrm{Col}_{LO}(H) \leq k$ for every finite subgraph $H \subseteq G$.

For now we omit the proof of the lemma, but note that it will follow from a compactness argument that we will give later.

To show (classically) that $\mathrm{Col}_{LO}(G) \leq k$ does not imply $\mathrm{Col}^W_\omega(G) \leq k$, we direct the reader to examples constructed by Erdös and Hajnal in [2], which can be found in Appendix A. These examples were originally used to show that the following result is sharp.

Theorem 1.3.12. (Erdös, Hajnal) If every finite subgraph of a graph G has coloring number at most n ($2 \leq n < \omega$), then the coloring number of G is at most $2n - 2$.

That is, for each $k \geq 1$, there is a graph G such that for every finite subgraph H of G, $\mathrm{Col}(H) = k + 1$, but $\mathrm{Col}(G) > 2k - 1$ (and so by the theorem it must be the case that $\mathrm{Col}(G) = 2k = 2n - 2$ for $n = k + 1$). These examples are given explicitly.

Notice that, together with Lemma 1.3.11, Theorem 1.3.12 proves that if $\mathrm{Col}_{LO}(G) \leq k$, then classically we have that $\mathrm{Col}_\omega(G) \leq 2k - 2$ (where $\mathrm{Col}_\omega(G)$ denotes the classical coloring number where we consider only well-orderings of V of type ω).

So classically, linear order coloring number and omega coloring number are not entirely different. At least they are both either finite or infinite.

While it is evident classically that $\mathrm{Col}^S_\omega(G) \leq k \iff \mathrm{Col}^W_\omega(G) \leq k$, we note that the equivalence between strong and weak ω-type linear orders requires nontrivial axioms in the sense of Reverse Mathematics analysis, as indicated by the following theorem.

Theorem 1.3.13. ($\mathsf{RCA}_0 + \Sigma^0_2$ Induction) The following are equivalent:

1. ACA_0
2. Every linear order of weak ω-type has strong ω-type.

Proof. ($1 \to 2$) We reason within ACA_0. Let $(\mathcal{L}, \leq_\mathcal{L})$ be a linear order of weak ω-type. So we know

$$(\forall x \in \mathcal{L})(\exists n \in \mathbb{N})[|\{y \in \mathcal{L} : y \leq_\mathcal{L} x\}| = n].$$

Now define the set

$$\begin{aligned} X \quad := \quad & \{\langle x, n\rangle \in \mathcal{L} \times \mathbb{N} : (\exists \sigma \in \mathrm{Seq}_\mathcal{L})[|\sigma| = n \wedge \\ & (\forall i < j < |\sigma|)[\sigma(i) <_\mathcal{L} \sigma(j) <_\mathcal{L} x] \wedge \\ & (\forall y \in \mathcal{L})[y <_\mathcal{L} x \to (\exists i \leq |\sigma|)[\sigma(i) = y]]]\}. \end{aligned}$$

Since the set X is arithmetical (i.e. it contains no set quantifiers, only number quantifiers and quantifiers over finite sequences of numbers) it exists in ACA_0.

Now we define a function $g : \mathcal{L} \to \mathbb{N}$ which witnesses $(\mathcal{L}, \leq_{\mathcal{L}})$ is of strong ω-type. Say

$$g(x) = n \iff \langle x, n \rangle \in X.$$

Define a function $f : \mathbb{N} \to \mathcal{L}$ by $f := g^{-1}$. It is clear that f explicitly gives the ordering $\leq_{\mathcal{L}}$ on $\mathcal{L}$.

$(2 \to 1)$ Fix a one-to-one function $f : \mathbb{N} \to \mathbb{N}$. We wish to show that $ran(f)$ exists. We want to define a computable order of weak ω-type $(\mathcal{L}, \leq_{\mathcal{L}})$ so that the strong order we get from statement 2 allows us to determine $ran(f)$. The elements of our linear order will be from the set

$$\mathcal{L} = \{a_{-1}\} \cup \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}.$$

To keep $\mathcal{L}$ computable, we will identify the a_i with the evens and the b_i with the odds. The idea will be that if we see $f(n) = m$, then a_m will have at least n many elements below it.

First we say that $a_i \leq_{\mathcal{L}} a_j \leftrightarrow i \leq j$, i.e.

$$a_{-1} <_{\mathcal{L}} a_0 <_{\mathcal{L}} a_1 <_{\mathcal{L}} a_2 <_{\mathcal{L}} \cdots .$$

Now define the function $F : \mathbb{N} \to \mathbb{N}$ by primitive recursion as

- $F(0) = 0$
- $F(k + 1) = F(k) + k + 1$

Note that at this point, for each $n \in \mathbb{N}$, there are exactly n elements in the order below a_n, in other words, $|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_n\}| = n$. Depending on f, we will insert elements from $\{b_n : n \in \mathbb{N}\}$ into the order

$$a_0 <_{\mathcal{L}} a_1 <_{\mathcal{L}} a_2 <_{\mathcal{L}} \cdots$$

at various places, ensuring throughout that $(\forall n)(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_n\}| < m]$. Because of this fact, we will see that the order $(\mathcal{L}, <_{\mathcal{L}})$ is of weak ω-type. We can then apply 2 to show $(\mathcal{L}, <_{\mathcal{L}})$ is also of strong ω-type.

For each $n \geq 1$, put

$$a_{f(n)-1} <_{\mathcal{L}} b_{F(n-1)} <_{\mathcal{L}} b_{F(n-1)+1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} b_{F(n)-1} <_{\mathcal{L}} a_{f(n)}$$

Notice that $F(n) - F(n-1) = n$, so that we have inserted exactly n many of the elements from $\{b_n : n \in \mathbb{N}\}$ directly below $a_{f(n)}$ and above $a_{f(n)-1}$.

We show an example of how the original ordering

$$a_0 <_{\mathcal{L}} a_1 <_{\mathcal{L}} a_2 <_{\mathcal{L}} \cdots$$

would change, given a few values of f, for an example. Say we have $f(0) = 4$, $f(1) = 7$, $f(2) = 0$, (etc.). The fact that $f(0) = 4$ tells us to place the first b (namely $b_{F(1)-1} = b_0$) between a_3 and a_4. The fact that $f(1) = 7$ tells us to place the next two b's (namely $b_{F(2)-2} = b_1$ and $b_{F(2)-1} = b_2$) between a_6 and a_7. The fact that $f(2) = 0$ tells us to place the next three b's (namely $b_{F(3)-3} = b_3$, $b_{F(3)-2} = b_4$ and $b_{F(3)-1} = b_5$) between a_{-1} and a_0. Given only this information, the ordering then becomes

$$a_{-1} <_{\mathcal{L}} b_3 <_{\mathcal{L}} b_4 <_{\mathcal{L}} b_5 <_{\mathcal{L}} a_0 <_{\mathcal{L}} a_1 <_{\mathcal{L}} a_2 <_{\mathcal{L}} a_3 <_{\mathcal{L}} b_0 <_{\mathcal{L}}$$

$$<_{\mathcal{L}} a_4 <_{\mathcal{L}} a_5 <_{\mathcal{L}} a_6 <_{\mathcal{L}} b_1 <_{\mathcal{L}} b_2 <_{\mathcal{L}} a_7 <_{\mathcal{L}} \cdots$$

The idea is that we want to have

$$(\forall n)(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_n\}| < m] = (\forall a \in V)(\exists m)(\forall k \geq_{\mathbb{N}} m)[\neg(a_k <_{\mathcal{L}} a)].$$

We prove this fact by induction on n. Note that the level of induction used here is Σ^0_2, and is therefore safe to employ in $\mathsf{RCA}_0 + \Sigma^0_2$ Induction.

Base Case. First, note that a_{-1} is the least element of $(\mathcal{L}, <_{\mathcal{L}})$, and that there is no way for any of the b's to be placed below it. Now, for $n = 0$ the formula becomes $(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_0\}| < m]$. The only way for any of the b's to be placed below a_0 (and above a_{-1}) is if there is some $\ell \in \mathbb{N}$ such that $f(\ell) = 0$. We have two cases: either there is such an ℓ or there is no such ℓ. In the latter case, the witness to the formula $(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_0\}| < m]$ is $m = 2$ (as a_{-1} is below). If the former holds, then there are exactly ℓ many of the b's below a_0, and the witness to the formula is $\ell + 2$ and we are done.

Induction Case. We wish to show that $(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_{n+1}\}| < m]$. By the inductive hypothesis we have $(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_n\}| < m]$. Let $\hat{m} \in \mathbb{N}$ be a witness to this, so that $|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_n\}| < \hat{m}$. Now, if there is no $\ell \in \mathbb{N}$ such that $f(\ell) = n + 1$, then no new elements will be placed between a_n and a_{n+1}, and so the witness for $(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_{n+1}\}| < m]$ is $\hat{m} + 1$. Suppose

$f(\ell) = n + 1$. Then the only new elements we will put between a_n and a_{n+1} are

$$b_{F(\ell-1)} <_{\mathcal{L}} b_{F(\ell-1)+1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} b_{F(\ell)-1}.$$

In other words, there are $F(\ell) - F(\ell-1) = \ell$ many of the b's placed below a_{n+1} and above a_n. Therefore our witness to the formula $(\exists m)[|\{y \in \mathcal{L} : y <_{\mathcal{L}} a_{n+1}\}| < m]$ is $\hat{m} + \ell + 1$.

Hence, by induction, the order we have constructed $(\mathcal{L}, <_{\mathcal{L}})$ is of weak ω-type. We apply the statement 2 to get that $(\mathcal{L}, <_{\mathcal{L}})$ is also of strong ω-type. Let $g : \mathbb{N} \to \mathcal{L}$ be a witnessing bijection for $(\mathcal{L}, <_{\mathcal{L}})$ being of strong ω-type. Thus we have the ordering explicitly as

$$g(0) <_{\mathcal{L}} g(1) <_{\mathcal{L}} g(2) <_{\mathcal{L}} \cdots$$

Then

$$m \in ran(f) \iff (\exists n)[f(n) = m] \iff (\exists n < g^{-1}(a_m))[f(n) = m],$$

the last of which may be checked in RCA_0, and we are done.

Qed

Classically, every forest has coloring number ≤ 2. The proof of this fact is indeed quite simple; the idea is that, given a forest $G = (V, E)$, we choose a vertex $a \in V$ as the least element and then order the rest of V by levels. In the remainder of the chapter, we show that ACA_0 is strong enough to prove this fact and that, in the restricted case of trees, RCA_0 suffices.

Theorem 1.3.14. (RCA_0) If $G = (V, E)$ is a countably infinite tree, then $\mathrm{Col}^S_\omega(G) \leq 2$.

Proof. We notice that the classical proof of this statement can be carried out effectively. Assume RCA_0 and let $G = (V, E)$ be a tree. Furthermore suppose that $V = \{v_0, v_1, v_2, \dots\}$. We wish to define a sequence of finite subsets $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$ of V and a sequence of orderings $\leq_0 \subseteq \leq_1 \subseteq \leq_2 \subseteq \cdots$ on the finite sets of vertices $V_0, V_1, V_2, \dots$ respectively, such that

1. Each V_i is finite, connected, and $\{v_0, \dots, v_i\} \subseteq V_i$ (so that $V = \bigcup_{i \in \mathbb{N}} V_i$).
2. Each $\leq_i$ is a 2-order of V_i.
3. $\leq_{i+1}$ is an end-extension of $\leq_i$ (i.e. for all $x \in V_i$ and $y \in V_{i+1} \setminus V_i$, $x \leq_{i+1} y$).

<u>Stage 0</u>: Define $V_0 = \{v_0\}$ and $v_0 \leq_0 v_0$.

<u>Stage $s+1$</u>: Suppose that we have already defined V_s and $\leq_s$. To get V_{s+1} and $\leq_{s+1}$, we do the following.

- If $v_{s+1} \in V_s$, then let $V_{s+1} = V_s$ and $\leq_{s+1} = \leq_s$.
- If $v_{s+1} \notin V_s$, then there is a unique path from v_{s+1} to V_s. Recall our discussion at the end of section 1.2; the path given by $P(V_s, v_{s+1})$ is this unique path. Let $\sigma = P(V_s, v_{s+1})$. Say that the vertices in this path given by σ are $\sigma(0) = u_0, \sigma(1) = u_1, \dots, \sigma(k) = u_k$. Then, by definition of P, $u_0 \in V_s$ and $\{u_1, u_2, \dots u_k\} \cap V_s = \emptyset$, while $E(u_i, u_{i+1})$ holds for each $i < k$ and $u_k = v_{s+1}$.

Now define $V_{s+1} = V_s \cup \{u_1, \ldots, u_k\}$ and extend $\leq_s$ to $\leq_{s+1}$ by taking $\leq_{s+1}$ to be an end-extension of $\leq_s$, where additionally,

$$u_1 \leq_{s+1} u_2 \leq_{s+1} \cdots \leq_{s+1} u_k = v_{s+1}.$$

The fact that each V_s is finite, connected and contains $\{v_0, \ldots, v_s\}$ follows by induction.

The bijection $f : \mathbb{N} \to V$ that gives a 2-order of V strong ω-type of is determined in the following way: Let $f(0) = v_0$. Now consider the induction step in the above. Suppose we have f for the set V_s, and that the last number f has been defined on is $m - 1$. If we are in the first case, we do not extend the definition of f. If we are in the second case, we let $f(m) = u_1$, $f(m+1) = u_2, \ldots, f(m+k-2) = u_{k-1}$ and $f(m+k-1) = u_k = v_1$. ***Qed***

Theorem 1.3.15. (RCA_0) If $G = (V, E)$ is a forest with finitely many components, then $\mathrm{Col}^S_\omega(G) \leq 2$.

Proof. Let $X \in \mathrm{Fin}_V$ be a set such that X contains exactly one vertex from each of the finitely many components of G. In other words, let X be a finite set of component representatives of G, as per our definition of a graph having finitely many components. Suppose $X = \{x_0, \ldots, x_k\}$. Define the set

$$\begin{aligned} P &:= \{\langle x, v\rangle \in X \times V : x = v \vee \mathrm{Path}_G^{x,v} \neq \emptyset\} \\ &= \{\langle x, v\rangle \in X \times V : x = v \vee (\exists \sigma \in \mathrm{Path}_G) \\ &\quad [\sigma(0) = x \wedge \sigma(|\sigma| - 1) = v]\} \\ &= \{\langle x, v\rangle \in X \times V : x = v \vee (\forall y \in X) \\ &\quad [y \neq x \to \neg \exists \sigma \in \mathrm{Path}_G[\sigma(0) = y \wedge \sigma(|\sigma| - 1) = v]]\} \end{aligned}$$

Notice that we have found a form of P which is Σ^0_1 and a form which is Π^0_1. Thus P is Δ^0_1, and so RCA_0 proves it is a set.

Now we define

$$T_i := \{v \in V : \langle x_i, v\rangle \in P\} \text{ for } 0 \leq i \leq k.$$

Then each T_i is Δ^0_1, and therefore exists in RCA_0.

Let $V = \{v_0, v_1, v_2, \ldots\}$. We define the function $g : \mathbb{N} \to V$ which is a 2-order of V of strong ω-type in the following way. Let $g(0) = v_0$. Now assume that g is defined up to m and let v_j be the vertex of least index that is not in the range of g so far. Let T_i be the component of G which contains v_j.

Case 1: If $T_i \cap \{g(0), \ldots, g(m)\} = \emptyset$, then let $g(m + 1) = v_j$. Notice that we have $\neg E(v_j, g(a))$ for all $a \leq m$ since for each $g(a)$ with $a \leq m$, $g(a) \notin T_i$.

Case 2: If $T_i \cap \{g(0), \ldots, g(m)\} \neq \emptyset$, then let $\emptyset \neq A = T_i \cap \{g(0), \ldots, g(m)\}$. By induction, we will have that A is connected. Let $\sigma = P(A, v_j)$ (i.e., let σ be the unique path from A to v_j), and say $|\sigma| = n$. Let $v_{k_1} = \sigma(1)$, $v_{k_2} = \sigma(2), \ldots,$

$v_{k_{n-1}} = \sigma(n-1) = v_j$. Now set $g(m+1) = v_{k_1}$, $g(m+2) = v_{k_2}, \ldots, g(m+n-1) = v_{k_{n-1}}$. Notice that $\sigma(0) \in A$, so we already have $\sigma(0) = g(k)$ for some $k \leq m$, and that $T_i \cap \{g(0), \ldots, g(m+n+1)\}$ remains connected. In addition, $E(v_{k_{n-2}}, v_j)$ holds, but $\neg E(x, v_j)$ holds for each $x \in P(A, v_j) \cup T_i$ with $x \neq v_{k_{n-2}}$.

The following picture represents a possible scenario for Case 2. Note that if two vertices are within the same dotted-line closed curve, that there may be connections (or some type of path) between them. Of course, no cycles allowed!

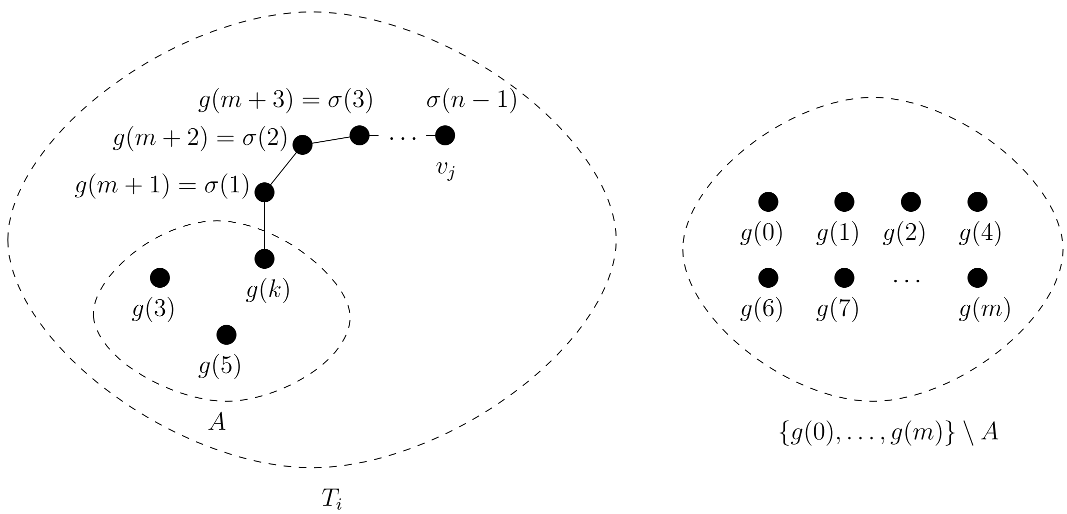

Fig. 1.1: Case 2 when G is a forest with finitely many components

We claim that the function $g : \mathbb{N} \to V$ we just built must be a 2-order. Suppose it is not. Then

$$(\exists i)(\exists j, k < i)[j \neq k \wedge E(g(i), g(j)) \wedge E(g(i), g(k))].$$

Run the construction of g up to i and decide which case we were in when we defined $g(i)$. If we were in Case 1, then by above we have $\neg E(g(i), g(j))$ and

$\neg E(g(i), g(k))$, a contradiction. If we were in Case 2, then $g(i) = \sigma(\ell)$ for some $1 \leq \ell \leq n-1$ (we can assume that $g(i) \notin X$ because we are looking at the stage at which $g(i)$ is defined, and if $g(i)$ were already in X, then it would have been defined at an earlier stage). Then $E(g(i-1), g(i))$ and $\neg E(g(\ell), g(i))$ for $\ell < i-1$, so that either $E(g(i), g(j))$ and $\neg E(g(i), g(k))$, or $\neg E(g(i), g(j))$ and $E(g(i), g(k))$, again a contradiction. Thus g is a 2-order of G. ***Qed***

As a special case of a more general result which we will prove in Chapter 3, we prove that ACA_0 suffices to show $\mathrm{Col}^S_\omega(G) \leq 2$ where G is a disjoint union of infinitely many trees. Later, we will give a reversal to show that ACA_0 is actually necessary for this result.

Theorem 1.3.16. (ACA_0) If $G = (V, E)$ is a disjoint union of infinitely many trees, then $\mathrm{Col}^S_\omega(G) \leq 2$.

Proof. Assume ACA_0 and let $G = (V, E)$ be a disjoint union of infinitely many trees, where $V = \{v_0, v_1, v_2, \ldots\}$. We proceed in a similar way as in the proof of the preceding theorem, except that we do not have a finite set of component representatives, since G is a disjoint union of infinitely many trees.

Using Σ^0_1 comprehension, we are able to define the transitive closure of the edge relation E by

$$\mathrm{tr\ cl}(E) := \{(u, v) : u = v \vee \exists \sigma \in \mathrm{Path}^{u,v}_G\}.$$

Since we now have the relation $\mathrm{tr\ cl}(E)$ at our disposal, we are able to tell whether two vertices u and v are in the same component. The vertices $v_i \in V$ and $v_j \in V$ are in the same component of $G \iff$ tr cl$(E)(v_i, v_j)$.

Define the function $f : \mathbb{N} \to \mathbb{N}$ by

- $f(0) = 0$,
- $f(i+1) = j$, where j is least such that $(\forall k \leq i)[\neg \text{tr cl}(E)(v_j, v_{f(k)})]$.

Intuitively, the function f is naming a sequence of indices of component representatives of G. From the perspective of Computability Theory, we think of f as doing the following.

1. $f(0)$ names v_0 as the first representative. Set counter $n = 1$.
2. f asks whether v_n is in the same component as any of the vertices $v_{f(0)}, \ldots, v_{f(i)}$, where $i \leq n$ is greatest such that $f(i)$ is defined.
   - If it is not, it outputs $f(i+1) = n$. Increment n and go to 2.
   - If it is, increment n and go to 2.

We think of f as continually guessing about what the next component representative will be. Since G has infinitely many components, f will always eventually find the answer. In Reverse Mathematics, this procedure is captured by the definition of $f(i+1)$ as given above.

Now define the sequence of trees given by $T_i = \{v \in V : \text{tr cl}(E)(v_{f(i)}, v)\}$. We think of T_i as giving us a complete list of components of G, as the function f gave us a complete list of representatives.

We now define the function $g : \mathbb{N} \to V$ giving a 2-order of V of strong ω-type exactly as in the proof of Theorem 1.3.15. ***Qed***

1.4 Summary of Results

In Chapter 2, I examine computability theoretic and reverse mathematical aspects of the linear order coloring number, most of which are related to the union of forests Theorem 3.1.3 of Komjáth and Milner, which is stated as the following.

Theorem 1.4.1. (Komjáth, Milner) [6] If a graph G is a union of $n < \omega$ forests, then $\mathrm{Col}(G) \leq 2n$.

First we show Theorem 2.1.1, which is stated as the following.

Theorem 1.4.2. (WKL_0) For any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq 2$.

Of course, showing that a given theorem T is provable using WKL_0 does not exclude the existence of a simpler proof that goes through in a less powerful system (such as RCA_0, for example). Therefore a reversal of some sort is necessary. Frequently, we do not proceed directly to the reversal; usually it is somewhat easier at first to find a computable counterexample to T. That is, we show that T does not hold in REC, the minimal ω-model of RCA_0 consisting of all recursive (or computable) subsets of ω.[1] Note that there are many ω-models of RCA_0.

[1] Note the important distinction between ω and $\mathbb{N}$. Within ACA_0, the set $\mathbb{N}$ denotes the

Finding a counterexample to T in REC suffices to show that RCA_0 is not strong enough to prove the theorem since $REC \models \mathsf{RCA}_0$. This counterexample will usually give us insight about what a reversal might look like. That was certainly the case when we proved Theorems 2.2.1 and 2.2.2, the latter of which is stated as the following.

Theorem 1.4.3. For any fixed $k \in \omega$, there is a computable forest G such that no computable linear ordering realizes $\mathrm{Col}_{LO}(G) \leq k$.

We also have the following corollary.

Corollary 1.4.4. There is a computable forest $G = (V, E)$ such that no computable linear ordering realizes $\mathrm{Col}_{LO}(G) \leq k$ for any $k \in \omega$.

The computable counterexample as given by the above theorem led us to find a proof for the backwards direction, or the reversal of, Theorem 2.1.1, which is a special case of the forward direction of the more general result of Theorem 2.3.3, which is stated as the following.

Theorem 1.4.5. (RCA_0) For any $k \in \mathbb{N}$, $k \geq 2$, the following are equivalent.

1. WKL_0

2. For any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq k$.

unique set X which is defined by $(\forall n)[n \in X]$. We could therefore conceivably have $\mathbb{N}$ be some nonstandard version of what we consider to be the natural numbers. We use ω to denote the set of "natural numbers" in the sense of the metatheory over which we are working.

In Chapter 3, I investigate similar questions concerning the strong and weak ω-coloring numbers of forests. We begin with an exposition of the classical proof of Theorem 3.1.3 of Komjáth and Milner. We then go on to prove Theorem 3.2.1, stated as the following.

Theorem 1.4.6. (RCA_0) the following are equivalent:

1. ACA_0
2. For any forest $G = (V, E)$, the set $\{A \subseteq V : A \text{ is good}\}$ exists.
3. For any $k \in \mathbb{N}$, $k \geq 2$, and any forest $G = (V, E)$, $\mathrm{Col}_\omega^S(G) \leq k$.

The motivation behind the proof of this theorem comes from first seeing that ACA_0 is required to know the set of good subsets of the vertex set of an arbitrary forest exists.

We go on to prove Theorem 3.2.2, stated as the following.

Theorem 1.4.7. There is a computable forest $G = (V, E)$ such that

$$REC \models \mathrm{Col}_{LO}(G) \leq 2 \text{ and } REC \models \mathrm{Col}_\omega^W(G) \leq 2$$

but $REC \not\models \mathrm{Col}_\omega^S(G) \leq k$, for any $k \in \omega$. That is, $REC \models \mathrm{Col}_\omega^S(G) = \omega$.

This theorem shows us that the strong ω coloring number is indeed a strong definition, at least computably speaking. It provides us an example of a computable forest G such that we can make the computable linear order coloring number of

G small (i.e., less than or equal to 2), but computable strong ω-coloring number infinite (in fact, as we show, we can even make the computable weak ω-coloring number of G less than or equal to 2 as well).

The next theorem we prove is a result related to the weak coloring number.

Theorem 1.4.8. (RCA_0) The following are equivalent:

1. ACA_0
2. For any forest $G = (V, E)$, $\mathrm{Col}^W_\omega(G) \leq 2$.

It would be interesting if we could prove some sort of reversal for the preceding theorem for $k \geq 2$.

In Chapter 4 I investigate the proof theoretic strength of a theorem of Erdös and Hajnal for two notions of coloring number. This theorem is stated as the following.

Theorem 1.4.9. (RCA_0) The following are equivalent.

1. ACA_0
2. For all graphs G and all $n \geq 2$, if every finite subgraph H of G has $\mathrm{Col}_{LO}(H) \leq n$, then $\mathrm{Col}^S_\omega(G) \leq 2n - 2$.
3. For all graphs G and all $n \geq 2$, if every finite subgraph H of G has $\mathrm{Col}_{LO}(H) \leq n$, then $\mathrm{Col}^W_\omega(G) \leq 2n - 2$.

To find similar reverse mathematics results relating to chromatic number, I would direct the reader to the paper of Gasarch and Hirst [3].

Finally, the appendix includes useful classical examples of computable graphs due to Erdös and Hajnal. These examples provide us a way to differentiate between the classical notions of linear order coloring number and ω-coloring number (or just coloring number).

Chapter 2

Linear Order Coloring Number

2.1 Upper Bound

Theorem 2.1.1. (WKL_0) For any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq 2$.

Proof. Assume WKL_0. Let $G = (V, E)$ be a forest with $V = \{v_0, v_1, v_2, \dots\}$. Let $T \subseteq \omega^{<\omega}$ be a bounded tree defined by

$$\sigma \in T \iff (\forall n < |\sigma|)[\sigma(n) \leq n + 1].$$

Now define a function

$$f : T \to \{\text{orderings of elements of } \mathrm{Fin}_V\}$$

in the following way.

- Let $f(\emptyset) =$ the ordering of $\{v_0\}$ given by $v_0 \leq v_0$.
- Let $f(\sigma * k) =$ the ordering of $\{v_0, \dots, v_{|\sigma|}, v_{|\sigma|+1}\}$ which agrees with the ordering defined by $f(\sigma)$ on $\{v_0, \dots, v_{|\sigma|}\}$ and inserts $v_{|\sigma|+1}$ into the k-th position in the ordering defined by $f(\sigma)$.

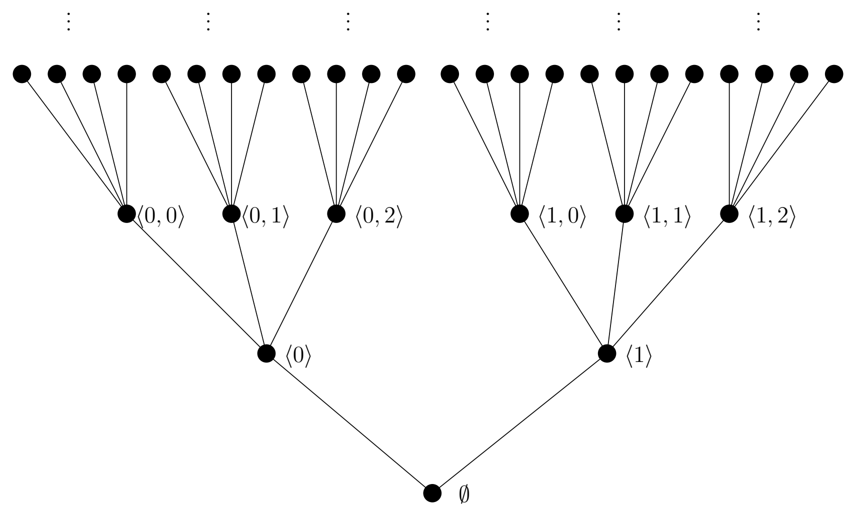

Fig. 2.1: The tree T

(As a bit of notation, define $\leq_\sigma := f(\sigma)$.)

For example, $f(\langle 0\rangle)$ = the ordering defined by $f(\emptyset)$ with v_1 inserted into the 0-th position, i.e. $f(\langle 0\rangle)$ = the ordering given by $v_1 \leq v_0$, while $f(\langle 1\rangle)$ = the ordering given by $v_0 \leq v_1$. We have that $f(\langle 0, 1\rangle)$ tells us to take the ordering given by $v_1 \leq v_0$ and put v_2 into the 1 position, obtaining $v_1 \leq v_2 \leq v_0$. The figure shows what the tree T with labels given by $f(\sigma) = \leq_\sigma$ looks like.

Here is a property of $\leq_\sigma$, and a definition.

1. $\sigma \subseteq \tau \implies \leq_\tau \restriction \{v_0, \ldots, v_{|\sigma|}\} = \leq_\sigma$.

2. If g is an infinite path in T, then we define $\leq_g$ by

$$x \leq_g y \leftrightarrow (\exists \sigma \in T)[\sigma \subset g \wedge x \leq_\sigma y].$$

Property 1 is clear from the definition of $\leq_\tau$. We show that if g is an infinite path

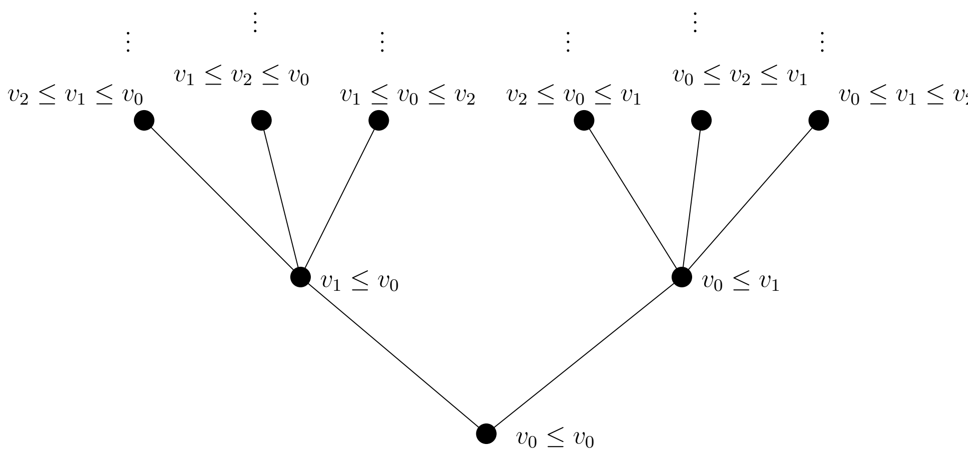

Fig. 2.2: The tree T with labels given by $f(\sigma)$

in T, then $\leq_g$ defines a linear order on V by verifying the axioms of a linear order.

1. Let $x, y \in V$ and suppose that $x \leq_g y$ and $y \leq_g x$. Then let $\sigma \subset g$ be such that $x \leq_\sigma y$ and let $\tau \subset g$ be such that $y \leq_\tau x$. Since both σ and τ are initial segments of g, assume (without loss of generality) that $\sigma \subseteq \tau$. Note that $x, y \in \{v_0, \ldots, v_{|\sigma|}\}$, otherwise $x \leq_\sigma y$ would be undefined. Then by property 1, $\leq_\tau \restriction \{v_0, \ldots, v_{|\sigma|}\} = \leq_\sigma$. Thus $x \leq_\tau y$, and since $\leq_\tau$ satisfies the axioms of linear order on its domain $\{v_0, \ldots, v_{|\tau|}\}$, we have that $x = y$.

2. Let $x, y, z \in V$ and suppose that $x \leq_g y$ and $y \leq_g z$. Fix $\sigma, \tau \subset g$ such that $x \leq_\sigma y$ and $y \leq_\tau z$. As before, assume $\sigma \subseteq \tau$. Note that $x, y, z \in \{v_0, \ldots, v_{|\tau|}\}$. Again, by property 1, $\leq_\tau \restriction \{v_0, \ldots, v_{|\sigma|}\} = \leq_\sigma$. Thus $x \leq_\tau y$, and since $\leq_\tau$ satisfies the axioms of linear order on its domain, we have that $x \leq_\tau z$. Thus there is a $\tau \in T$ such that $\tau \subset g$ and $x \leq_\tau z$, and therefore, by the definition of $\leq_g$, we have that $x \leq_g z$.

3. Let $x, y \in V$ and suppose $x \not\leq_g y$. We need to show $y \leq_g x$. Since $x \not\leq_g y$, we have, by definition, that $(\forall \sigma \in T)[\sigma \not\subset g \vee x \not\leq_\sigma y]$ holds. Suppose $x = v_s$ and $y = v_t$. Let $m = \max\{s, t\}$. Let $\sigma \in T$ be such that $|\sigma| = m$ and $\sigma \subset g$. Then we must have $x \not\leq_\sigma y$. Since $\leq_\sigma$ satisfies the axioms of linear order on its domain $\{v_0, \ldots, v_{|\sigma|}\}$, we have that $y \leq_\sigma x$. Thus, by definition, we have $y \leq_g x$.

Now we define another tree $S \subseteq T$ such that

$$\sigma \in S$$

$$\iff \text{ the ordering } \leq_\sigma \text{ on } \{v_0, \ldots, v_{|\sigma|}\} \text{ given by } f(\sigma) \text{ is a 2-order.}$$

Formally, S is defined using Σ^0_0 comprehension by

$$\begin{aligned} \sigma \in S \quad \leftrightarrow \quad & (\forall n < |\sigma|)(\neg \exists i \neq j < |\sigma|) \\ & [v_i \leq_\sigma v_n \wedge v_j \leq_\sigma v_n \wedge E(v_i, v_n) \wedge E(v_j, v_n)]. \end{aligned}$$

Since T is a bounded tree, we must also have that S is a bounded tree. We wish to show that S is infinite.

Lemma 2.1.2. (RCA_0) Every finite forest F has $\mathrm{Col}_{LO}(F) \leq 2$.

Proof of lemma. Fix a finite forest $F = (V_F, E_F)$. Suppose $V_F = \{v_0, \ldots, v_k\}$. To define a 2-order on F, first let $X = \{x_0, \ldots, x_j\}$ be a finite set of component representatives, and proceed as in the proof of Theorem 1.3.15. This proves the lemma.

By Lemma 2.1.2, S is infinite, and by WKL_0, S has a path. Let g be such a path in S. We verify that $\leq_g$ is a 2-order.

Suppose g is not a 2-order. Then there are distinct i, j, k such that $(v_i \leq_g v_k) \wedge (v_j \leq_g v_k) \wedge E(v_i, v_k) \wedge E(v_j, v_k)$. Let $\sigma \in S$ be a witness to $(v_i \leq_g v_k) \wedge (v_j \leq_g v_k)$, (i.e., σ is a witness to both $(v_i \leq_g v_k)$ and $(v_j \leq_g v_k)$—as we discussed before, we can use the single string σ to witness both inequalities). Thus $v_i \leq_\sigma v_k$ and $v_j \leq_\sigma v_k$, but this is a contradiction, as $\sigma \in S$ implies that $\leq_\sigma$ is a 2-order on V. ***Qed***

2.2 Computable Counter-examples

The following theorem shows that RCA_0 does not prove that for any forest G, $\mathrm{Col}_{LO}(G) \leq 2$, as this statement fails in REC.

Theorem 2.2.1. There is a computable forest G such that no computable linear order realizes $\mathrm{Col}_{LO}(G) \leq 2$.

Proof. Throughout the following construction I will present as clearly as possible the intuition, using pictures, needed to understand it completely. The ideas and gadgets used for this proof are fundamental for the understanding of much of what follows, therefore it would be prudent to make those ideas perfectly clear with at times a somewhat informal discussion.

We diagonalize against all partial computable functions $\{\varphi_e : e \in \mathbb{N}\}$ (which may or may not actually be linear orderings) to construct our forest G. To do this, we satisfy infinitely many requirements

$$\mathcal{R}_e : \varphi_e \text{ is not a 2-order of } V.$$

As notation, we define $\leq_e$ to be the order (if there is one) given by φ_e. Clearly, if $\mathcal{R}_e$ is satisfied for all $e < \omega$, then no computable linear order realizes $\mathrm{Col}_{LO}(G) \leq 2$. Let $e \in \mathbb{N}$ be fixed. The strategy to meet a single requirement $\mathcal{R}_e$ is given below as setting up and springing a trap for φ_e. To defeat φ_e, we do the following. Place three vertices in our graph u_0, u_1, u_2 with no edges between them, then wait for the opponent φ_e to order them.

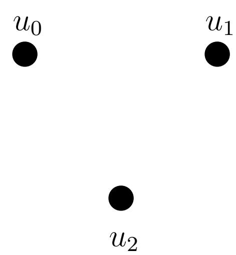

Fig. 2.3: Trap for φ_e

If he never orders them (i.e., we wait for φ_e forever), then we defeat φ_e trivially (i.e. $\mathcal{R}_e$ is satisfied), as he is not an ordering of V at all. Suppose that in the case that he does order them, we have $u_0 <_e u_1 <_e u_2$, without loss of generality. Once he has decided on this ordering, he may never change his mind, as he is computable. Then we spring the trap! We place two more vertices in the graph a and b and add the edges $E(u_0, a), E(a, u_2), E(u_2, b), E(b, u_1)$. (In general, we make the vertex that φ_e made the $\leq_e$-largest centrally located within the gadget).

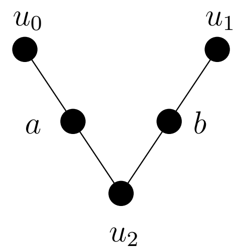

Fig. 2.4: Sprung trap for φ_e

Then it is our opponent's turn to place a, b in the $\leq_e$ order. (We think of φ_e, the alleged computable 2-ordering of V, as our opponent, for it is he against whom we are trying to diagonalize.)

Case 1: He places either a or b higher than u_2, in which case we have

$$u_0 <_e u_1 <_e u_2 <_e a \quad \textbf{or} \quad u_0 <_e u_1 <_e u_2 <_e b.$$

(In the first case it does not matter where b is and in the second case it does not matter where a is.) Then we win ($\mathcal{R}_e$ is satisfied), as a is connected to u_0 and u_2 (while they are lower in the ordering), or b is connected to u_2 and u_1, while they are lower in the ordering, in either case. So $\leq_e$ is not a 2-order.

Case 2: He places both a and b lower than u_2, in which case we have something like

$$u_0 <_e a <_e u_1 <_e b <_e u_2.$$

Then u_2 is connected to both a and b, which are lower in the ordering than u_2. Again, $\leq_e$ is not a 2-order, and we win! ($\mathcal{R}_e$ is satisfied).

We dovetail this process for each φ_e, and notice that the graph we produce is a forest since we never complete a cycle, and computable since we do not later change our minds by connecting vertices at a stage later than the stage by which they have been placed in the graph. Furthermore, notice that there is no interaction between the requirements, as we have dedicated to each φ_e its own set of three vertices, and it does not matter how φ_e orders the vertices that are not dedicated to it, so $\mathcal{R}_e$ causes no injury to the other requirements.

Formally, the construction runs as follows.

At stage $s = 0$, let $u_0^e = 6e, u_1^e = 6e + 2$ and $u_2^e = 6e + 4$.

At stage $s > 0$, we do the following. We say that $\mathcal{R}_e$ is *active* if φ_e orders $u_0^e = 6e, u_1^e = 6e + 2, u_2^e = 6e + 4$. Let $i_e \in \{0, 1, 2\}$ be such that $u_{i_e}^e$ is largest in the ordering given by φ_e. Now let $e \leq s$ be the least number such that $\mathcal{R}_e$ is active but not yet satisfied. Then let a_e and b_e be the least odd numbers not used so far in the construction. Make the connections

$$E(u_{i_e}^e, a_e) \wedge E(u_{i_e}^e, b_e) \wedge E(a_e, u_{i_e+1}^e) \wedge E(b_e, u_{i_e+2}^e),$$

where the addition is done modulo 3. This ends the construction.

We claim that all the requirements are satisfied. Suppose for a contradiction that there is some requirement $\mathcal{R}_e$ that is not satisfied. Run the above construction and wait for $\mathcal{R}_e$ to be active. If we wait forever, then $\mathcal{R}_e$ is satisfied trivially as φ_e never orders $u_0^e = 6e, u_1^e = 6e + 2$ and $u_2^e = 6e + 4$. Once $\mathcal{R}_e$ is active, we have $u_{i_e}^e, u_{i_e+1}^e, u_{i_e+2}^e$, where $u_{i_e}^e$ is largest in the ordering given by φ_e (and the addition is done modulo 3). We then add odd numbers a_e and b_e, and edge connections

$$E(u^e_{i_e}, a_e) \wedge E(u^e_{i_e}, b_e) \wedge E(a_e, u^e_{i_e+1}) \wedge E(b_e, u^e_{i_e+2}).$$

As we argued before, since we have these connections, the ordering given by φ_e is not a 2-order, which is a contradiction. Thus $\mathcal{R}_e$ is satisfied, and we are done. ***Qed***

Theorem 2.2.2. For any fixed $k \in \omega$, there is a computable forest G such that no computable linear ordering realizes $\mathrm{Col}_{LO}(G) \leq k$.

Proof. For fixed k, the proof is similar to that of the above proof for $k = 2$. The gadget we use starts off with $k^2 - k + 1$ vertices that are unconnected by edges. Then the trap that we spring is introducing k many new vertices, connecting them with the other vertices in a way that ensures φ_e does not witness the linear order coloring number of G being at most k.

The following pictures illustrate a trap and sprung trap for $k = 3$ and $k = 4$. Note that in each case we place as the central vertex whichever vertex (among the vertices from the original trap) φ_e puts greatest in its ordering. To explain why the trap for $k = 3$ works, suppose that φ_e orders u_6 largest in the ordering that it gives. As for the when $k = 2$, there are two possibilities. The first is that after we put the vertices b_0, b_1 and b_2 in the graph, φ_e places some b_i with $0 \leq i \leq 2$ larger than u_6. Without loss of generality suppose it is b_0. But then φ_e would fail to be a 3-order of G as b_0 is connected to u_0, u_1 and u_6 while it is larger than them in its ordering. The only other case is that all of the vertices b_0, b_1 and b_2 are below u_6. But then φ_e would again fail to be a 3-order, as there are 3 elements connected to u_6 that are lower than it in its ordering. The traps for $k = 4$ and higher work similarly.

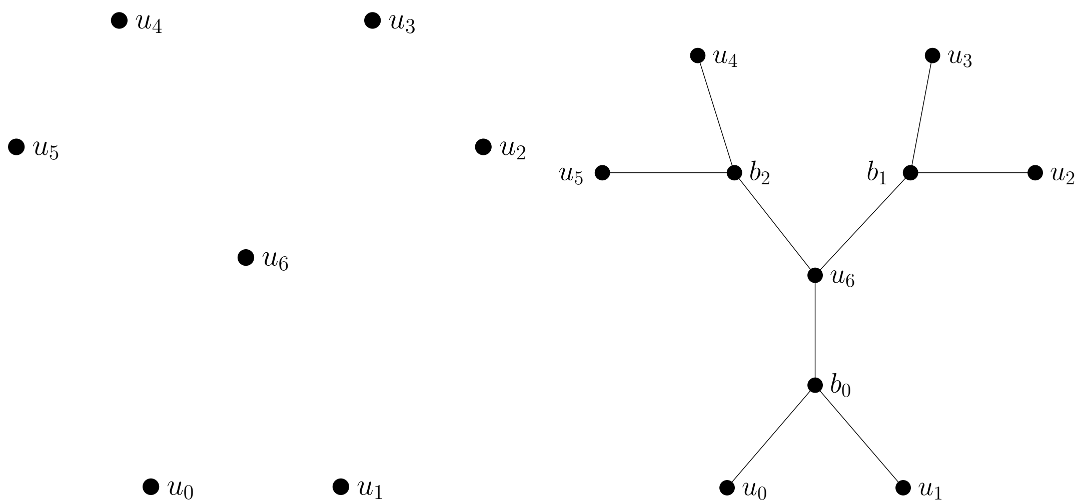

Fig. 2.5: Trap and sprung trap for $k = 3$

Qed

In fact, Theorem 2.2.2 can be modified to show that there is a single computable forest G that works for any $k < \omega$ (we could say that the computable linear order coloring number of G is ω).

Corollary 2.2.3. There is a computable forest $G = (V, E)$ such that no computable linear ordering realizes $\mathrm{Col}_{LO}(G) \leq k$ for any $k \in \omega$.

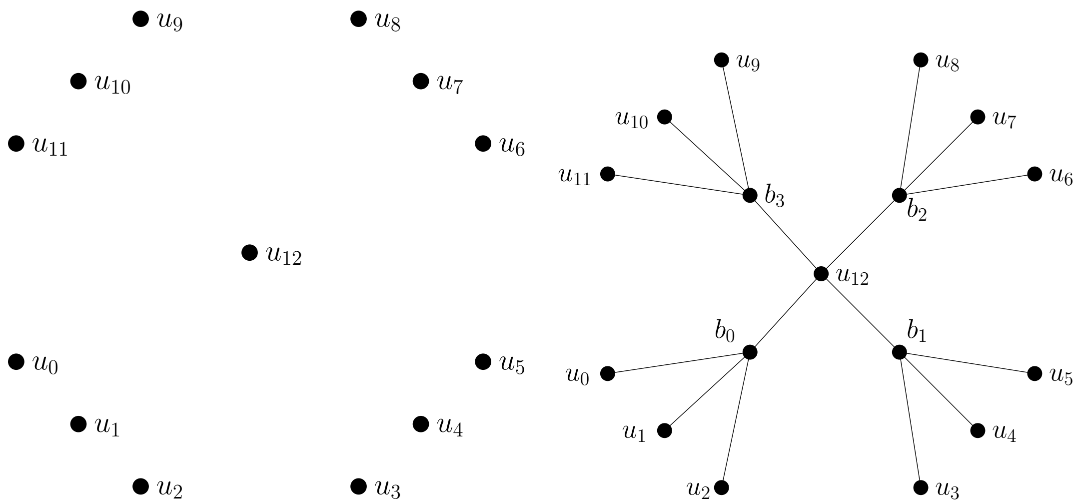

Fig. 2.6: Trap and sprung trap for $k = 4$

Proof. The requirements for the construction would look like

$$\mathcal{R}_{\langle e,k\rangle} : \varphi_e \text{ is not a } k\text{-order of } V.$$

To satisfy a single requirement $\mathcal{R}_{\langle e,k\rangle}$, we spring a trap for φ_e and k as described in Theorem 2.2.2. Since this requirement will be satisfied for each $k \in \omega$, we have ensured that φ_e does not witness any finite linear order coloring number.

More intuitively, we can think of creating the desired computable forest G by taking an effective disjoint union, over all $k \in \omega$, of the forests from Theorem 2.2.2. ***Qed***

2.3 Reverse Math Results

The following lemma will be extremely useful to us in the proof of the theorem that follows.

Lemma 2.3.1. (Lemma 2.2 from Schmerl [7]) *Let $2 \leq n \in \omega$. Then the following statement is provable from $\mathsf{RCA}_0 + \neg\, \mathsf{WKL}_0$: There are pairwise disjoint Σ^0_1 subsets $A_0, A_1, \ldots, A_{n-1} \subseteq \mathbb{N}$ such that whenever $f : \mathbb{N} \to n$ is a function, there is x such that $x \in A_{f(x)}$.*

Theorem 2.3.2. (RCA_0) The following are equivalent.

1. WKL_0

2. For any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq 2$.

Proof. First notice that we already proved $(1 \to 2)$ as Theorem 2.1.1.

$(2 \to 1)$ We work in RCA_0. Assume that for any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq 2$. In other words, we assume that for any forest G, there is a 2-order of G.

It is sufficient to prove the negation of the statement in Lemma 2.3.1. We will use the formulation the lemma for $n = 3$. That is, we will end up showing that for all pairwise disjoint Σ^0_1 subsets $A_0, A_1, A_2 \subseteq \mathbb{N}$ there is a function $f : \mathbb{N} \to 3$ such that for all x, $x \notin A_{f(x)}$. We realize that since we are working over RCA_0, we cannot actually talk about Σ^0_1 sets as if they exist, since they might not. Talking about them as sets in this context is really shorthand for talking about the corresponding collections of numbers defined by Σ^0_1 formulas.

Fix Σ^0_1 formulas

$$(\exists s)[\varphi_0(x, s)]$$

$$(\exists s)[\varphi_1(x, s)]$$

$$(\exists s)[\varphi_2(x,s)]$$

which are disjoint. That is, for each $0 \leq i < 3$, we have

$$(\forall x)[\exists s\varphi_i(x,s) \to (\neg\exists s\varphi_{i+1}(x,s) \wedge \neg\exists s\varphi_{i+2}(x,s))]$$

(The addition in the subscripts for the formula above is done modulo 3.) We think of the formulas above as corresponding to pairwise disjoint Σ^0_1 sets $A_0, A_1, A_2 \subseteq \mathbb{N}$, respectively, from Lemma 2.3.1.

We define the graph $G = (V, E)$ in the following way. Let the set of vertices V be defined by

$$V := \{u^i_x : 0 \leq i < 3, x \in \mathbb{N}\} \cup \{a_{\langle x,s\rangle} : x, s \in \mathbb{N}\} \cup \{b_{\langle x,s\rangle} : x, s \in \mathbb{N}\}.$$

Let the edge relation E be defined in the following way. For $0 \leq i < 3$, $x, s \in \mathbb{N}$:

$$E(u^i_x, a_{\langle x,s\rangle}) \wedge E(u^i_x, b_{\langle x,s\rangle}) \wedge E(u^{i+1}_x, a_{\langle x,s\rangle}) \wedge E(u^{i+2}_x, b_{\langle x,s\rangle})$$

$$\iff \varphi_i(x,s) \wedge (\forall t < s)[\neg\varphi_i(x,t)],$$

where the addition $i+1$ and $i+2$ is modulo 3.

The associated picture will aid the reader in seeing exactly what the edge connections look like in the graph G.

We see that the edge relation E is definable in RCA_0, as only bounded quantifiers were used in its definition.

We can see that if $\leq_V$ witnesses $\mathrm{Col}_{LO}(G) \leq 2$ and $(\exists s)[\varphi_i(x,s)]$ holds, then

$$u^i_x \neq \max\{u^0_x, u^1_x, u^2_x\}$$

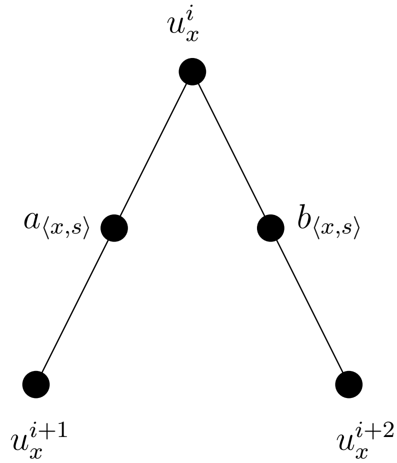

Fig. 2.7: The edge connections in G for fixed $0 \le i < 3$, $x, s \in \mathbb{N}$

where the maximum is taken relative to $\le_V$. (As we explained in the proof of Theorem 2.2.1.)

Now we define the function $f : \mathbb{N} \to 3$ by $f(x) = i$, where $u_x^i = \max\{u_x^0, u_x^1, u_x^2\}$, and the maximum is taken in the order $\le_V$. Then, since $u_x^i \ne \max\{u_x^0, u_x^1, u_x^2\}$, and $(\exists s)[\varphi_i(x, s)]$ holding corresponds to (in the sense of Lemma 2.3.1) x entering (or already being in) the Σ_1^0 set A_i at stage s, we see that, for all $x \in \mathbb{N}$, $x \notin A_{f(x)}$, and we are done. ***Qed***

Theorem 2.3.3. For any $k \in \omega$ such that $k \ge 2$, RCA_0 proves that the following are equivalent.

1. WKL_0

2. For any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \le k$.

Proof. Notice that we have already proved the theorem for $k = 2$ as Theorem

2.3.2. It will not be difficult to modify that proof to get the result for any $k \in \omega$ with $k \geq 2$.

$(1 \to 2)$

As noted above, by Theorem 2.3.2, we have in WKL_0 that for any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq 2$. Thus it is clear that for any forest $G = (V, E)$, $\mathrm{Col}_{LO}(G) \leq k$ also holds in WKL_0 for $k \geq 2$.

$(2 \to 1)$

By Schmerl's lemma it suffices to show that for all pairwise disjoint Σ^0_1 subsets $A_0, A_1, \ldots, A_{k^2-k} \subseteq \mathbb{N}$, there is a function $f : \mathbb{N} \to k^2 - k + 1$ such that $(\forall x)[x \notin A_{f(x)}]$. Again, we realize that the collections above we call sets do not necessarily exist as sets in RCA_0.

Fix Σ^0_1 formulas

$$(\exists s)[\varphi_0(x, s)], (\exists s)[\varphi_1(x, s)], \ldots, (\exists s)[\varphi_{k^2-k}(x, s)]$$

which are disjoint. That is, for each $0 \leq i < k^2 - k + 1$, we have

$$(\forall x)[(\exists s)\varphi_i(x, s) \to \bigwedge_{0 \leq \ell < k^2-k+1, \ell \neq i} \neg(\exists s)\varphi_\ell(x, s)].$$

We think of the formulas above as corresponding to pairwise disjoint Σ^0_1 sets $A_0, A_1, \ldots, A_{k^2-k} \subseteq \mathbb{N}$, respectively, from Lemma 2.3.1.

We define the graph $G = (V, E)$ in the following way. Let the set of vertices V be defined by

$$V := \{u^i_x : 0 \leq i < k^2 - k + 1, x \in \mathbb{N}\} \cup \bigcup_{0 \leq i < k} \{a^i_{\langle x,s \rangle} : x, s \in \mathbb{N}\}.$$

Let the edge relation E be defined in the following way. For $0 \leq i < k^2 - k + 1$, $x, s \in \mathbb{N}$:

$$\bigwedge_{0 \leq \ell \leq k} E(u_x^i, a_{\langle x,s \rangle}^{\ell}) \wedge \bigwedge_{0 \leq j < k} \left(\bigwedge_{\ell = i + j(k-1)+1}^{i+(j+1)(k-1)} E(u_x^{\ell}, a_{\langle x,s \rangle}^{j}) \right)$$
$$\iff \varphi_i(x, s) \wedge (\forall t < s)[\neg \varphi_i(x, t)].$$

where all of the addition and multiplication is done modulo $k^2 - k + 1$.

We have included pictures, for the cases when $k = 3$ and $k = 4$, which will aid the reader in seeing exactly what the edge connections look like in the graph G for values of k larger than 2 (note the similarity to the gadgets we used in the proof of Theorem 2.2.2).

We see that the edge relation E is definable in RCA_0, as only bounded quantifiers were used in its definition.

We can see that if $\leq_V$ witnesses $\mathrm{Col}_{LO}(G) \leq k$ and $(\exists s)[\varphi_i(x, s)]$ holds, then

$$u_x^i \neq \max \{u_x^j : 0 \leq j < k^2 - k + 1\},$$

where the maximum is taken relative to $\leq_V$. (As we explained in the proof of Theorem 2.2.2.)

Now we define the function $f : \mathbb{N} \to k^2 - k + 1$ by

$$f(x) = i, \text{ where } u_x^i = \max \{u_x^j : 0 \leq j < k^2 - k + 1\},$$

and the maximum is taken in the order $\leq_V$. Then, since

$$u_x^i \neq \max \{u_x^j : 0 \leq j < k^2 - k + 1\},$$

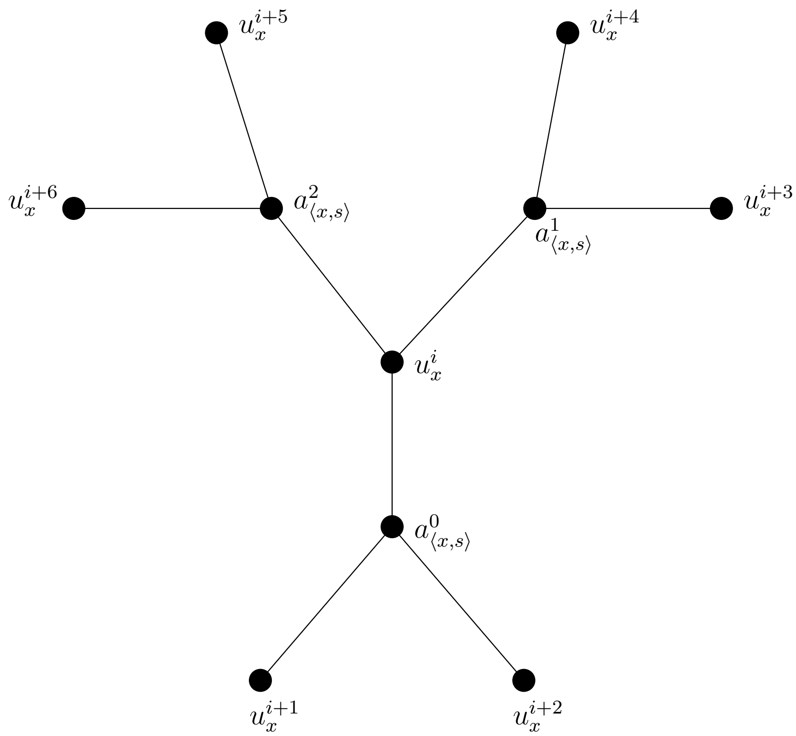

Fig. 2.8: The edge connections in G in the case $k = 3$ for fixed $0 \leq i < 7$, $x, s \in \mathbb{N}$, where any addition is modulo 7

and $(\exists s)[\varphi_i(x, s)]$ holding corresponds to (in the sense of Lemma 2.3.1) x entering (or already being in) the Σ^0_1 set A_i at stage s, we see that, for all $x \in \mathbb{N}$, $x \notin A_{f(x)}$, and we are done.

Qed

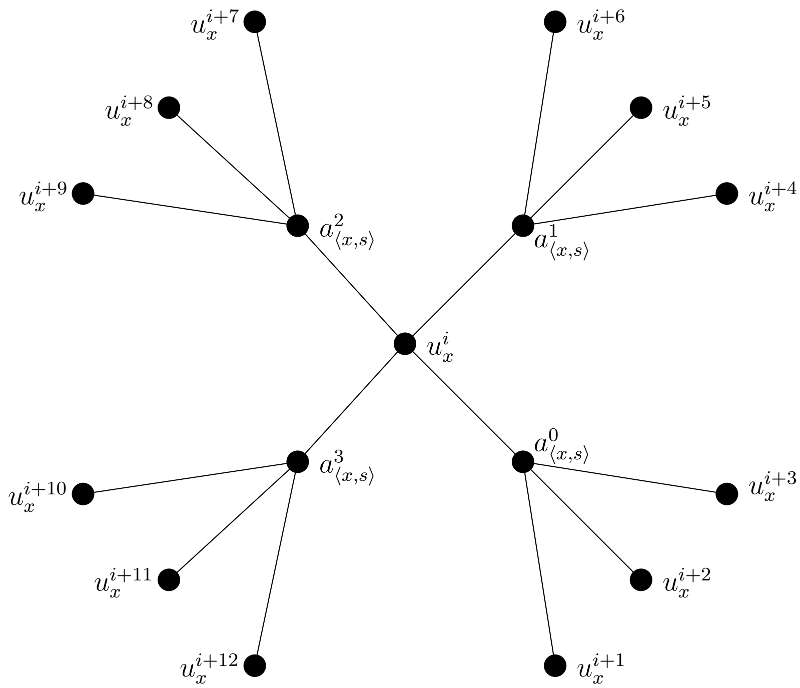

Fig. 2.9: The edge connections in G in the case $k = 4$ for fixed $0 \leq i < 13$, $x, s \in \mathbb{N}$, where any addition is modulo 13

Chapter 3

Strong and Weak ω-Coloring Numbers

3.1 Classical Proof

In this section we give an exposition of the classical proof of a theorem of Komjáth and Milner, but first we need a couple definitions. We keep in mind throughout the proof that we want the theorem to be provable in ACA_0.

Definition 3.1.1. (RCA_0) **(End Extension, [6])** Suppose $A \subseteq V$ is a finite subset of vertices from V, and let $\leq_A$ be an ordering of A. We call an ordering $\leq_B$ on a finite set $B \supset A$ an *end extension* of $\leq_A$ if $\leq_B\restriction_A = \leq_A$ and

$$(\forall a \in A)(\forall b \in B \setminus A)[a \leq_B b].$$

If $A \subseteq V$ is finite and $\leq_A$ is an ordering on A, then we say that $\leq_A$ can be *end extended* to an ordering $\leq_B$ of a finite $B \supset A$ if $\leq_B$ is an end extension of $\leq_A$.

Definition 3.1.2. (Good Subset, [6]) We call a finite subset of vertices $A \subseteq V$ *good* (with parameter k) if every k-ordering of A can be end-extended to a k-

ordering of any finite $B \supseteq A$, i.e.,

$$(\forall \text{ finite } B \supseteq A)(\forall k\text{-orders } \leq_A)(\exists k\text{-order } \leq_B)(\leq_B \text{ is an end extension of } \leq_A).$$

When we call a set good it will usually be understood what the parameter k is from the context.

Notice that the above definition is Π^0_1 (the second and third quantifiers are bounded and the predicate is computable as A and B are finite subsets of V). Thus, for a computable graph $G = (V, E)$, the set of all good subsets of V, $\{A \subseteq V : A \text{ is good}\}$, is Π^0_1.

Theorem 3.1.3. (Komjáth, Milner [6]) *If a graph G is a union of $n < \omega$ forests, then $\mathrm{Col}(G) \leq 2n$.*

Proof. The following is the proof from [6].

Let $n < \omega$ and suppose $G = (V, E)$ is a union of n forests. We will prove by induction on n that for any vertex $a \in V$, there is a $2n$-order of G in order type $|V|$ in which a is the least element.

Case $|V| < \omega$: It suffices to show there is a vertex $x \in V \setminus \{a\}$ with degree $d(x) < 2n$. Note that the *degree* of a vertex x is the number of y such that $E(x, y)$. Then we use the Induction Hypothesis to get a $2n$-order of $V \setminus \{x\}$ with a as the least element. Then we place x as the greatest vertex in the order.

For a contradiction, suppose there is no vertex $x \in V \setminus \{a\}$ with $d(x) < 2n$. Thus

$$(\forall x \in V \setminus \{a\})[d(x) \geq 2n].$$

For any finite graph $G = (V, E)$, it is well known that

$$e(V) = \frac{1}{2} \sum_{g \in V} d(g)$$

where $e(V)$ is the number of edges connecting vertices from V.

Thus by the previous two facts we have

$$e(V) \geq n(|V| - 1) + \frac{1}{2} d(a).$$

But $e(V) \leq n(|V| - 1)$ because G is a union of n forests. Corollary 1.5.3 from [1] states that for a finite tree $T = (V, E)$, $e(V) = |V| - 1$. So it follows that for a forest $F = (V, E)$, $e(V) \leq |V| - 1$. By the previous inequalities, we must have $d(a) = 0$, so a is an isolated vertex. Since $V \setminus \{a\}$ is a union of n forests as well, $e(V \setminus \{a\}) \leq n(|V \setminus \{a\}| - 1) = n(|V| - 2)$. But since a is isolated, $e(V \setminus \{a\}) = e(V)$, so $e(V) \leq n(|V| - 2) < n(|V| - 1)$, which is a contradiction.

Case $|V| = \omega$: It suffices to show any finite subset of V is contained in a good subset. Then we can find an increasing sequence of good subsets $A_0 \subseteq A_1 \subseteq \cdots$ so that $V = \bigcup_{i \in \omega} A_i$ with $a \in A_0$ and we define $2n$-orderings $<_i$ of the A_i so that a is the first element of $<_0$ and $<_{i+1}$ is an end-extension of $<_i$.

For a contradiction, suppose there is a finite set A which has no good extension.

Claim. For any finite subset of vertices $X \subseteq V$, we have

$$n|X| - e(X) \geq 0.$$

Proof of Claim. First consider the case when X is the vertex set of a finite tree

$T = (X, E)$. Then we have

$$e(X) = \frac{1}{2}\sum_{g\in X} d(g) \leq \frac{1}{2}(2|X| - 2) = |X| - 1.$$

So if we made a forest out of the same vertex set X, then it would have fewer edges (being possibly disconnected in places) than the corresponding tree, so we still would have $e(X) \leq |X| - 1$. Now, if X were the vertex set of a union of n forests, then we would have

$$e(X) \leq n(|X| - 1) = n|X| - n$$

Therefore

$$n|X| - e(X) \geq n \geq 0$$

and the claim is proved.

Claim. By extending A if necessary, we may assume A is such that

$$n|A| - e(A) \leq n|B| - e(B)$$

for any finite $B \supseteq A$.

Proof of Claim. Let

$$k = \min\{n|C| - e(C) : A \subseteq C \text{ finite}\}$$

We know there is an $\hat{A}$ such that $A \subseteq \hat{A}$ where $n|\hat{A}| - e(\hat{A}) = k$ because the set $\{n|C| - e(C) : A \subseteq C \text{ finite}\}$ has a minimum. Then for all finite B with $B \supseteq \hat{A}$, we have

$$n|B| - e(B) \geq n|\hat{A}| - e(\hat{A}).$$

Thus the claim is proved.

Let B be minimal such that B witnesses that A is not good. Then there must be an edge between A and $B \setminus A$; otherwise any $2n$-ordering of A followed by any $2n$-ordering of $B \setminus A$ would give an end-extension which is a $2n$-ordering of B. Also, any vertex $x \in B \setminus A$ has $d(x) \geq 2n$ in the subgraph of G induced on B. Otherwise, by the minimality of B, any $2n$-ordering of A can be end-extended to a $2n$-ordering of $B \setminus \{x\}$ and then x can be placed at the end. Counting the edges in B and not in A, we have $e(B) - e(A) = \frac{1}{2}\sum_{b \in B} \hat{d}(b)$, where $\hat{d}(b)$ denotes the number of vertices $y \in B$ such that $E(b, y)$ and at least one of b or y is not in A. Since $\hat{d}(b) = d(b)$ for all $b \in B \setminus A$, we have $\hat{d}(b) \geq 2n$ for all $b \in B \setminus A$. In addition, there is at least one $b \in A$ which is connected to a $y \in B$. Therefore

$$e(B) - e(A) \geq \frac{1}{2}(1 + 2n|B \setminus A|) = \frac{1}{2} + n|B \setminus A|$$

It follows that

$$n|A| - e(A) \geq \frac{1}{2} + n|B \setminus A| + n|A| - e(B)$$

Since $n|B \setminus A| + n|A| = n|B|$, we get $n|A| - e(A) \geq \frac{1}{2} + n|B| - e(B)$, and therefore $n|A| - e(A) > n|B| - e(B)$, which is a contradiction. ***Qed***

3.2 Strong ω-Coloring Number Results

Theorem 3.2.1. (RCA_0) For each $k \in \mathbb{N}$, $k \geq 2$, the following are equivalent:

1. ACA_0

2. For any forest $G = (V, E)$, the set $\{A \subseteq V : A \text{ is good}\}$ exists.

3. For any forest $G = (V, E)$, $\mathrm{Col}^S_\omega(G) \leq k$.

Proof. $(1 \to 2)$ We reason within ACA_0. Let $G = (V, E)$ be a forest. Let $\psi(X)$ be the predicate which states that X is a good subset of V, i.e.,

$$\psi(X) = X \in \mathrm{Fin}_V \wedge (\forall \text{ finite } B \supseteq X)(\forall \text{ 2-order } \leq_X)(\exists \text{ 2-order } \leq_B)$$

$$[\leq_B \text{ is an end extension of } \leq_A]$$

As we previously noted, $\psi(X)$ is a Π^0_1 predicate. Therefore, using arithmetical comprehension, we may define within ACA_0 the set of all good subsets of V as

$$\{A \in \mathrm{Fin}_V : \psi(A)\}.$$

$(1 \to 3)$ This implication follows by a formalization of the classical proof with $k = 2$. The induction when $|V|$ is finite is arithmetical, and ACA_0 can define the collection of good subsets when $|V|$ is infinite. The remainder of the classical proof consists of finitary counting arguments which can clearly be formalized within ACA_0. Since it holds for $k = 2$, it clearly must also hold for $k \geq 2$ as well.

We prove $(3 \to 1)$ and $(2 \to 1)$ simultaneously with a single construction.

Fix a one-to-one function $f : \mathbb{N} \to \mathbb{N}$. Also fix $k \in \mathbb{N}$ with $k \geq 2$. We build a forest $G = (V, E)$ in RCA_0. Let

$$V := \{a^i_n : n \in \mathbb{N}, 0 \leq i < k\} \cup \{c_n : n \in \mathbb{N}\}$$

The only edge relations that hold are $E(c_n, a^i_{f(n)})$ for $0 \leq i < k$ and $n \in \mathbb{N}$. Note that this is equivalent to making connections $E(c_{f^{-1}(m)}, a^i_m)$ for $0 \leq i < k$ and $n \in \mathbb{N}$, where $f(n) = m$. This ends the construction.

The following picture illustrates an example of what the edge connections in G will be if, for instance, $k = 3$ and $f(0) = 3$, $f(1) = 1$, but 0 and 2 are not in the range of f. Notice that in this example, the sets $\{a^0_0, a^1_0, a^2_0\}$ and $\{a^0_2, a^1_2, a^2_2\}$ are good, while the sets $\{a^0_1, a^1_1, a^2_1\}$ and $\{a^0_3, a^1_3, a^2_3\}$ are not good.

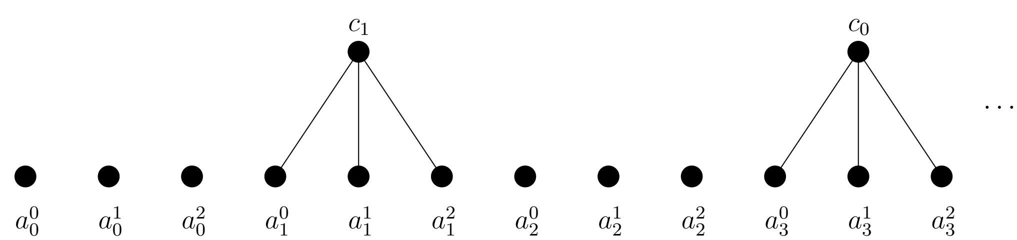

Fig. 3.1: Edge connections in G for $k = 3$ if $f(0) = 3$, $f(1) = 1$, but 0 and 2 are not in the range of f

The next picture illustrates an example of what the edge connections in G will be if, for instance, $k = 4$ and $f(0) = 0$, $f(1) = 2$, but 1 is not in the range of f. Notice that in this example, the set $\{a^0_1, a^1_1, a^2_1, a^3_1\}$ is good, while the sets $\{a^0_0, a^1_0, a^2_0, a^3_0\}$ and $\{a^0_2, a^1_2, a^2_2, a^3_2\}$ are not good.

Then we see that for our graph G,

$$m \in ran(f) \iff \{a^0_m, a^1_m, \dots, a^{k-1}_m\} \text{ is not good.}$$

For if m never appears in the range of f, then we will never connect any of the

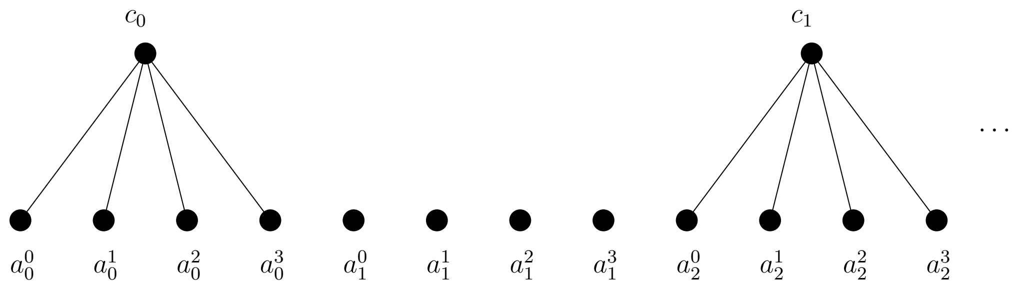

Fig. 3.2: Edge connections in G for $k = 4$ if $f(0) = 0$, $f(1) = 2$, but 1 is not in the range of f

vertices from $\{a_m^i : 0 \leq i < k\}$ to any of the vertices from $\{c_n : n \in \mathbb{N}\}$ (also note that none of the a's are connected by an edge).

Suppose $B \supseteq \{a_m^i : 0 \leq i < k\}$ is a finite extension of $\{a_m^i : 0 \leq i < k\}$. Let $\leq_A$ be a 2-order of $\{a_m^i : 0 \leq i < k\}$ and let $\leq_B$ be a 2-order of B. Define $\leq_B'$ by $\leq_B' \restriction B \setminus \{a_m^i : 0 \leq i < k\} = \leq_B$, $\leq_B' \restriction \{a_m^i : 0 \leq i < k\} = \leq_A$ and $a_m^i \leq_B' b$ for each $b \in B \setminus \{a_m^i : 0 \leq i < k\}$, i.e., $\leq_B'$ is an end-extension of $\leq_A$. We claim that $\leq_B'$ is a 2-order of B. To prove the claim, suppose $\leq_B'$ is not a 2-order of B. Then there is a $b \in B$ and there are $v_0, v_1 \in B$ such that $v_0 \leq_B' b$, $v_1 \leq_B' b$ and $E(v_0, b) \wedge E(v_1, b)$.

<u>Case 1</u>: $b \in \{a_m^i : 0 \leq i < k\}$. Then $v_0, v_1 \in \{a_m^i : 0 \leq i < k\}$, $v_0 \leq_A b$ and $v_1 \leq_A b$, as $\leq_B'$ is defined so that it agrees with $\leq_A$ on $\{a_m^i : 0 \leq i < k\}$. This is impossible since $\leq_A$ is a 2-order of $\{a_m^i : 0 \leq i < k\}$.

<u>Case 2</u>: $b, v_0, v_1 \in B \setminus \{a_m^i : 0 \leq i < k\}$. Since $\leq_B'$ agrees with $\leq_B$ on $B \setminus \{a_m^i : 0 \leq i < k\}$, we have that $v_0 \leq_B b$ and $v_1 \leq_B b$, which is impossible as $\leq_B$ is a

2-order of B.

Case 3: $b \in B \setminus \{a_m^i : 0 \le i < k\}$ and at least one of v_0 or v_1 is in $\{a_m^i : 0 \le i < k\}$. Without loss of generality suppose that $v_0 \in \{a_m^i : 0 \le i < k\}$. Then v_0 is not connected to anything in the graph so $E(v_0, b)$ does not hold.

Conversely, if m is in the range of f, then we will connect $E(c_{f^{-1}(m)}, a_m^i)$ for each $0 \le i < k$, so if we take $B \supseteq \{a_m^i : 0 \le i < k\}$ to be a finite extension of $\{a_m^i : 0 \le i < k\}$, which has a k-ordering $\le_A$, where B contains the vertex $c_{f^{-1}(m)}$, then any end extension given by $\{a_m^i : 0 \le i < k\} \le_B B \setminus \{a_m^i : 0 \le i < k\}$ is not a k-order since $a_m^i <_B c_{f^{-1}(m)}$ for $0 \le i < k$ and $E(a_m^i, c_{f^{-1}(m)})$ holds. By statement 2 we have that the set $\{A \subseteq V : A \text{ is good}\}$ exists for our G, and so $\{A \subseteq V : A \text{ is not good}\}$ exists. Therefore, by the above biconditional, the range of f exists, and we have proven $(2 \to 1)$.

Now we prove $(3 \to 1)$. Let $g : \mathbb{N} \to V$ be a bijection witnessing that $\mathrm{Col}_\omega^S(G) \le k$ for the graph G we just constructed. Thus g defines a k-order $\le_V$ on the vertex set V where

$$g(0) \le_V g(1) \le_V g(2) \le_V \ldots .$$

By the construction and the above argument,

$$m \in ran(f) \iff (\exists c \in V)\left[\bigwedge_{0 \le i < k} E(c, a_m^i)\right] \iff (\exists c \in V)[E(c, a_m^0)].$$

We claim that

$$(\exists c \in V)[E(c, a_m^0)] \iff (\exists j \le \max\{g^{-1}(a_m^\ell) : 0 \le \ell < k\})[E(g(j), a_m^0)]$$

and therefore

$$m \in ran(f) \iff (\exists j \leq \max\{g^{-1}(a_m^\ell) : 0 \leq \ell < k\})[E(g(j), a_m^0)].$$

The last of this string can be checked in RCA_0 due to the bounded quantifier.

To show the forward direction of the claim, suppose that $(\exists c \in V)[E(c, a_m^0)]$, but $\neg(\exists j \leq \max\{g^{-1}(a_m^\ell) : 0 \leq \ell < k\})[E(g(j), a_m^0)]$. Fix j such that $g(j) = c$. Then since $j > \max\{g^{-1}(a_m^\ell) : 0 \leq \ell < k\}$, we have $a_m^\ell <_V c$ for $0 \leq \ell < k$. However, if $E(c, a_m^0)$ holds, then $E(c, a_m^\ell)$ holds for all $0 \leq \ell < k$, contradicting that $\leq_V$ is a k-order.

Conversely, suppose that $\neg(\exists c \in V)[E(c, a_m^0)]$. Then $\neg(\exists j \in \mathbb{N})[E(g(j), a_m^0)]$ and hence $\neg(\exists j \leq \max\{g^{-1}(a_m^\ell) : 0 \leq \ell < k\})[E(g(j), a_m^0)]$.

Thus we have proven $(3 \to 1)$. ***Qed***

Theorem 3.2.2. There is a computable forest $G = (V, E)$ such that

$$REC \models \mathrm{Col}_{LO}(G) \leq 2 \text{ and } REC \models \mathrm{Col}_\omega^W(G) \leq 2$$

but $REC \not\models \mathrm{Col}_\omega^S(G) \leq k$, for any $k \in \omega$. That is, $REC \models \mathrm{Col}_\omega^S(G) = \omega$.

Proof. The construction essentially employs the idea of the proof of Theorem 3.2.1 for each instance of k in the statement of that theorem. We define a graph $G = (V, E)$. First we place as vertices all of the even numbers in increasing order

$$a_0 < a_1 < a_2 < a_3 < a_4 < \ldots.$$

We want to satisfy the infinitely many requirements

$$\mathcal{R}_{\langle e,k\rangle} : \varphi_e \text{ does not witness } \mathrm{Col}^S_\omega(G) \leq k.$$

Formally, the requirement $\mathcal{R}_{\langle e,k\rangle}$ is that (assuming φ_e is a bijection from $\mathbb{N}$ onto V) there is an $n_k \in V$ and $\ell_0, \ldots, \ell_{k-1} \in V$ such that $E(n_k, \ell_i)$ holds for all $0 \leq i < k$ and $\varphi_e^{-1}(n_k) > \varphi_e^{-1}(\ell_i)$ for $0 \leq i < k$.

We claim that if all of the requirements are satisfied, then no computable well-ordering realizes $\mathrm{Col}^S_\omega(G) \leq k$, for any $k \in \mathbb{N}$. Suppose there were such a computable strong ω-type k-order. Then it must be a computable bijection φ_e for some $e < \omega$. Since, for each $k < \omega$, $\mathcal{R}_{\langle e,k\rangle}$ is satisfied, we have that there is an $n_k \in V$ and $\ell_0, \ldots, \ell_{k-1} \in V$ such that $E(n_k, \ell_i)$ holds for all $0 \leq i < k$ and $\varphi_e^{-1}(n_k) > \varphi_e^{-1}(\ell_i)$ for $0 \leq i < k$. Thus φ_e fails to be a k-order of V for all k, which is exactly what we want.

Fix a well ordering of the requirements $\mathcal{R}_{\langle e,k\rangle}$ given by

$$\mathcal{R}_{\langle e_0,k_0\rangle} < \mathcal{R}_{\langle e_1,k_1\rangle} < \mathcal{R}_{\langle e_2,k_2\rangle} < \cdots .$$

and say that $\mathcal{R}_{\langle e_i,k_i\rangle}$ *has higher priority than* $\mathcal{R}_{\langle e_j,k_j\rangle}$ if and only if $\langle e_i, k_i\rangle < \langle e_j, k_j\rangle$.

To ensure that a single requirement $\mathcal{R}_{\langle e,k\rangle}$ is satisfied, do the following to construct the forest $G = (V, E)$. Assign the first k many even numbers $a_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}$ that have so far not been assigned to any requirement, to the highest priority requirement without an assignment. Wait for $a_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}$ to enter the range of φ_e. If we wait forever, then $\mathcal{R}_{\langle e,k\rangle}$ is satisfied trivially, since in that case φ_e fails to be a bijection. Suppose

$$\varphi_e(\ell_0) = a_{i_0}, \varphi_e(\ell_1) = a_{i_1}, \ldots, \varphi_e(\ell_{k-1}) = a_{i_{k-1}}.$$

Next, we wait for a stage s by which φ_e has converged on all numbers in $\mathbb{N}$ which are $\leq_{\mathbb{N}} \max\{\ell_0, \ldots, \ell_{k-1}\}$. If φ_e fails to converge on any of these numbers, then $\mathcal{R}_{\langle e,k\rangle}$ is satisfied for all k, as φ_e is not total, and therefore not a bijection. Once we have found this stage s, let $c_{\langle e,k\rangle}$ be the least odd number greater than s and greater than all numbers in the range of φ_e on the domain $\mathbb{N} \upharpoonright \max\{\ell_0, \ldots, \ell_{k-1}\}$. Thus if $\varphi_e(m) = c_{\langle e,k\rangle}$, then m is greater than each of $\ell_0, \ldots, \ell_{k-1}$.

Put $c_{\langle e,k\rangle}$ into V and make the edge connections $\bigwedge_{0\leq j<k} E(c_{\langle e,k\rangle}, a_{i_j})$. With these edge connections, if φ_e is a bijection and $\varphi_e(m) = c_{\langle e,k\rangle}$, then there are $\ell_0, \ldots, \ell_{k-1}$ such that $E(c_{\langle e,k\rangle}, \ell_j)$ and $\varphi_e(\ell_j) < \varphi_e(m)$ for each $0 \leq i < k$. Therefore φ_e is not a k-order.

The following picture illustrates a case when we have a priority ordering

$$\mathcal{R}_{\langle 42,7\rangle} < \mathcal{R}_{\langle 14,2\rangle} < \mathcal{R}_{\langle 1,5\rangle} < \cdots$$

and the sets of even numbers assigned to $\mathcal{R}_{\langle 42,7\rangle}, \mathcal{R}_{\langle 14,2\rangle}, \mathcal{R}_{\langle 1,5\rangle}$ all enter the ranges of $\varphi_{42}, \varphi_{14}$ and φ_1, respectively.

The following picture illustrates the same situation as that of the above, except that one of the numbers from $\{a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$ never enters the range of φ_1, and we win trivially because φ_1 is not a bijection.

Notice that the vertex set V that we have defined for our graph $G = (V, E)$ is

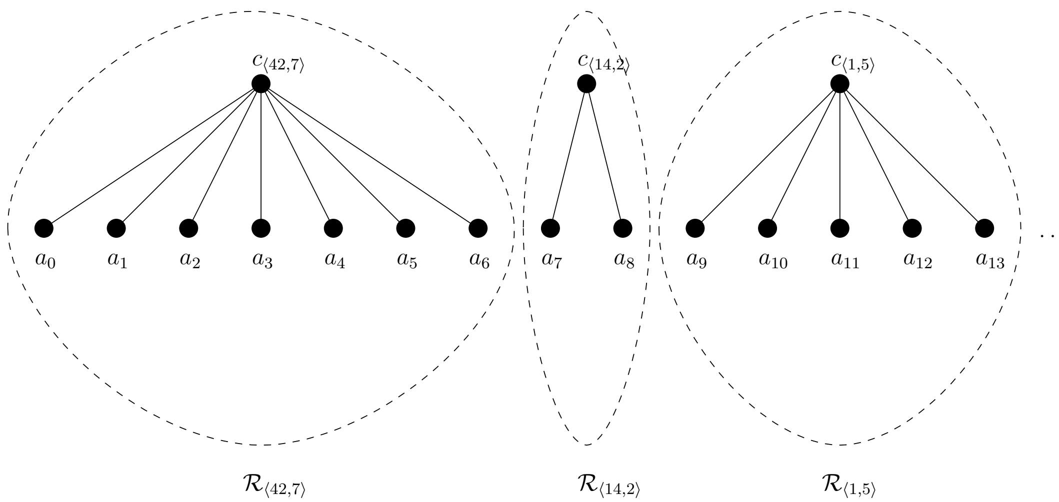

Fig. 3.3: A case when we have a priority ordering $\mathcal{R}_{\langle 42,7\rangle} < \mathcal{R}_{\langle 14,2\rangle} < \mathcal{R}_{\langle 1,5\rangle} < \cdots$.

computable, as V contains all the even numbers, and if an odd number c is in V, then we will know by stage c of the construction.

Now we define the computable 2-order of V that witnesses $\mathrm{Col}_{LO}(G) \leq 2$. For $u, v \in V$, define $\leq_V$ by $u \leq_V v$ if and only if u is odd (and is actually a vertex) and v is even, or u and v are both even and $u <_{\mathbb{N}} v$, or u and v are both odd (and actually vertices) and $u <_{\mathbb{N}} v$. Note that this 2-order has type $\omega + \omega$. In fact, we could even define a computable 2-order that has weak ω-type in the following way. Let $A_{\langle e,k\rangle}$ be the set of even numbers assigned to the requirement $\mathcal{R}_{\langle e,k\rangle}$. Define the weak ω-type 2-order $\leq_V$ by

$$A_{\langle e_0,k_0\rangle} \leq_V A_{\langle e_1,k_1\rangle} \leq_V A_{\langle e_2,k_2\rangle} \leq_V \cdots$$

(what essentially amounts to the natural ordering on the even numbers) with the

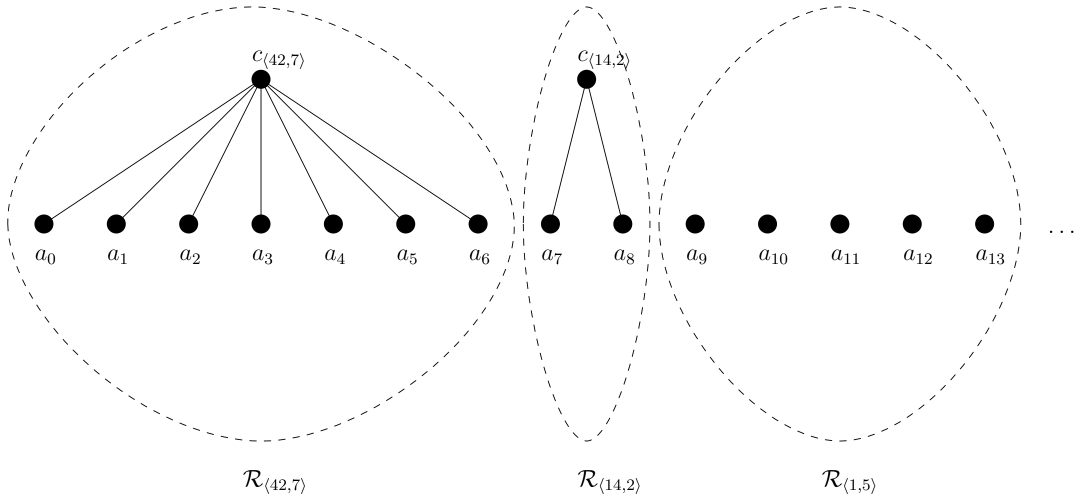

Fig. 3.4: A case when we have a priority ordering $\mathcal{R}_{\langle 42,7\rangle} < \mathcal{R}_{\langle 14,2\rangle} < \mathcal{R}_{\langle 1,5\rangle} < \cdots$, but one of the even numbers from $\{a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$ never enters the range of φ_1

addition of placing the odd number $c_{\langle e,k\rangle}$ as an immediate predecessor to $A_{\langle e,k\rangle}$ (that is, if we ever put the odd number $c_{\langle e,k\rangle}$ into V). ***Qed***

3.3 Weak ω-Coloring Number Results

Theorem 3.3.1. (RCA_0) The following are equivalent:

1. ACA_0

2. For any forest $G = (V, E)$, $\mathrm{Col}^W_\omega(G) \leq 2$.

Proof. ($1 \to 2$) This direction follows from Theorem 3.2.1 since $\mathrm{Col}^S_\omega(G) \leq 2$ implies $\mathrm{Col}^W_\omega(G) \leq 2$ over RCA_0.

$(2 \to 1)$ Suppose that for any forest $G = (V, E)$, $\mathrm{Col}^W_\omega(G) \leq 2$. Fix a one-to-one function $f : \mathbb{N} \to \mathbb{N}$. We wish to show that the range of f exists.

We construct a forest $G = (V, E)$ as follows. The vertex set is

$$V := \{a^e_n : e \in \mathbb{N} \wedge (\forall m < n)[f(m) \neq e]\} \cup \{b^e_n : e \in \mathbb{N} \wedge (\forall m < n)[f(m) \neq e]\}.$$

The edge relation is given by

$$E(a^e_n, a^e_{n+1}) \wedge E(b^e_n, b^e_{n+1}) \iff \neg(\exists m \leq n)[f(m) = e].$$

and

$$E(a^e_n, b^e_n) \iff f(n) = e.$$

This ends the construction of G.

Now fix a 2-order $\leq_V$ witnessing $\mathrm{Col}^W_\omega(G) \leq 2$. We claim that

$$e \notin ran(f) \iff (\exists k)[a^e_k <_V a^e_{k+1} \wedge b^e_k <_V b^e_{k+1}].$$

Notice that this suffices to get the range of f, since we also have

$$e \notin ran(f) \iff (\forall m)[f(m) \neq e],$$

which is a Π^0_1 condition, and thus there is a Δ^0_1 way to define the range of f. Hence by Δ^0_1 comprehension, the range of f exists.

For the forward direction of the claim, assume that $e \notin ran(f)$. Notice V contains every element from $\{a^e_n : n \in \mathbb{N}\}$ and $\{b^e_n : n \in \mathbb{N}\}$. If $(\forall k)[a^e_{k+1} <_V a^e_k]$, then every a^e_k for $k \geq 1$ is below a^e_0 in the ordering $\leq_V$, which contradicts the fact that

$\leq_V$ is a weak ω-type order. Thus $\neg(\forall k)[a^e_{k+1} <_V a^e_k]$. Thus we can fix $k \in \mathbb{N}$ such that $a^e_k <_V a^e_{k+1}$.

Now, we also have $a^e_\ell <_V a^e_{\ell+1}$ for all $\ell \geq k$. For if $\ell > k$ were least such that $a^e_\ell >_V a^e_{\ell+1}$, then we would have $E(a^e_{\ell+1}, a^e_\ell) \wedge E(a^e_\ell, a^e_{\ell-1})$ with $a^e_\ell >_V a^e_{\ell-1}$ (whether e is in the range of f or not) and $a^e_\ell >_V a^e_{\ell+1}$, contradicting the fact that $\leq_V$ is a 2-order.

Similarly to the case for the a's, if $(\forall k)[b^e_{k+1} <_V b^e_k]$, then every b^e_k for $k \geq 1$ is below b^e_0 in the ordering $\leq_V$, which contradicts the fact that $\leq_V$ is a weak ω-type order. Thus $\neg(\forall k)[b^e_{k+1} <_V b^e_k]$. Thus we can fix $k \in \mathbb{N}$ such that $b^e_k <_V b^e_{k+1}$.

Now, we also have $b^e_\ell <_V b^e_{\ell+1}$ for all $\ell \geq k$. For if $\ell > k$ were least such that $b^e_\ell >_V b^e_{\ell+1}$, then we would have $E(b^e_{\ell+1}, b^e_\ell) \wedge E(b^e_\ell, b^e_{\ell-1})$ with $b^e_\ell >_V b^e_{\ell-1}$ (whether e is in the range of f or not) and $b^e_\ell >_V b^e_{\ell+1}$, again contradicting the fact that $\leq_V$ is a 2-order. Therefore the forward direction of the claim holds.

Conversely, assume $(\exists k)[a^e_k <_V a^e_{k+1} \wedge b^e_k <_V b^e_{k+1}]$. For a contradiction, suppose $e \in ran(f)$. So we can let n be such that $f(n) = e$. Notice we must have $n \geq k+1$, for otherwise a^e_{k+1} and b^e_{k+1} would not be defined as vertices in V.

Then, using the fact that $a^e_\ell <_V a^e_{\ell+1}$ and $b^e_\ell <_V b^e_{\ell+1}$ for all $\ell \geq k$, we have

$$(a^e_{n-1} <_V a^e_n) \wedge (b^e_{n-1} <_V b^e_n) \wedge E(a^e_{n-1}, a^e_n) \wedge E(b^e_{n-1}, b^e_n) \wedge E(a^e_n, b^e_n).$$

We have two cases: either $a^e_n <_V b^e_n$ or $b^e_n <_V a^e_n$. Either case violates the fact that $\leq_V$ is a 2-order. Hence $e \notin ran(f)$, and we have proven the claim. Thus the theorem follows.

The following picture illustrates the contradiction we obtain in proving the backwards direction of the claim that

$$e \notin ran(f) \iff (\exists k)[a^e_k <_V a^e_{k+1} \wedge b^e_k <_V b^e_{k+1}].$$

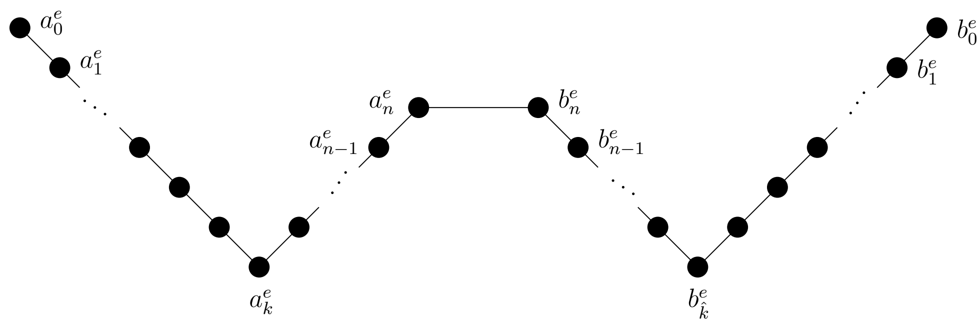

Fig. 3.5: A contradiction when $f(n) = e$

Qed

An interesting open question involves the classification of Theorem 3.3.1 for values of $k \geq 2$. In other words, can we get a reversal from the statement, "For any $k \in \mathbb{N}$, $k \geq 2$, and any forest $G = (V, E)$, $\mathrm{Col}^W_\omega(G) \leq k$" to one of the major subsystems. At the very least, we already know that this statement is provable in ACA_0, by Theorem 3.3.1. It would appear as though the method of proof used for Theorem 3.3.1, however, does not translate into a reversal to ACA_0 for any case when $k > 2$.

Chapter 4

Results on Subgraphs

4.1 Finite Subgraphs

In this section we will analyze the following theorem, due to Erdös and Hajnal.

Theorem 4.1.1. (Erdös, Hajnal) If every finite subgraph of a graph G has coloring number at most n ($2 \leq n < \omega$), then the coloring number of G is at most $2n - 2$.

We state without proof a theorem from [1] about finite graphs. The finitary nature of the theorem makes it intuitively clear that it is likely provable in RCA_0. However, since we have not yet gone through all the formal details, it is indeed safe to say that the following theorem is provable at least in ACA_0, which is all we need.

Theorem 4.1.2. (Nash-Williams, from [1]) A finite graph $G = (V, E)$ can be partitioned into at most k forests if and only if for every non-empty set $U \subseteq V$, $e(U) \leq k(|U| - 1)$.

The Nash-Williams theorem will be useful to us as we prove the following theorem

Theorem 4.1.3. (RCA_0) The following are equivalent.

1. ACA_0

2. For all graphs G and all $n \geq 2$, if every finite subgraph H of G has $\mathrm{Col}_{LO}(H) \leq n$, then $\mathrm{Col}^S_\omega(G) \leq 2n-2$.

3. For all graphs G and all $n \geq 2$, if every finite subgraph H of G has $\mathrm{Col}_{LO}(H) \leq n$, then $\mathrm{Col}^W_\omega(G) \leq 2n-2$.

Proof. $(2 \rightarrow 3)$ This direction follows easily, since $\mathrm{Col}^S_\omega(G) \leq 2n-2$ implies $\mathrm{Col}^W_\omega(G) \leq 2n-2$ over RCA_0.

$(3 \rightarrow 1)$ Let $G = (V, E)$ be a forest and let $n = 2$. By Lemma 2.1.2, RCA_0 suffices to prove that every finite forest F has $\mathrm{Col}_{LO}(F) \leq 2$. Thus the hypothesis for statement 3 is satisfied, so we may apply it to get $\mathrm{Col}^W_\omega(G) \leq 2$; but this implies ACA_0 by Theorem 3.3.1.

$(1 \rightarrow 2)$ Let $G = (V, E)$ be a graph such that $V = \{v_0, v_1, v_2, \dots\}$. Furthermore, let $k = n - 1 \geq 1$. By hypothesis every finite subgraph $H = (V_H, E_H)$ of G has $\mathrm{Col}_{LO}(H) \leq k+1$. Now, since H is finite and $\mathrm{Col}_{LO}(H) \leq k+1$, we claim that H is a union of at most k forests. Let $V_H = \{v_0, v_1, \dots, v_j\}$. Fix a $k+1$-ordering $\leq_{V_H}$ of V_H such that

$$v_{\ell_0} \leq_{V_H} \cdots \leq_{V_H} v_{\ell_{k+1}} \leq_{V_H} \cdots \leq_{V_H} v_{\ell_j}.$$

Since $\mathrm{Col}_{LO}(H) \leq k+1$, we know that for each $0 \leq i \leq k$, there are at most i many vertices connected to v_{ℓ_i} which are lower than it in the ordering $\leq_{V_H}$. For $k < i \leq j$, there are at most k vertices connected to v_{ℓ_i} which are lower than it in the ordering $\leq_{V_H}$. Therefore the number of possible connections in H has the following upper bound:

$$\begin{aligned} e(V_H) &\leq 0+1+\cdots+k+k((j+1)-k) \\ &= \frac{k(k-1)}{2} + k|V_H| - k^2 \\ &= k\left(\frac{k}{2} - \frac{1}{2} + |V_H| - k\right) \\ &= k\left(|V_H| - \left(\frac{k+1}{2}\right)\right) \\ &\leq k(|V_H|-1). \end{aligned}$$

Notice that we have indeed counted every edge in the finite subgraph H of G, and possibly more.

Fix a finite subgraph $\hat{H} = (V_{\hat{H}}, E_{\hat{H}})$ of G. By the above, for $U \subseteq V_{\hat{H}}$ we have the bound on the number of edges in the subgraph of $\hat{H}$ induced by U: $e(U) \leq k(|U|-1)$. Now by the result of Nash-Williams, $\hat{H}$ is a union of at most k forests. Thus every finite subgraph of G is a union of at most k forests.

Now we claim that the entire graph G is also a union of k forests. (Note that this requires no more than WKL_0.) To prove this claim, we build an k branching tree, which, intuitively, guesses at level i which forest to put v_{i-1} into, and stops building above a node of the tree whose guess includes a cycle in one of the alleged forests. We work with the tree $T = k^{<\mathbb{N}}$. Label the nodes of T in following way. Define the function $f : T \to (\mathrm{Fin}_V)^k$ by

- $f(\emptyset) := \langle \emptyset, \ldots, \emptyset \rangle$, where there are k many components, each consisting of $\emptyset$.
- $f(\sigma) := \langle F_0^\sigma, \ldots, F_{k-1}^\sigma \rangle$, where $F_i^\sigma = \{v_j : j < |\sigma| \wedge \sigma(j) = i\}$ for $0 \leq i < k$.

Now we define a subtree $S \subseteq T$ by

$$\sigma \in S \iff \text{ each } F_i^\sigma \text{ is a forest, for } 0 \leq i < k.$$

Since $S \subseteq T$, and each node in T has k many successors, each node in S has no more than k many successors, so it is bounded. Since we have the property that every finite subgraph of G is a union of k forests, we have that, for each j, $\{v_0, \ldots, v_j\}$ is a union of k forests, so that there is some path such that each of the elements from $\{v_0, \ldots, v_j\}$ fits into one of the F_i^σ along that path without creating a cycle. Therefore we can conclude that S is infinite, and so by weak König's lemma, that S has a path. Let g be such a path in S. It is clear that g gives us k many forests $\langle F_0^g, \ldots, F_{k-1}^g \rangle$ such that $G = \bigcup_{i=0}^{k-1} F_i^g$, because each v_i was included in the union at some finite level of the tree, and we are guaranteed that the tree will continue above that level, ensuring that each v_i will be included in the limit. Thus the claim is proved. Since G is a union of k forests, and $k = n-1$, we have that G is a union of at most $n-1$ forests, so that $\mathrm{Col}_\omega^S(G) \leq 2n-2$, as Theorem 3.1.3 goes through in ACA_0. ***Qed***

Appendix A

Erdös-Hajnal Examples

A.1 A Few Definitions

Recall the following theorem of Erdös and Hajnal we analyzed in Chapter 4.

Theorem A.1.1. (Erdös, Hajnal, [2]) If every finite subgraph of a graph G has coloring number at most n ($2 \leq n < \omega$), then the coloring number of G is at most $2n - 2$.

In this chapter we present a few of the examples that show the previous result is sharp.

In [2], the relation $R(\alpha, \beta, \gamma, \delta)$ is defined.

Definition A.1.2. The relation $R(\alpha, \beta, \gamma, \delta)$ holds if for every graph $G = (V, E)$ with $|V| = \alpha$ and if every subgraph H of G with $|H| < \gamma$ has coloring number $\leq \beta$, then $\text{Col}(G) \leq \delta$.

The case we are interested in is when $\gamma = \omega$, and we see that the above theorem amounts to the statement saying that $R(\omega, \beta, \omega, 2\beta - 2)$ holds. This result is actually sharp, i.e., there are examples showing that $R(\omega, \beta, \omega, 2\beta - 3)$ does not hold.

We will give a few of these examples, but first we need a definition.

Definition A.1.3. (Erdös, Hajnal, [2]) We define graphs $G(k, l) = (V(k, l), E(k, l))$ for $l \geq 3$ if $k = 2$ and for $l \geq 2$ if $k \geq 3$.

1. $V(k, l) = \omega$.

2. We define $B(k, l)$ as a disjoint partition of type ω of ω.

   $j \in B_i(k, l)$ if $j = l\binom{k}{2}i + s,\ 0 \leq s < l\binom{k}{2}$

3. We define the set of edges $E(k, l)$ for $k \geq 3$, $l \geq 2$. First we define a partition of type $l - 1$ of each $B_i(k, l)$.

   Assume $j \in B_i(k, l)$, $j = \binom{k}{2}i + s,\ 0 \leq s < l\binom{k}{2}$.

4. $j \in B_{i,r}(k, l)$ for $0 \leq r < l - 2$ if $r\binom{k}{2} \leq s < (r + 1)\binom{k}{2}$.

   $j \in B_{i,l-2}(k, l)$ if $(l - 2)\binom{k}{2} \leq r < l\binom{k}{2}$

5. $\{j, j'\} \in E(k, l)$ if $j, j' \in B_{i,r}(k, l)$ for some i and $r < l - 1$ and $j < j'$, $j' - j \leq k - 1$.

   $\left\{l\binom{k}{2}i + s, l\binom{k}{2}i + \binom{k}{2} + s\right\} \in E(k, l)$ for $(l-2)\binom{k}{2} \leq s < (l-1)\binom{k}{2}$.

   If $p \neq li - 1$ for some i, $p > 0$, then for every $0 \leq w < k - 1$,

$$\left\{(p - 1)\binom{k}{2} + v, p\binom{k}{2} + w\right\} \in E(k, l)$$

$$\text{for } w(k-1) - \binom{w}{2} \leq v < (w+1)(k-1) - \binom{w+1}{2}.$$

A.2 The Examples

To show that $R(\omega, 3, \omega, 3)$ fails, we construct a graph G such that every finite subgraph of G has coloring number ≤ 3, but $\mathrm{Col}(G) > 3$ (classically). We obtained this example from [2].

Define $G = (V, E)$ in the following way. Let $V = \omega \cup \{a_0\}$, where a_0 is an element disjoint to ω. Say $A = \{a_0\}$. Let $G(\omega) = G(2, 3)$.

Let $j = 3i + s$, $0 \leq s < 3$. For $i \in \omega$, let $v(j, A, G) = \emptyset$ if $s = 0$, $v(j, A, G) = \{a_0\}$ if $s = 1$ or $s = 2$.

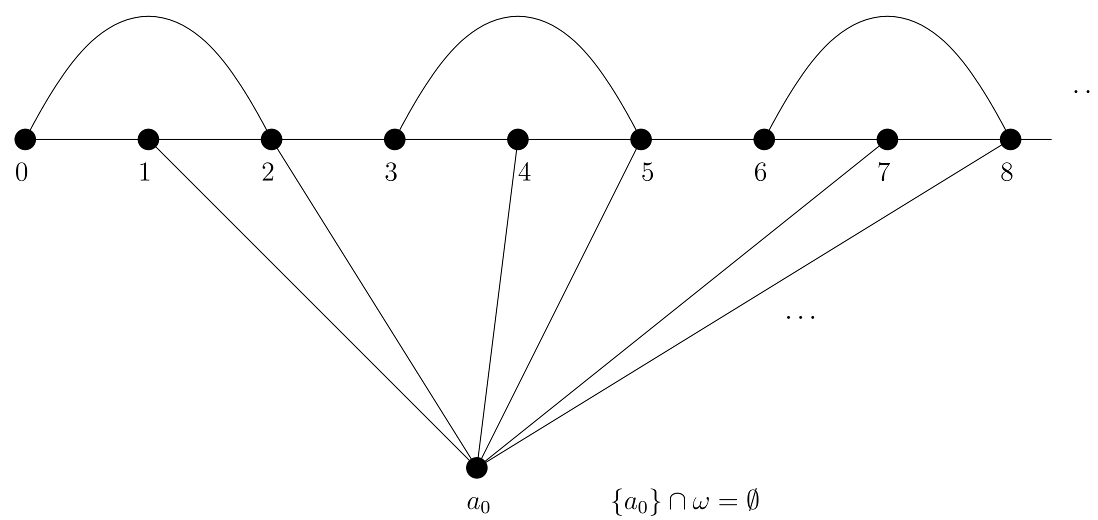

Fig. A.1: The graph G such that $\mathrm{Col}(H) \leq 3$ for every finite $H \subseteq G$, but $\mathrm{Col}(G) > 3$.

Now we construct a graph to show that $R(\omega, 4, \omega, 5)$ fails. That is, we construct a graph G such that every finite subgraph of G has coloring number ≤ 4, but $\mathrm{Col}(G) > 5$ (classically). We obtained this example from [2].

Define $G = (V, E)$ in the following way. Let $V = \omega \cup \{a_0, a_1\}$, where a_0, a_1 are disjoint to ω. Say $A = \{a_0, a_1\}$. Let $G(\omega) = G(3, 2)$ and $G(A) = \emptyset$.

Now we define the vertices from A to $G(3, 2)$. Let $j = 6i + s$. Say $v(j, A, G) = \emptyset$ if $0 \leq s < 4$. Say $v(j, A, G) = \{a_0, a_1\}$ if $4 \leq s < 6$.

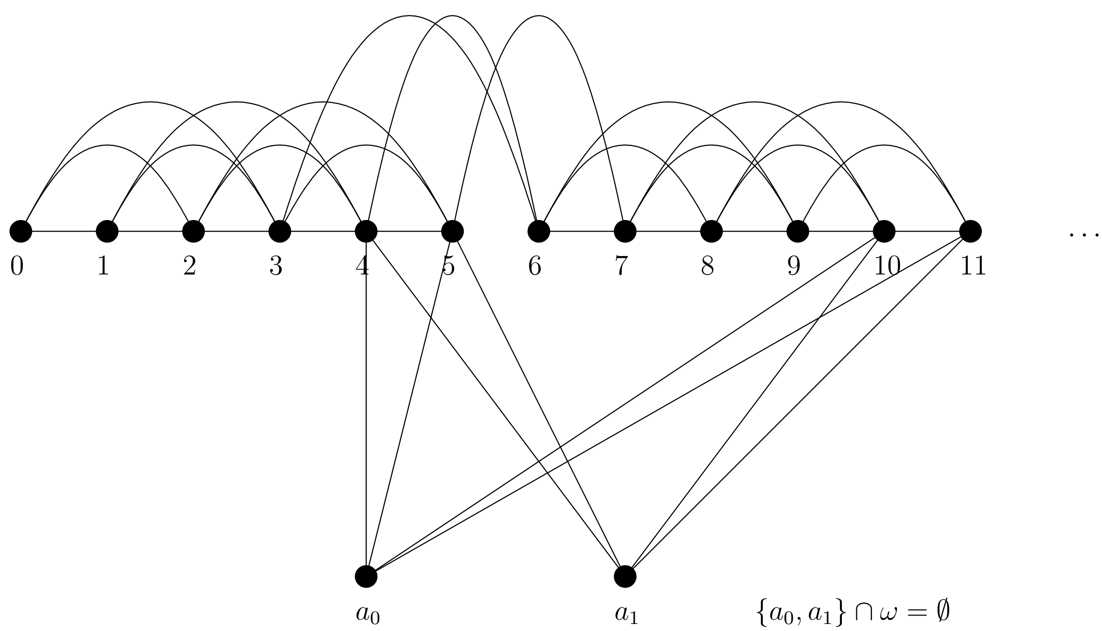

Fig. A.2: The graph G such that $\mathrm{Col}(H) \leq 4$ for every finite $H \subseteq G$, but $\mathrm{Col}(G) > 5$.

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