Ordered Groups, Computability and Cantor-Bendixson Rank

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Abstract

We study ordered groups in the context of both algebra and computability. Ordered groups are groups that admit a linear order that is compatible with the group operation. We explore some properties of ordered groups and discuss some related topics. We prove results about the semidirect product in relation to orderability and computability. In particular, we give a criteria for when a semidirect product of orderable groups is orderable and for when a semidirect product is computably categorical. We also give an example of a semidirect product that has the halting set coded into its multiplication structure but it is possible to construct a computable presentation of this semidirect product.

We examine a family of orderable groups that admit exactly countably many orders and show that their space of orders has arbitrary finite Cantor-Bendixson rank. Furthermore, this family of groups is also shown to be computably categorical, which in particular will allow us to conclude that any computable presentation of the groups does not admit any noncomputable orders. Lastly, we construct an example of an orderable computable group with no computable Archimedean orders but at least one computable non-Archimedean order.

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Chapter 1

Introduction

1.1 Preliminaries

In computable structure theory, we study mathematical structures such as linear orders, graphs, groups, etc. from the perspective of computability theory and analyze their effective content. One of the main themes of computable structure theory is to understand the relationship between structural and computational properties of a structure. This study begins with defining what we mean by a computable structure. We say a structure is *computable* if its domain is a computable set, and all of its functions, relations and constants are uniformly computable. We will be interested in computable groups and so we give a precise definition.

Definition 1.1.1. A group $(G, \cdot_G)$ is called a *computable group* if its domain G is a computable set and the group operation $\cdot_G \colon G \times G \to G$ is a computable function. A *computable presentation* or a *computable copy* of a group G is a computable group H such that $H \cong G$.

We will be primarily interested in infinite computable groups, in which case we will usually identify the domain of our group with ω. A class of structures that is of great interest and a central theme of our study are ordered groups.

Definition 1.1.2. Let G be a group and let $<$ be a strict total order on G. The pair $(G, <)$

is called an *ordered group* if the order $<$ is both left and right invariant, that is, $a < b$ implies $ca < cb$ and $ac < bc$ for all $a, b, c \in G$. A group is called an *orderable group* if it admits a strict total order which is bi-invariant, that is, both left and right invariant.

In the literature, these groups are sometimes more aptly called bi-ordered groups to distinguish from left-ordered groups (or right-ordered groups). It is worth noting that the theories of bi-ordered groups and left-ordered groups are not the same, in an informal sense and a precise logical sense in that their first-order theories do not coincide. We will only be concerned with groups with bi-invariants orders and so we will simply say an ordered group or an orderable group without the qualifier "bi".

As a piece of terminology, an *orderable computable group* is a computable group which admits an invariant order but the order is not necessarily computable.

In this dissertation, we will be studying results about ordered groups both in the context of algebra and computability. A good deal of work has been done on ordered groups both from the side of algebra and computability. For example, we have the following well-known result about ordered abelian groups. See Theorem 1.2.2 for a proof.

Theorem 1.1.3 ([Lev42]). *An abelian group is orderable if and only if it is torsion-free.*

In contrast, it turns out that the effective analog of this result does not hold.

Theorem 1.1.4 ([DK86]). *There is a computable torsion-free abelian group that is isomorphic to $\bigoplus_\omega \mathbb{Z}$ with no computable orders.*

This result shows that how a group is computably presented can affect what computational properties it has. We will prove a result in similar spirit to this in Chapter 4.

We follow standard computability notation. We will write φ_e to denote the partial computable function with index e and degree will always mean Turing degree. For a reference on computability theory, see [Soa16; Web12].

1.2 Ordered groups

We will usually write 1 to denote the identity of a group. For abelian groups, we will sometimes use additive notation and write 0 for the identity. If $(G, <)$ is an ordered group, we will sometimes write $g \leq h$ to mean $g < h$ or $g = h$. An element g in an ordered group is called *positive* if $1 < g$ and *negative* if $g < 1$.

We start with the following simple result.

Proposition 1.2.1. *Every nontrivial ordered group is torsion-free and hence infinite.*

Proof. Suppose $g \in G$ and $1 < g$. Then $g < g^2$, $g^2 < g^3$ and so forth, and by transitivity, $1 < g^n$ for all positive integers n. The case when $g < 1$ is proved similarly. □

Proposition 1.2.1 shows that being torsion-free is a necessary condition for a group to be orderable; however, in general, this is not a sufficient condition. There do exist finitely generated torsion-free groups which are not orderable. See Example 1.2.11 for an example of such a group. There is one class of groups for which being torsion-free is enough to conclude that they are orderable and these are precisely the abelian groups.

Theorem 1.2.2 ([Lev42])**.** *An abelian group is orderable if and only if it is torsion-free.*

Proof. Let G be a torsion-free abelian group. We can embed G into a torsion-free divisible group D (see Appendix A.1). It suffices to show that D is orderable. To not overcomplicate notation, we describe the order of D by declaring its set of positive elements under the order. (See Section 1.3 for why this is sufficient.)

Since D is torsion-free and divisible, it is a $\mathbb{Q}$-vector space and thus $D \cong \mathbb{Q}^\lambda$ for some cardinal number λ. Pick a basis $\{b_\nu\}_{\nu \in \lambda}$ for D and linearly order the basis set. Every nonzero element $g \in D$ can be expressed uniquely as $g = q_1 b_{\nu_1} + \cdots + q_n b_{\nu_n}$ for some nonzero rational coefficients q_i and $\nu_1 < \cdots < \nu_n$. Declare g to be positive if q_1 is a positive rational number. Note our order on D is just a lexicographic order with respect to the ordered basis. It can be checked that this definition makes D into an ordered group. □

We mention in passing that Theorem 1.2.2 is also true for nilpotent groups, see [KK74, Corollary 4, p. 16] for a proof.

Example 1.2.3. The additive groups $(\mathbb{Z},+)$, $(\mathbb{Q},+)$ and $(\mathbb{R},+)$ with their usual orders are ordered groups.

Example 1.2.4. It is easy to see that orderability is preserved under taking subgroups—thus every subgroup of an orderable group is orderable.

Example 1.2.5. If $(G,<)$ is an ordered group, then we can define a new order on G by $g <^* h$ if and only if $h < g$. The order $<^*$ is called the dual or opposite order, it is also an invariant order on G.

Example 1.2.6. If $(G,<_G)$ and $(H,<_H)$ are ordered groups, then so is their direct product $G \times H$ under the lexicographical ordering, which declares that

$$(g,h) < (g',h') \text{ if and only if } g <_G g', \text{ or } g = g' \text{ and } h <_H h'.$$

More generally, the following is true.

Proposition 1.2.7. *Let Γ be a possibly infinite set and let $\{(G_\gamma,<_\gamma) \mid \gamma \in \Gamma\}$ be a collection of ordered groups indexed by Γ.*

(i) *Let $<_\Gamma$ be a well-order on the index set Γ. Define an order on the direct product $G = \prod_{\gamma\in\Gamma} G_\gamma$ by $g <_G g'$ if and only if $g_\lambda <_\lambda g'_\lambda$ where $\lambda \in \Gamma$ is the $<_\Gamma$-least component at which g and g' differ. Then $(G,<_G)$ is an ordered group.*

(ii) *Let $<_\Gamma$ be a linear order on the index set Γ. Define an order on the direct sum $G = \bigoplus_{\gamma\in\Gamma} G_\gamma$ by $g <_G g'$ if and only if $g_\lambda <_\lambda g'_\lambda$ where $\lambda \in \Gamma$ is the $<_\Gamma$-least component at which g and g' differ. Then $(G,<_G)$ is an ordered group.*

Example 1.2.8. The additive group $\mathbb{Z}^2$ can be ordered in uncountably many different ways as follows. Choose a vector $\vec{v} \in \mathbb{R}^2$ with irrational slope. We can order the elements of $\mathbb{Z}^2$ by considering their dot product with $\vec{v}$. For $\vec{m}, \vec{n} \in \mathbb{Z}^2$, define

$$\vec{m} < \vec{n} \text{ if and only if } \vec{m} \cdot \vec{v} <_{\mathbb{R}} \vec{n} \cdot \vec{v}$$

where $<_{\mathbb{R}}$ denotes the usual order of $\mathbb{R}$.

Example 1.2.9. Every free group is orderable and more so free groups with more than one generator have uncountably many orderings. See [CR16, §3.2] for a construction of an explicit ordering.

Example 1.2.10. The fundamental groups of surfaces (i.e. two-dimensional manifolds) are orderable with the exceptions of the Klein bottle and the real projective plane. More information can be found in [CR16, §3.3].

Example 1.2.11 (Non-orderable group)**.** The fundamental group of the Klein bottle cannot be given an invariant ordering. The fundamental group can be expressed as

$$\pi_1(\text{Klein bottle}) \cong \mathbb{Z} \rtimes \mathbb{Z} \cong \langle x \rangle \rtimes \langle y \rangle \cong \langle x, y \mid yxy^{-1} = x^{-1} \rangle.$$

Notice that $1 < x$ if and only if $x^{-1} < 1$. However, the defining relation implies that $1 < x$ if and only if $1 < yxy^{-1} = x^{-1}$ and we have a contradiction.

We have the following classification of the number of orderings of an ordered abelian group. This can be deduced from a more general result due to Linnell [Lin11] along with a theorem of Sikora [Sik04, Proposition 1.7]. See also [Min72; Teh61] for a more general discussion of the orderings of an abelian group and how to construct all possible different orderings of an abelian group. Recall that the rank of an abelian group is the cardinality of a maximal linearly independent subset.

Theorem 1.2.12. *Let A be a torsion-free abelian group.*

(1) *If A has rank one, that is, A is a subgroup of $(\mathbb{Q}, +)$, then A has exactly two orders.*

(2) *If A has rank greater than one, then A has uncountably many orders.*

Corollary 1.2.13. *The additive group of rational numbers* $(\mathbb{Q}, +)$ *has only two orders.*

We mention a few algebraic facts that hold for ordered groups. Recall that the commutator of two elements g and h in a group is defined as $[g, h] = ghg^{-1}h^{-1}$. The commutator of g and h is 1 if and only if g and h commute.

Proposition 1.2.14. *Let* $(G, <)$ *be an ordered group.*

(i) *If* $g < h$*, then* $h^{-1} < g^{-1}$*.*

(ii) *If* $g < h$ *and* $g' < h'$*, then* $gg' < hh'$*.*

(iii) *If* $g^n = h^n$ *for some nonzero integer* n*, then* $g = h$*.*

(iv) *If* $[g^m, h^n] = 1$ *for some nonzero integers* m *and* n*, then* $[g, h] = 1$*.*

(v) *If* $h \neq 1$*, then* $(g_1hg_1^{-1}) \cdots (g_nhg_n^{-1}) \neq 1$*. In other words, ordered groups do not have generalized torsion.*

(vi) *The equation* $x^n = g$ *has at most one solution for any* $g \in G$ *and any positive integer* n*.*

Proof. We prove only (iv), (v) and (vi). To prove (iv), assume g and h do not commute. Say $gh < hg$. Then $g < hgh^{-1}$ and we can multiply this inequality by itself m times to conclude $g^m < hg^mh^{-1}$. From this we derive $h < g^{-m}hg^m$ and now we can multiple this inequality by itself n times to get $h^n < g^{-m}h^ng^m$. Thus, $g^mh^n < h^ng^m$, and g^m and h^n do not commute.

To prove (v), assume $h \neq 1$ and say $1 < h$. Then $1 < g_ihg_i^{-1}$ and by multiplying inequalities we get that $1 < (g_1hg_1^{-1}) \cdots (g_nhg_n^{-1})$.

To prove (vi), note if x and y are two different solutions and, say $x < y$, then $x^n < y^n$, a contradiction. □

In an ordered group $(G, <)$, the *absolute value* $|g|$ of an element $g \in G$ is defined as:

$$|g| = \begin{cases} 1, & \text{if } g = 1, \\ g, & \text{if } 1 < g, \\ g^{-1}, & \text{if } g < 1. \end{cases}$$

The following summarizes some of the properties of the absolute value. The proofs of these can be found in [Che58].

Proposition 1.2.15. *Let $(G, <)$ be an ordered group.*

(i) *For any integer n, $|g^n| = |g|^{|n|}$.*

(ii) *For all $g, h \in G$, $|ghg^{-1}| = g|h|g^{-1}$.*

(iii) *If g and h are positive, then $|gh| = |g||h| = gh$.*

(iv) *If g and h are negative, then $|gh| = |h||g| = h^{-1}g^{-1}$.*

(v) *If $g < 1$, $1 < h$ and $|g| < |h|$, then*

$$|gh| = |g|^{-1}|h| = gh \quad \textit{and} \quad |hg| = |h||g|^{-1} = hg.$$

(vi) *If $1 < g$, $h < 1$ and $|g| < |h|$, then*

$$|gh| = |h||g|^{-1} = h^{-1}g^{-1} \quad \textit{and} \quad |hg| = |g|^{-1}|h| = g^{-1}h^{-1}.$$

Next we want to prove a result about order-preserving maps on $\mathbb{R}$.

Definition 1.2.16. Let $(G, <_G)$ and $(H, <_H)$ be ordered groups. A group isomorphism $\varphi \colon G \to H$ is called an *order-isomorphism* if $x <_G y$ implies $\varphi(x) <_H \varphi(y)$ for all $x, y \in G$.

Proposition 1.2.17 (Hion's Lemma)**.** *Suppose A and B are subgroups of $(\mathbb{R}, +)$ endowed with the usual order, and $f \colon A \to B$ is an order-isomorphism. Then there is a positive real number r such that $f(x) = rx$ for all $x \in A$.*

Proof. Assume towards a contradiction, there exists $x, y \in A$ with $f(x)/x \neq f(y)/y$. Assume $x/y < f(x)/f(y)$. Fix a rational number m/n such that $x/y < m/n < f(x)/f(y)$. Then we have that $nx < my$ and $mf(y) < nf(x)$ but this is a contradiction since f is an order-isomorphism. Thus $f(x)/x$ is constant for all $x \neq 0$. □

1.3 Positive cone and the space of orders

A helpful tool for studying ordered groups is the notion of the positive cone of an ordering.

Definition 1.3.1. Let $(G, <)$ be an ordered group. The *positive cone* of G is defined to be the set $P = \{g \in G \mid 1 < g\}$.

The positive cone P of an ordering satisfies the following properties:

(1) for every $g \in G$ exactly one of $g = 1$, $g \in P$ or $g^{-1} \in P$ holds;

(2) P is a semigroup, i.e., $P \cdot P \subseteq P$; and

(3) P is self-conjugate, i.e., $gPg^{-1} \subseteq P$ for all $g \in G$.

Conversely, given a subset $P \subseteq G$ that satisfies the above three conditions, we can define an invariant strict total order on G by the recipe $g < h$ if and only if $g^{-1}h \in P$. In summary, we have the following.

Theorem 1.3.2. *A subset P of a group G is the positive cone of an order on G if and only if P satisfies the following conditions:*

(i) $P \cap P^{-1} = \emptyset$*;*

(ii) $P \cup P^{-1} = G \backslash \{1\}$*;*

(iii) $P \cdot P \subseteq P$*; and*

(iv) $gPg^{-1} \subseteq P$ *for all* $g \in G$*.*

Thus we see that when describing an ordering of an orderable group, we need only specify the positive cone and verify that it satisfies conditions (i)–(iv) above. This is a very useful characterization of the orderings of a group because it allows us to shift perspective to thinking of orderings as certain special subsets of G instead of as binary relations. With this in mind, we can make the following definition.

Definition 1.3.3. The *space of orders* of an orderable group G, denoted $\mathbb{X}(G)$, is the set of all positive cones of G. In notation,

$$\mathbb{X}(G) = \{P \subseteq G \mid P \text{ is the positive cone of an order on } G\}.$$

A very useful fact about the space of orders is that it is a topological space. We now show how to put a topology on the set of all orderings of an orderable group.

For any set X, the product topology on its power set $\mathcal{P}(X)$ is the topology generated by the subbasis consisting of the sets

$$U_x = \{A \subseteq X \mid x \in A\} \quad \text{and} \quad V_x = \{A \subseteq X \mid x \notin A\}$$

for each $x \in X$. Note that the sets U_x and V_x are complements of each other. A basis for this topology can be obtained by taking finite intersections of various U_x and V_x. We can identity the power set $\mathcal{P}(X)$ with the set of all functions from X to $\{0,1\}$, namely, the set $2^X = \{f \colon X \to \{0,1\}\}$. In this identification, our subbasis now corresponds to the sets

$$U_x = \{f \colon X \to \{0,1\} \mid f(x) = 1\} \quad \text{and} \quad V_x = \{f \colon X \to \{0,1\} \mid f(x) = 0\}.$$

We will freely identity $\mathcal{P}(X)$ with 2^X and represent elements of 2^X as both subsets of X and functions $X \to \{0,1\}$. With respect to the product topology, the space 2^X is totally disconnected, compact, Hausdorff and has a basis of clopen sets. Thus 2^X is a Boolean space.

We can now define a topology on $\mathbb{X}(G)$. For any orderable group G, we can regard $\mathbb{X}(G)$ as a subspace of 2^G and so the topology on $\mathbb{X}(G)$ has as a subbasis all sets of the form

$$U_g \cap \mathbb{X}(G) = \{P \in \mathbb{X}(G) \mid g \in P\} \quad \text{and} \quad V_g \cap \mathbb{X}(G) = \{P \in \mathbb{X}(G) \mid g^{-1} \in P\}$$

for all $g \in G$. As we will prove in Theorem 1.3.4, this topology makes $\mathbb{X}(G)$ a closed subset of 2^G and therefore a Boolean space.

Here is another useful way to describe the topology of $\mathbb{X}(G)$. Suppose G is an orderable group and $<$ is an ordering of G. Consider a finite string of inequalities $g_1 < \cdots < g_n$ for

$g_i \in G$. The set of orderings of G that satisfy these inequalities forms an open neighborhood of $<$ in $\mathbb{X}(G)$. More so, the set of all such neighborhoods forms a basis for the topology of $\mathbb{X}(G)$. Equivalently, one can multiply the inequalities as necessary and see that a basis for $\mathbb{X}(G)$ consists of all sets of orderings in which some specified finite set of elements of G are all positive.

Theorem 1.3.4. *If G is an orderable group, then $\mathbb{X}(G)$ is a closed subset of 2^G. Thus $\mathbb{X}(G)$ is a Boolean space.*

Proof. We will argue that the complement $2^G \setminus \mathbb{X}(G)$ is an open set. We define a collection of subsets U_1, U_2, U_3, and U_4 of 2^G such that each will satisfy conditions (i)–(iv) from Theorem 1.3.2, respectively. Define

$$U_1 = \{Y \subseteq G \mid Y \cap Y^{-1} = \emptyset\} = \{Y \subseteq G \mid \forall g \in G \; (g \notin Y \vee g^{-1} \notin Y)\}.$$

Then $Y \in U_1$ if and only if Y satisfies condition (i). Notice that we can express the complement of U_1 as

$$2^G \setminus U_1 = \bigcup_{g \in G} U_g \cap U_{g^{-1}}.$$

Therefore $2^G \setminus U_1$ is a union of open sets and so is open. Next, define

$$U_2 = \{Y \subseteq G \mid \forall g \in G \backslash \{1\} \; (g \in Y \vee g^{-1} \in Y)\},$$

$$U_3 = \{Y \subseteq G \mid \forall g, h \in G \; (g, h \in Y \Rightarrow gh \in Y)\},$$

and

$$U_4 = \{Y \subseteq G \mid \forall g, h \in G \; (h \in Y \Rightarrow ghg^{-1} \in Y)\}.$$

Likewise, we can express their complements as

$$2^G \setminus U_2 = \bigcup_{g \in G \backslash \{1\}} V_g \cap V_{g^{-1}},$$

$$2^G \setminus U_3 = \bigcup_{g, h \in G} U_g \cap U_h \cap V_{gh},$$

and

$$2^G \setminus U_4 = \bigcup_{g,h \in G} U_h \cap V_{ghg^{-1}}.$$

Then each of $2^G \setminus U_2, 2^G \setminus U_3$ and $2^G \setminus U_4$ is an open set because they are a union of open sets. Finally, observe that

$$2^G \setminus \mathbb{X}(G) = 2^G \setminus U_1 \cup 2^G \setminus U_2 \cup 2^G \setminus U_3 \cup 2^G \setminus U_4.$$

Hence we can conclude that $2^G \setminus \mathbb{X}(G)$ is an open set and therefore $\mathbb{X}(G)$ is closed. $\square$

1.4 Trees and Π^0_1 classes

Let $2^{<\omega}$ denote the set of finite binary strings. A *tree* T is a subset of $2^{<\omega}$ that is closed under initial segments. In symbols, if $\sigma \in T$ and $\tau \subseteq \sigma$, then $\tau \in T$. A *path* through T is a function $f : \omega \to \{0, 1\}$ such that for all n,

$$f \restriction n = \langle f(0), \ldots, f(n-1) \rangle \in T.$$

We write $[T]$ to denote the set of all paths through T and $[T]$ is called the *set of paths* through T. A set $P \subseteq 2^\omega$ is called a Π^0_1 *class* if $P = [T]$ for some computable tree $T \subseteq 2^{<\omega}$.

We have already seen that 2^ω is a Boolean space and thus $[T]$ is also a topological space endowed with the subspace topology. When working with trees, it is helpful to have an alternative characterization of this topology. For each string $\sigma \in 2^{<\omega}$, define

$$B_\sigma = \{f \in 2^\omega \mid \sigma \subset f\}.$$

The topology generated by all sets of the form B_σ is equivalent to the product topology on 2^ω and the collection of clopen sets $\{B_\sigma \mid \sigma \in 2^{<\omega}\}$ is a basis for the topology of 2^ω. Similarly, if T is a tree, then for all $\sigma \in 2^{<\omega}$,

$$B_\sigma \cap [T] = \{f \in [T] \mid \sigma \subset f\}$$

is a basic open set in the subspace topology of $[T]$.

Proposition 1.4.1.

(i) *If $T \subseteq 2^{<\omega}$ is a tree, then $[T]$ is a closed subset of 2^ω.*

(ii) *For every closed set $C \subseteq 2^\omega$, there is a tree $T \subseteq 2^{<\omega}$ such that $C = [T]$.*

Proof. For part (i), let $T \subseteq 2^{<\omega}$ be a tree. Let

$$A = \{\sigma \in 2^{<\omega} \mid \sigma \text{ is not on a path through } T\}$$

and let $U = \bigcup_{\sigma \in A} B_\sigma$. We claim $U = 2^\omega \setminus [T]$. If $X \in U$, then $X \in B_\sigma$ for some $\sigma \in A$. So for all $Y \in [T]$ there exists some $n < |\sigma|$ such that $\sigma(n) \neq Y(n)$. Since $\sigma \subset X$, it follows that for all $Y \in [T]$ there exists some n such that $X(n) \neq Y(n)$. So $X \notin [T]$. Conversely, if $X \notin [T]$, then there exists some n such that $X \upharpoonright n \notin T$. Let $\tau = X \upharpoonright n$. Then $\tau \in A$ and so $B_\tau \subseteq U$. Thus $X \in B_\tau \subseteq U$. Hence $U = 2^\omega \setminus [T]$ and since U is an open set, we get that $[T]$ is a closed set.

To prove part (ii), suppose $C \subseteq 2^\omega$ is closed. Define a subset T of $2^{<\omega}$ by $\sigma \in T$ if and only if there exists an $X \in C$ with $\sigma \subset X$. The set T is easily seen to be a tree. We claim that $C = [T]$. First assume $X \in C$. Then $X \upharpoonright n \in T$ for all n by definition of T and so $X \in [T]$. Next assume $X \in [T]$. We show that $X \in \overline{C}$ (the closure of C in 2^ω). Fix some B_σ such that $\sigma \subset X$; so $X \in B_\sigma$. Now $\sigma \in T$ because $X \in [T]$. So there exists a $Y \in C$ with $\sigma \subset Y$. Then $Y \in B_\sigma$ and therefore $X \in \overline{C}$. But $\overline{C} = C$ since C is closed. Therefore $X \in C$ and it follows that $C = [T]$. □

Proposition 1.4.1 tells us that we can view closed sets in 2^ω as sets of paths through trees. Since a Π^0_1 class is defined as the set of paths through a computable tree, we sometimes refer to a Π^0_1 class as an effectively closed subset of 2^ω.

In connection with ordered groups, we have the following important result about computable groups.

Theorem 1.4.2. *If G is an orderable computable group, then $\mathbb{X}(G)$ is a Π^0_1 class.*

Proof. Let G be an orderable computable group. Assume the domain of G is ω and fix an enumeration $G = \{g_0, g_1, g_2, \ldots\}$ with g_0 representing the identity element. We describe how to build a computable tree $T \subseteq 2^{<\omega}$ which will have the property that $[T] = \mathbb{X}(G)$. We will build T in stages such that $T_0 \subseteq T_1 \subseteq \cdots$ and $T = \bigcup_{s\in\omega} T_s$. Each T_s denotes part of T build at the end of stage s and includes all nodes of T of length at most s. To each $\sigma \in T$, we assign a finite set $P_\sigma \subseteq \omega$ which will consist of elements σ "thinks" are part of the positive cone of some order.

Construction:

Stage 0: Set $T_0 = \langle\rangle = \{\lambda\}$ and $P_\lambda = \emptyset$.

Stage 1: Set $T_1 = \{\langle 0\rangle\}$ (this ensures g_0 is not on any path through T) and set $P_{\langle 0\rangle} = \emptyset$.

Stage $s+1$: Assume we have defined T_s. Define T_{s+1} to include T_s. For each $\sigma \in T_s$ with $|\sigma| = s$: if $g_0 \in P_\sigma$, then σ has no extensions in T_{s+1}; else, put $\sigma 0$ and $\sigma 1$ into T_{s+1}. We define the sets

$$P_{\sigma 0} = P_\sigma \cup \{g_{s+1}^{-1}\} \cup \{gh \mid g, h \in P_\sigma\} \cup \{g_i h g_i^{-1} \mid h \in P_\sigma \text{ and } i \leq s+1\}$$

and

$$P_{\sigma 1} = P_\sigma \cup \{g_{s+1}\} \cup \{gh \mid g, h \in P_\sigma\} \cup \{g_i h g_i^{-1} \mid h \in P_\sigma \text{ and } i \leq s+1\}.$$

This completes the construction of T.

We can view a path through T as a choice of signs for each $g \in G \backslash \{g_0\}$ and the elements of $[T]$ are exactly the positive cones of G. Hence $[T] = \mathbb{X}(G)$. □

Theorem 1.4.2 along with the various basis theorems for Π^0_1 classes imply the existence of orders with different computability theoretic properties for every orderable computable group. For example, by the Low Basis Theorem, every orderable computable group has an order of low degree.

1.5 Archimedean relations

In this section we introduce the Archimedean property of an ordering and its generalization.

Definition 1.5.1. An ordered group $(G, <)$ is called *Archimedean* if for every pair of positive elements $g, h \in G$ there exists a positive integer n such that $g < h^n$.

Example 1.5.2. The usual orderings of $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are Archimedean.

Example 1.5.3. Every subgroup of an Archimedean group is Archimedean.

Example 1.5.4. The orderings of $\mathbb{Z}^2$ described in Example 1.2.8 are Archimedean. On the other hand, the lexicographic ordering of $\mathbb{Z}^2$ is not Archimedean.

The following well-known result of Hölder is indispensable in the study of ordered groups (see [CR16, Theorem 2.6] for a proof). To put it succinctly, it states that the group $(\mathbb{R}, +)$ is universal for Archimedean ordered groups. Thus all Archimedean ordered groups are abelian, a fact that is not at all obvious from the definition.

Theorem 1.5.5 (Hölder's theorem)**.** *Every Archimedean ordered group is order-isomorphic to a subgroup of the naturally ordered additive group of real numbers.*

In fact, an even more general result holds for ordered abelian groups, due to Hans Hahn. This is regarded to be one of the deepest results in the theory of ordered abelian groups.

Theorem 1.5.6 (Hahn's Embedding Theorem)**.** *Every ordered abelian group is order-isomorphic to a subgroup of some lexicographic product of the naturally ordered additive reals.*

In the case of Archimedean groups, Hahn's Embedding Theorem reduces to Hölder's theorem. See [Cli54], [Fuc63, §IV.5] and [KK74, §VII.3] for more details and proofs.

We can generalize the Archimedean property in the following way. Recall that the absolute value $|g|$ of an element g is defined to be $|g| = \max(g, g^{-1})$.

Definition 1.5.7. Let $(G,<)$ be an ordered group and let $g,h \in G$. We define $g \sim h$ if there exist positive integers m and n such that $|g| < |h|^m$ and $|h| < |g|^n$. We say g and h are *Archimedean equivalent* if $g \sim h$. We say g is *Archimedean less than* h, denoted by $g \ll h$, if $|g|^n < |h|$ for all positive integers n.

Note that an ordered group is Archimedean if and only if all nonidentity elements are Archimedean equivalent. This observation along with Proposition 1.6.15 together give another proof of the fact that every Archimedean ordered group is abelian. The following properties of the relations $\sim$ and $\ll$ are routine.

Proposition 1.5.8. *Let $(G,<)$ be an ordered group, and let $\sim$ and $\ll$ be as defined above.*

(i) *For all $g,h \in G$, exactly one of the following holds: $g \ll h$, $g \sim h$ or $h \ll g$.*

(ii) *$\sim$ is an equivalence relation and $\ll$ is a transitive relation.*

(iii) *$g \sim g^n$ for all nonzero integers n.*

(iv) *$g \ll h$ if and only if $g^n < |h|$ for all integers n.*

(v) *$h \sim h'$ implies $ghg^{-1} \sim gh'g^{-1}$ for all $g \in G$.*

(vi) *$h \ll h'$ implies $ghg^{-1} \ll gh'g^{-1}$ for all $g \in G$.*

(vii) *$g \ll h$, $g \sim g'$ and $h \sim h'$ imply $g' \ll h'$.*

For an ordered group $(G,<)$, we will refer to the equivalence classes of G under the Archimedean equivalent relation as the *Archimedean classes* of G. We will write $[g]$ to denote the Archimedean class of $g \in G$. We can define a linear order on the set of Archimedean classes of G by declaring

$$[g] \ll [h] \text{ if and only if } g \ll h$$

for $g,h \in G$. (We will abuse notation and denote the order on the Archimedean classes also by $\ll$.) Proposition 1.5.8(vii) tells us that this induced linear order on the set of

Archimedean classes is well-defined. It follows from the definitions that the identity of G forms an Archimedean class by itself—moreover, it is the least class under the prescribed ordering of the Archimedean classes.

We collect several useful results about $\sim$ and $\ll$.

Proposition 1.5.9. *Let $(G, <)$ be an ordered group and let $g, h \in G$.*

(i) *If $g \not\sim h$, then $|g| < |h|$ if and only if $|g| \ll |h|$.*

(ii) *If $g \ll h$, then h, gh and hg all have the same sign, that is to say, all elements are either positive or negative.*

Proof. (i) Assume g and h are positive. Suppose first that $g < h$. If $h < g^n$ for some positive integer n, then $g \sim h$, a contradiction to our assumption. So $g^n < h$ for all positive integers n and $g \ll h$. Conversely, if $g \ll h$, then, by definition, $g^n < h$ for all positive integers n and so $g < h$.

(ii) Suppose $g \ll h$, and h and gh do not have the same sign. Consider the case when h is positive and gh is negative. So $gh < 1$ and $h < g^{-1}$. Then we must have that either $h \sim g^{-1}$ or $h \ll g^{-1}$. But since $g \sim g^{-1}$, this would imply that either $h \sim g$ or $h \ll g$, the desired contradiction. Therefore, h and gh must have the same sign. □

Proposition 1.5.10. *Let $(G, <)$ be an ordered group.*

(i) *If $a \ll g$ and $b \ll g$, then $ab \ll g$.*

(ii) *If $a \ll b$, then $b \sim ab \sim ba$.*

(iii) *If $g \not\sim h$, $a \ll g$ and $b \ll h$, then $ab \ll gh$.*

(iv) *Let $a_1, \ldots, a_n \in G$. Suppose there exists an i such that $a_j \ll a_i$ for all $j \neq i$. Then $a_1 \cdots a_n \sim a_i$.*

Proof. (i) Suppose $|a| < |b|$. Then $|a||b| < |b|^2$ and $|b||a| < |b|^2$. Note that either $|ab| \leq |a||b|$ or $|ab| \leq |b||a|$. In either case, we get that $|ab| < |b|^2$ and so $|ab|^n < |b|^{2n}$. Since $b \ll g$, $|b|^{2n} < |g|$ for all $n \geq 1$. Thus $|ab|^n < |g|$ for all $n \geq 1$ and $ab \ll g$.

Similarly, if $|b| < |a|$, then we will get that $|ab| < |a|^2$. It follows that $|ab|^n < |a|^{2n} < |g|$ for all $n \geq 1$ and $ab \ll g$.

(ii) First, assume a and b are positive. Then $1 < a$ implies $b < ab$ and $a < b$ implies $ab < b^2$. So we get $b \sim ab$ and $b \sim ba$ follows by a similar argument. Next, assume a and b are negative. Then a^{-1} and b^{-1} are positive and $b^{-1} \sim a^{-1}b^{-1} \sim b^{-1}a^{-1}$. By taking inverses, $b \sim ab \sim ba$.

Our next case is when a is negative and b is positive. Note $|a| < |b|$ by our assumption and so we get $|ab| = |a|^{-1}|b| = ab$. We have $ab < b$ because $a < 1$. Our assumption $a \ll b$ implies $a^{-2} < b$ and so $1 < aba$. Thus, $b < abab = (ab)^2$ and $b \sim ab$. Likewise, we can get $b \sim ba$. The case when a is positive and b is negative can be proven similarly.

(iii) Assume $g \not\sim h$ and say $g \ll h$. Then $a \ll g \ll h$ implies $a \ll h$. So we have $a \ll h$ and $b \ll h$ and in turn $ab \ll h$ by (i). By (ii), $h \sim gh$ and we get $ab \ll gh$.

(iv) If $a_{i-1} \ll a_i$, then $a_{i-1}a_i \sim a_i$ by (ii). Next, $a_{i+1} \ll a_i$ and $a_{i-1}a_i \sim a_i$ imply $a_{i-1}a_ia_{i+1} \sim a_{i-1}a_i \sim a_i$. We can continue along like this to get that $a_1 \cdots a_n \sim a_i$. □

1.6 Convex subgroups

In this section we discuss an important concept in the theory of ordered groups. We start with the definition of a convex subgroup.

Definition 1.6.1. Suppose G is an ordered group with ordering $<$. A subgroup C of G is *convex relative to* $<$ if $c_1 < g < c_2$ implies $g \in C$ for all $c_1, c_2 \in C$ and $g \in G$.

Remark. Definition 1.6.1 is equivalent to saying that $g \in C$ whenever $1 < g < c$ for any $c \in C$ and $g \in G$.

Proposition 1.6.2. *Let $(G, <)$ be an ordered group, and suppose C and D are convex subgroups relative to $<$. Then either $C \subseteq D$ or $D \subseteq C$.*

Proof. Suppose $C \neq D$. Without loss of generality, assume there exists some $d \in D \setminus C$. We will argue that $C \subseteq D$. Let $c \in C$. There are two possibilities: $1 < |c| < |d|$ or $1 < |d| < |c|$. If $1 < |d| < |c|$, then we have that $d \in C$ by convexity of C, but this contradicts $d \notin C$. Thus it must be that $1 < |c| < |d|$ and so $c \in D$ by convexity of D. Hence, $C \subseteq D$. □

Corollary 1.6.3. *The convex subgroups of an ordered group are linearly ordered by inclusion. Furthermore, the union of any collection of convex subgroups is a convex subgroup and the intersection of any collection of convex subgroups is a convex subgroup.*

Example 1.6.4. The additive group $\mathbb{Z}^2$ with respect to the lexicographic order has the following chain of convex subgroups: $\{0\} \times \{0\} \subset \{0\} \times \mathbb{Z} \subset \mathbb{Z}^2$.

Convex subgroups allow us to give an alternative characterization of Archimedean ordered groups.

Proposition 1.6.5. *Let $(G, <)$ be an ordered group. Then G is Archimedean if and only if it has no nontrivial proper convex subgroups. In other words, an Archimedean ordered group has no convex subgroups other than the whole group and the trivial subgroup.*

Proof. First assume G is Archimedean. Let H be a nontrivial convex subgroup of G. We need to show that $H = G$. Suppose $g \in G$ and $h \in H \setminus \{1\}$. Then there exists some positive n such that $1 < |g| < |h|^n$. By convexity of H, we get that $g \in H$ and $H = G$.

Conversely, assume G is not Archimedean. Then there exists two positive elements $g, h \in G$ such that $h^n < g$ for all positive n. Define

$$H = \{x \in G \mid \exists\, n \in \mathbb{Z}\ (h^{-n} < x < h^n)\}.$$

It is straightforward to show that H is a convex subgroup of G and, furthermore, H is proper since $g \notin H$. Thus we have a nontrivial proper convex subgroup of G. □

Convex subgroups of an orderable group are closely related to the orderability of its quotients. In particular, a quotient by a convex normal subgroup has a naturally induced order on it.

Proposition 1.6.6. *Suppose $(G, <)$ is an ordered group and C is a convex normal subgroup of G. Then G/C is an orderable group and the following recipe defines an ordering of it: $gC \prec hC$ if and only if $g < h$.*

Proof. We show that the given ordering is well-defined and the rest follows easily from this. Suppose $gC = g'C, hC = h'C$ and $gC \neq hC$. Assume $g < h$. We have to show that $g' < h'$. We can write $g' = gc_1$ and $h' = hc_2$ for some $c_1, c_2 \in C$. Observe that $1 < g^{-1}h$ and $g^{-1}h \notin C$, therefore $c < g^{-1}h$ for all $c \in C$. In particular, $c_1c_2^{-1} < g^{-1}h$ and equivalently $gc_1 < hc_2$. Hence $g' < h'$, as desired. □

The next proposition is an easy consequence of the correspondence theorem for quotient groups. It says that the natural correspondence between subgroups of G/C and subgroups of G that contain C preserves convexity.

Proposition 1.6.7. *Suppose $(G, <)$ is an ordered group and C a convex normal subgroup of G. Then a subgroup H of G satisfying $C \subseteq H$ is convex relative to $<$ if and only if H/C is convex relative to the natural quotient order on G/C.*

We next define the notion of a convex jump.

Definition 1.6.8. If $C \subset D$ is a pair of distinct convex subgroups of an ordered group, we say that $C \subset D$ is a *convex jump* if there are no convex subgroups strictly between them. We will use the notation $C \sqsubset D$ to denote a convex jump.

Proposition 1.6.9. *Suppose $(G, <)$ is an ordered group.*

(i) *If C is a convex subgroup, then gCg^{-1} is a convex subgroup for all $g \in G$.*

(ii) *If $C \sqsubset D$ is a convex jump, then $gCg^{-1} \sqsubset gDg^{-1}$ is a convex jump for all $g \in G$.*

Proof. For (i), clearly gCg^{-1} is a subgroup. Let $g, h \in G$ and $c \in C$. Then $1 < h < gcg^{-1}$ holds if and only if $1 < g^{-1}hg < c$, and so $g^{-1}hg \in C$ implies $h \in gCg^{-1}$. To show (ii), let H be a convex subgroup of G and suppose $gCg^{-1} \subseteq H \subseteq gDg^{-1}$. Then $C \subseteq g^{-1}Hg \subseteq D$. Therefore either $C = g^{-1}Hg$ or $D = g^{-1}Hg$, and so $H = gCg^{-1}$ or $H = gDg^{-1}$. □

Recall the following algebraic notion: if H is a subgroup of a group G, the *normalizer of H in G* is the subgroup $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. The normalizer $N_G(H)$ contains H and it is the largest subgroup in which H is normal.

Proposition 1.6.10. *Suppose C and D are convex subgroups of an ordered group.*

(i) *If $C \sqsubset D$ is a convex jump, then $N_G(C) = N_G(D)$.*

(ii) *If $C \sqsubset D$ is a convex jump, then C is normal in D and the natural quotient order on D/C is Archimedean.*

Proof. To prove (i), since $C \sqsubset D$ implies $gCg^{-1} \sqsubset gDg^{-1}$, we can conclude that if g belongs to $N_G(C)$ or $N_G(D)$, then it belongs to the other one as well.

To prove (ii), first observe that $D \subseteq N_G(D) = N_G(C)$ shows that D normalizes C. Next to see that D/C is Archimedean, note that if D/C contained a nontrivial proper convex subgroup H/C, then it would follow that $C \subset H \subset D$. But this contradicts $C \sqsubset D$ is a jump. Thus D/C contains no nontrivial proper convex subgroups and must be Archimedean by Proposition 1.6.5. □

Corollary 1.6.11. *If an ordered group G has only finitely many convex subgroups, then each convex subgroup is normal in G.*

Proof. The finitely many convex subgroups of G form a chain, say $C_1 \subset \cdots \subset C_n = G$. Furthermore, $C_i \sqsubset C_{i+1}$ for all i. Then by Proposition 1.6.10(i), $N_G(C_1) = N_G(C_2) = \cdots = N_G(G) = G$ and C_i is normal in G for all i. □

Proposition 1.6.12. *Suppose $(G, <)$ is an ordered group and C is a convex subgroup. Let $g \in G$ and $c \in C$. If $g \notin C$, then $c \ll g$.*

Proof. If $g \notin C$, then for all $x \in C$, $|x| < |g|$. For otherwise, there would exist $x_0 \in C$ with $1 < |g| < |x_0|$, and this would imply $g \in C$ by convexity of C. Therefore $|x| < |g|$ for all $x \in C$. In particular, $|c|^n < |g|$ for all n and so $c \ll g$. □

In an ordered group every nonidentity element determines a convex jump as we now discuss.

Definition 1.6.13. Let g be an element of an ordered group G and $g \neq 1$. Define G^g to be the intersection of all convex subgroups of G that contain g and define G_g to be the union of all convex subgroups of G that do not contain g.

It is an immediate consequence of the definitions that the subgroups G_g and G^g form a convex jump, i.e., $G_g \sqsubset G^g$. The next proposition gives useful descriptions of G_g and G^g.

Proposition 1.6.14. *Suppose $(G, <)$ is an ordered group. Let $g \in G$ with $g \neq 1$. Then*

$$G_g = \{x \in G \mid x \ll g\} \quad \textit{and} \quad G^g = \{x \in G \mid x \ll g \textit{ or } x \sim g\}.$$

Proof. Let $g \in G$ be a nonidentity element and let $A = \{x \in G \mid x \ll g\}$. We first want to prove that $G_g \subseteq A$. Suppose $h \in G_g$. Then there exists a convex subgroup C such that $h \in C$ and $g \notin C$. By Proposition 1.6.12, $h \ll g$ and thus $h \in A$. Now assume $h \in A$. Define

$$H = \{x \in G \mid \exists\, n \in \mathbb{Z}\ (h^{-n} < x < h^n)\}.$$

Then H is a convex subgroup. Moreover, $g \notin H$ because otherwise we would have $h^{-n} < g < h^n$ but this contradicts $h \ll g$. Therefore, $H \subseteq G_g$ and since clearly $h \in H$, we can conclude $h \in G_g$. Hence, $A \subseteq G_g$ and so $A = G_g$, as desired.

To prove the second part, let $B = \{x \in G \mid x \ll g \text{ or } x \sim g\}$ and let $h \in G^g$. Suppose on the contrary $g \ll h$. Define

$$H = \{x \in G \mid \exists\, n \in \mathbb{Z}\ (g^{-n} < x < g^n)\}.$$

Once again H is a convex subgroup and $g \in H$. By definition of G^g, it follows that $G^g \subseteq H$.

Furthermore, it must be that $h \notin H$ since $g \ll h$. So we have $h \in G^g$ and $h \notin H$ but this contradicts $G^g \subseteq H$. Hence, either $h \ll g$ or $h \sim g$, and we get $h \in B$. Next, let $h \in B$. We have to show that $h \in G^g$. Let C be a convex subgroup that contains g. It suffices to show that $h \in C$. We know that either $h \ll g$ or $h \sim g$. If $h \ll g$, then $|h| < |g|$ and it follows $h \in C$. On the other hand, if $h \sim g$, then $|h| < |g|^m$ for some m and again it will follow that $h \in C$. This shows that $B \subseteq G^g$ and the result follows. □

Proposition 1.6.15. *Let* $(G, <)$ *be an ordered group and let* $a, b \in G$*. Then*

$$[a, b] \ll \max(|a|, |b|).$$

Proof. Without loss of generality, assume $|a| < |b|$. We have to show that $[a, b] \ll |b|$. Consider the subgroup H of G generated by a and b. Let D be the intersection of all convex subgroups in H containing b and let C be the union of all convex subgroups in H not containing b. In other words, let $C = H_b$ and $D = H^b$. In fact, $D = H$ because $a, b \in D$. Then $C \subset H$ is a convex jump and, by Proposition 1.6.10(ii), C is normal in H. Now C is a normal, convex subgroup of H, so we can form the quotient group H/C which will be orderable. Because $C \subset H$ is a jump, then H/C is Archimedean and, in particular, must be abelian. Therefore $[H, H] \subseteq C$ and so $[a, b] \in C$. (Here $[H, H]$ denotes the commutator subgroup of H.) By Proposition 1.6.14, we conclude that $[a, b] \ll |b|$. □

1.7 Cantor-Bendixson rank

Definition 1.7.1. Let X be a topological space. A point $x \in X$ is called an *isolated point* if $\{x\}$ is open. A point $x \in X$ is called a *limit point* if every neighborhood of x contains a point other than x. If $A \subseteq X$, we say that a point $x \in X$ is a *limit point of* A if every neighborhood of x contains a point of A other than x.

In the context of the space of orders, an isolated point in $\mathbb{X}(G)$ corresponds to an order on G that is the unique order satisfying some finite string of inequalities. Equivalently, an isolated order is the unique order in which some fixed finite set of elements of G are all positive. As for trees, if T is a tree, then a path $f \in [T]$ is an isolated path if there is an $n \in \omega$ such that f is the unique path in T through $f \upharpoonright n$; in other words, if there exists a $\sigma \in T$ with $B_\sigma \cap [T] = \{f\}$. Recall that an isolated path in a Π^0_1 class is computable.

Proposition 1.7.2. *Let $T \subseteq 2^{<\omega}$ be a tree. Then $[T]$ is finite if and only if $[T]$ contains only isolated paths.*

Proof. If $[T]$ is finite, then it is clear by the Hausdorff property that every path in $[T]$ is isolated. Now suppose $[T]$ contains only isolated paths. For each $f \in [T]$, fix a string $\sigma_f \in T$ such that f is the unique path through σ_f. Then the collection $\{B_{\sigma_f} \mid f \in [T]\}$ is an open cover of $[T]$. By compactness, it has a finite subcover, say $\{B_{\sigma_1}, \ldots, B_{\sigma_n}\}$. Then $[T]$ can only contain finitely many paths, else for some i there would need to be more than one path passing through σ_i, a contradiction. Hence $[T]$ has only finitely many paths. $\square$

Definition 1.7.3. Let X be a topological space. The *Cantor-Bendixson derivative* of X, denoted X', is the set of nonisolated points of X; equivalently, X' is the set of limit points of X. For each ordinal α, define $X^{(\alpha)}$ recursively as follows:

(1) $X^{(0)} = X$,

(2) $X^{(\alpha+1)} = (X^{(\alpha)})'$,

(3) $X^{(\alpha)} = \bigcap_{\gamma<\alpha} X^{(\gamma)}$ if α is a limit ordinal.

In general, if X is an arbitrary topological space, then X' is not necessarily closed. However, if X is a Hausdorff space, then X' is closed in X and therefore $X^{(\alpha)}$ is a closed subset of X for all α.

It is clear from the definition that the subspaces $X^{(\alpha)}$ form a nonincreasing sequence, that is, $X^{(\delta)} \subseteq X^{(\gamma)}$ whenever $\gamma < \delta$. As shown by the next proposition, this transfinite sequence of derivatives must eventually be constant.

Proposition 1.7.4. *Let X be a topological space. If $|X| = \kappa$ for some cardinal number κ, then there is an ordinal $\alpha < \kappa^+$ such that $X^{(\alpha)} = X^{(\alpha+1)}$.*

Proof. Assume for a contradiction, $X^{(\alpha)} \neq X^{(\alpha+1)}$ for all ordinals $\alpha < \kappa^+$. Then for all $\alpha < \kappa^+$, there exists some $x_\alpha \in X$ such that $x_\alpha \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. This gives a map from κ^+ into X via $\alpha \mapsto x_\alpha$ that is injective. This is a contradiction since by assumption $|X| = \kappa$. Hence, there must exist some $\alpha < \kappa^+$ such that $X^{(\alpha)} = X^{(\alpha+1)}$. □

Definition 1.7.5. Let X be a topological space. The *Cantor-Bendixson rank* of X, denoted $\mathrm{CB}(X)$, is the least ordinal α such that $X^{(\alpha)} = X^{(\alpha+1)}$. We say a point $x \in X$ has *Cantor-Bendixson rank* α if α is the least ordinal such that $x \in X^{(\alpha)}$ but $x \notin X^{(\alpha+1)}$. We write $\mathrm{CB}(x) = \alpha$ to denote x has Cantor-Bendixson rank α. If $x \in X^{(\alpha)}$ for all α, then we write $\mathrm{CB}(x) = \infty$.

Remark. If X has Cantor-Bendixson rank α, then by induction it follows that $X^{(\beta)} = X^{(\alpha)}$ for all $\beta > \alpha$.

Note that $\mathrm{CB}(X) = 0$ if every point of X is a limit point and for $x \in X$, $\mathrm{CB}(x) = 0$ if x is an isolated point. We make the observation that if A and B are subsets of a topological space, then $A \subseteq B$ implies $A' \subseteq B'$.

Proposition 1.7.6. *Let X be a topological space and let $A \subseteq X$. Suppose* $\mathrm{CB}(a) \geq \alpha$ *for all $a \in A$, or equivalently, $A \subseteq X^{(\alpha)}$. If $x \in X$ is a limit point of A, then* $\mathrm{CB}(x) \geq \alpha + 1$.

Proof. If $A \subseteq X^{(\alpha)}$, then $A' \subseteq (X^{(\alpha)})' = X^{(\alpha+1)}$. Since $x \in X$ is a limit point of A, we have $x \in A' \subseteq X^{(\alpha+1)}$ and therefore $\mathrm{CB}(x) \geq \alpha + 1$. □

Proposition 1.7.7. *If C is a countable closed subset of 2^ω, then the Cantor-Bendixson rank of C is a countable ordinal α such that α is a successor ordinal and $C^{(\alpha)} = \emptyset$.*

Proof. Let C be a nonempty countable closed subset of 2^ω. By Proposition 1.7.4, the Cantor-Bendixson rank of C is a countable ordinal, say α. We claim that $C^{(\alpha)} = \emptyset$. Since C is countable and closed, it must contain an isolated point. Because otherwise C would be a countable, perfect subset of 2^ω and that is impossible since perfect sets are uncountable. In addition, every nonempty derivative of C must also contain an isolated point because they are also closed, and either finite or countable. Thus $C^{(\alpha)} = C^{(\alpha+1)}$ implies that $C^{(\alpha)} = \emptyset$, as required.

Next we want to show that α is a successor ordinal. Suppose for a contradiction α is a limit ordinal. Then $C^{(\alpha)} = \bigcap_{\gamma<\alpha} C^{(\gamma)}$ where each $C^{(\gamma)}$ is a nonempty closed set. It follows by compactness that $\bigcap_{\gamma<\alpha} C^{(\gamma)}$ must be nonempty and we have a contradiction. Therefore α is a successor ordinal. $\square$

Remark. Observe that if $C \subseteq 2^\omega$ is a nonempty countable closed set and C has Cantor-Bendixson rank α, where $\alpha = \beta + 1$ for some ordinal β, then $C^{(\beta)}$ will be nonempty, finite and contain only isolated points.

Suppose $P \subseteq 2^\omega$ is a countable Π^0_1 class. Then P has isolated elements and, in turn, computable elements. Note that P has Cantor-Bendixson rank 1 if and only if P contains only isolated elements, and by Proposition 1.7.2, this is equivalent to P being finite. Thus we see that countable Π^0_1 classes of rank 1 are finite classes with all their members computable, so from a computational viewpoint they are not very interesting.

1.8 Open questions

We would like to take a small prelude into ordered fields to motivate some interesting questions about ordered groups. As with groups, there is a set of algebraic conditions that determines if a subset of an orderable field is the positive cone of an order. Likewise, one can define the space of orders of an orderable field to consist of all positive cones of orders on the field. It is well-known that the space of orders of an orderable field forms a topological space, in fact, a Boolean space (i.e. totally disconnected, compact and Hausdorff). Craven [Cra75] showed that the converse is also true.

Theorem 1.8.1 ([Cra75, Theorem 5]). *Every Boolean space is homeomorphic to the space of orders of some orderable field.*

Furthermore, Metakides and Nerode [MN79] showed that the effective versions of these results hold as well.

Theorem 1.8.2 ([MN79, Theorem 7.1]). *The space of orders of an orderable computable field is a Π^0_1 class.*

Theorem 1.8.3 ([MN79, Theorem 7.3]). *Every Π^0_1 class is homeomorphic to the space of orders of some orderable computable field and the homeomorphism is Turing degree preserving.*

These results motivate the question whether similar results hold for ordered groups. We have already seen in Theorem 1.4.2 that the analogue of Theorem 1.8.2 is true for orderable computable groups. However, unlike in the case of fields, it still remains an open question whether Theorems 1.8.1 and 1.8.3 are true for orderable groups and orderable computable groups, respectively.

Question 1.8.4. Given a closed subset $C \subseteq 2^X$ for some set X, is there an orderable group G such that $\mathbb{X}(G) \cong C$?

Question 1.8.5. Given a Π^0_1 class $P \subseteq 2^\omega$, is there an orderable computable group G such that $\mathbb{X}(G) \cong P$? Does the homeomorphism preserve the Turing degrees?

It is worth pointing out that Theorem 1.8.3 cannot hold in its full generality for computable groups. Notice that for any order of a computable group, the dual order is an order of the same Turing degree. So in particular, the corresponding Π^0_1 class of the space of orders always contains Turing comparable members. But it is known that there do exist Π^0_1 classes all of whose members are mutually Turing incomparable (see e.g. [JS72, Theorem 4.7]). Thus we cannot get exactly the same result as Theorem 1.8.3 for computable groups. Solomon [Sol02] has shown that the spaces of orders of computable torsion-free abelian and nilpotent groups cannot represent all Π^0_1 classes even in a weak manner. For more details and precise formulations of the results, see [Sol02].

In trying to address Question 1.8.5, an obstacle to deal with is to understand the computable presentations of orderable groups. As we saw in Theorem 1.1.4 there exists a computable presentation of $\bigoplus_\omega \mathbb{Z}$ that does not have any computable orders. At the same time, it is clear that there do exist computable copies of $\bigoplus_\omega \mathbb{Z}$ which will have computable orders. A natural question to ask is does an orderable computable group always have a computable presentation that admits a computable order?

Here is a remarkable result about computable groups recently shown by Darbinyan [Dar20]. It answers an open question for bi-orderable groups.

Theorem 1.8.6 ([Dar20, Corollary 1])**.** *There exists a bi-orderable computable group which does not have a computable presentation with a computable bi-order. Moreover, the group can be chosen to be a solvable group of derived length* 3*.*

Surprisingly the proof does not involve a lot of computability machinery but instead is proven using algebraic tools involving wreath products and embedding theorems. In contrast, Solomon [Sol02] has shown that every orderable computable abelian group has a computable presentation which admits a computable order. These two results lead to the following open question. Recall that a metabelian group is a solvable group of derived length 2.

Question 1.8.7. Does every bi-orderable computable metabelian group have a computable presentation with a computable bi-order?

In the study of Π^0_1 classes, there has been considerable work done studying the relationship between the degrees of members of Π^0_1 classes and the Cantor-Bendixson ranks of Π^0_1 classes. One known result in this area is the following theorem due to Cenzer, Downey, Jockusch, and Shore [Cen+93].

Theorem 1.8.8 ([Cen+93, Theorem 2.2])**.** *For every computable ordinal $\alpha \geq 1$, there is a countable (thin) Π^0_1 class P_α with Cantor-Bendixson rank $\alpha + 1$. Furthermore, for any*

$X \in P_\alpha$, X is an isolated path if and only if X is computable.

An interesting question to ask related to ordered groups is the following.

Question 1.8.9. For every computable ordinal $\alpha \geq 1$, is there an orderable computable group G such that $\mathbb{X}(G)$ is countable, has Cantor-Bendixson rank $\alpha + 1$ and the only computable elements of $\mathbb{X}(G)$ are the isolated orders?

In trying to address this question, we must first solve the problem of coming up with an example of an orderable group that has exactly countably many orders. This turns out to be not a trivial task and it is rather difficult to construct an example of such a group. The only known example of such a group in the literature is given by Buttsworth [But71]. In Section 3.1, we will prove that the group constructed in [But71], denoted $G(p, q)$, has exactly countably many orders. We will then carry out a further analysis of the space of orders of $G(p, q)$ to prove a new result that the space of orders $\mathbb{X}(G(p, q))$ has Cantor-Bendixson rank 2. From here, we will show that it is possible to generalize the construction that Buttsworth gives to deduce the following result.

Theorem 1.8.10. *For all $2 \leq n < \omega$, there is a countable orderable group G_n such that $\mathbb{X}(G_n)$ is countable and has Cantor-Bendixson rank n.*

In Section 3.3, we will show that the groups G_n all have a computable presentation. So we have a partial answer to part of our question from above, namely, we can build an orderable computable group such that its space of orders is countable and has Cantor-Bendixson rank n for all $2 \leq n < \omega$. There seems to be no obvious way to extend this group construction further to be able to achieve infinite Cantor-Bendixson ranks. We conjecture that to be able to get infinite ranks it will be necessary to come up with an entirely different example of an orderable group that has countably many orders.

As for the other part of the question on trying to make only the computable elements be isolated in the space of orders, we show that this method of constructing orderable groups cannot achieve this property. Each group G_n has a “nice” computable presentation in which all the orders are computable. More so, it turns out that the groups G_n are all computably categorical. This result implies that every order on every computable copy of G_n is computable. Thus the best we can do is make all the members of the corresponding Π^0_1 class of space of orders be computable. So it seems that once again it will be necessary to come up with a new example of an orderable group to be able to have a positive answer to this question.

1.9 Overview

In Chapter 2, we study the semidirect product in relation to orderability and computability. We give a criteria for when a semidirect product of two orderable groups is orderable. We give an example of a semidirect product that has the halting set coded into the homomorphism describing its multiplication structure but it is still possible to construct a computable presentation of this semidirect product. We prove a result giving sufficient conditions for when is a semidirect product computably categorical.

In Chapter 3, we construct a family of orderable groups such that they have exactly countably many orders and their space of orders has Cantor-Bendixson rank n for any $2 \leq n < \omega$. We will also prove that these groups have a computable presentation and they are all computably categorical.

In Chapter 4, we construct a computable torsion-free abelian group that is a computable copy of $\bigoplus_\omega \mathbb{Z}$ such that the group has no computable Archimedean orders but the group does admit at least one computable order—this computable order will necessarily be non-Archimedean.

Chapter 2

Semidirect products

2.1 Orderability of semidirect products

In this section, we discuss the semidirect product and make connections to orderability. We start by proving a general result about short exact sequences and extensions of groups. We mention in passing that the notion of group extensions and short exact sequences is the same. Any extension of groups can be viewed as a short exact sequence and any short exact sequence can be thought of as an extension of groups.

Suppose $(N, <_N)$ and $(H, <_H)$ are ordered groups and we have a short exact sequence

$$1 \to N \hookrightarrow G \xrightarrow{f} H \to 1.$$

Equivalently, suppose the group G is an extension of N by H. Consider the following recipe to define an ordering of G. Let $g, g' \in G$. Define

$$g <_G g' \text{ if and only if } f(g) <_H f(g'), \text{ or else } f(g) = f(g') \text{ and } 1 <_N g^{-1}g'. \tag{2.1}$$

Note that if $f(g) = f(g')$ then $f(g^{-1}g') = f(g)^{-1}f(g') = 1$ and so $g^{-1}g' \in \ker f = N$. The above recipe gives us a way to linearly order G. In the next proposition we give a necessary and sufficient condition for when this ordering is also invariant.

Proposition 2.1.1. *Suppose G is a group with normal subgroup N and quotient group $H \cong G/N$. In other words, suppose there is a short exact sequence*

$$1 \to N \hookrightarrow G \xrightarrow{f} H \to 1.$$

Suppose $(N, <_N)$ and $(H, <_H)$ are ordered groups. Then the recipe of (2.1) defines an invariant ordering of G if and only if the conjugation action of G upon N preserves the given ordering of N.

Proof. For the forward direction, let $g \in G$ and suppose $x, y \in N$ with $x <_N y$. Then $f(x) = f(y) = 1$ and $1 <_N x^{-1}y$. So by (2.1), $x <_G y$ and this implies $gxg^{-1} <_G gyg^{-1}$ since $<_G$ is invariant by assumption. But $f(gxg^{-1}) = f(gyg^{-1}) = 1$ since $N = \ker f$ is a normal subgroup. Thus by (2.1) it must be that $1 <_N (gxg^{-1})^{-1}(gyg^{-1})$ and so $gxg^{-1} <_N gyg^{-1}$.

For the reverse direction, suppose $g, g' \in G$ with $g <_G g'$. We first show that $xg <_G xg'$ holds for all $x \in G$. Assume $f(g) <_H f(g')$. Then

$$f(xg) = f(x)f(g) <_H f(x)f(g') = f(xg')$$

since $<_H$ is invariant. Next, assume $f(g) = f(g')$ and $1 <_N g^{-1}g'$. Then

$$f(xg) = f(x)f(g) = f(x)f(g') = f(xg')$$

and

$$1 <_N g^{-1}g' = g^{-1}x^{-1}xg' = (xg)^{-1}xg'.$$

Thus in either case, it follows by (2.1) that $xg <_G xg'$.

Lastly, we want to show that $gx <_G gx'$ for all $x \in G$. First assume $f(g) <_H f(g')$. Then

$$f(gx) = f(g)f(x) <_H f(g')f(x) = f(g'x)$$

and so by (2.1) it follows that $gx <_G g'x$. Next assume $f(g) = f(g')$ and $1 <_N g^{-1}g'$. Then

$$f(gx) = f(g)f(x) = f(g')f(x) = f(g'x)$$

and by our hypothesis

$$1 = x^{-1}1x <_N x^{-1}g^{-1}g'x = (gx)^{-1}g'x.$$

So by (2.1) we have that $gx <_G g'x$. □

Remark. Notice that the order $<_G$ in (2.1) is an extension of the order $<_N$ of N. In fact, the positive cone for the prescribed ordering of G is the union of the positive cone of N and the pullback of the positive cone of H. That is, $P_G = P_N \cup f^{-1}(P_H)$.

With this result in hand, let us now turn towards the semidirect product. A semidirect product of two groups can be viewed as an extension of the two groups. Even more so, a short exact sequence of groups splits exactly when the group in the middle is a semidirect product. Since semidirect products are a special case of group extensions, we can reformulate Proposition 2.1.1 for semidirect products and make some important conclusions. But first, let us interpret the recipe in (2.1) for a semidirect product.

Suppose $(N, <_N)$ and $(H, <_H)$ are ordered groups. Suppose we have a semidirect product $G = N \rtimes_\varphi H$ where $\varphi\colon H \to \mathrm{Aut}(N)$ is a group homomorphism. As a piece of notation, we will sometimes write φ_h to denote the automorphism $\varphi(h)\colon N \to N$ for $h \in H$. Any element $g \in G$ can be uniquely written as $g = nh$ for some $n \in N$ and $h \in H$. Also, to be precise, since different authors use different notations when working with semidirect products, we will write our group multiplication as

$$gg' = (nh)(n'h') = n(hn'h^{-1})hh' = n\varphi_h(n')hh'.$$

So in particular, we will freely interpret the group homomorphism φ as inducing a conjugation action inside of G, i.e., $\varphi_h(n) = hnh^{-1}$. We will say that φ is *order-preserving* by which we will mean that the automorphism φ_h is order-preserving for all $h \in H$. (Really, the conjugation action of H on N is order-preserving.) It is important to note here that when we say order-preserving, we mean order-preserving with respect to some fixed order of N. Here, for example, we mean the order $<_N$ of N.

We can express our semidirect product as a short exact sequence

$$1 \to N \hookrightarrow G \xrightarrow{f} H \to 1$$

where f is the projection function. So $f(g) = f(nh) = h$. We can now rewrite (2.1) as

$$g <_G g' \text{ if and only if } h <_H h', \text{ or else } h = h' \text{ and } \varphi_{h^{-1}}(n) <_N \varphi_{h^{-1}}(n') \tag{2.2}$$

for any $g, g' \in G$ with $g = hn$ and $g' = h'n'$. To see where the last inequality comes from, observe that if $h = h'$, then

$$g^{-1}g' = (nh)^{-1}n'h = h^{-1}n^{-1}n'h = (h^{-1}n^{-1}h)(h^{-1}n'h) = \varphi_{h^{-1}}(n^{-1})\varphi_{h^{-1}}(n').$$

Since $<_N$ is invariant, the inequality

$$1 <_N \varphi_{h^{-1}}(n^{-1})\varphi_{h^{-1}}(n')$$

is equivalent to

$$\varphi_{h^{-1}}(n) <_N \varphi_{h^{-1}}(n').$$

We make two observations related to the recipe described in (2.2) to order $G = N \rtimes_\varphi H$. First, if we are checking the condition described in Proposition 2.1.1 for G, namely, whether the conjugation action of G upon N preserves the given ordering of N, it is sufficient to only check conjugation by all $h \in H$. To see this, let $x, y \in N$ with $x <_N y$. Then if we conjugate this inequality by some $nh \in G$, we have

$$(nh)x(nh)^{-1} = n\varphi_h(x)n^{-1} <_N n\varphi_h(y)n^{-1} = (nh)y(nh)^{-1}.$$

But

$$n\varphi_h(x)n^{-1} <_N n\varphi_h(y)n^{-1} \text{ if and only if } \varphi_h(x) <_N \varphi_h(y).$$

Thus we see that we only need to check conjugation by $h \in H$. The second observation is that in the case when φ is order-preserving, the recipe described in (2.2) reduces to a reverse lexicographic ordering on $G = N \rtimes_\varphi H$. This is easy to see, for if φ is order-preserving, then

$$\varphi_{h^{-1}}(n) <_N \varphi_{h^{-1}}(n') \text{ if and only if } n <_N n'.$$

Above we are composing with the map φ_h (which we are assuming is order-preserving) to simplify the inequality. We collect our results related to the orderability of semidirect products.

Proposition 2.1.2. *Let $(N, <_N)$ and $(H, <_H)$ be ordered groups. Let $G = N \rtimes_\varphi H$ be a semidirect product with $\varphi\colon H \to \mathrm{Aut}(N)$. The recipe of* (2.2) *defines an invariant ordering of G if and only if φ is order-preserving with respect to $<_N$.*

Proof. See Proposition 2.1.1 and the above discussion. □

Proposition 2.1.3. *Let $(N, <_N)$ and $(H, <_H)$ be ordered groups. Let $G = N \rtimes_\varphi H$ be a semidirect product with $\varphi\colon H \to \mathrm{Aut}(N)$. Suppose $\varphi_h\colon N \to N$ is order-preserving with respect to $<_N$ for all $h \in H$. Then G is orderable and the following recipe defines an ordering*

$$1 <_G nh \text{ if and only if } 1 <_H h, \text{ or else } h = 1 \text{ and } 1 <_N n.$$

Proof. Follows from Proposition 2.1.2 and the observation mentioned above. □

Not surprisingly, it turns out that if a semidirect product is orderable, then it always has a reverse lexicographic ordering.

Proposition 2.1.4. *Let $G = N \rtimes_\varphi H$ be a semidirect product of orderable groups and suppose G is orderable. Then there exists orderings $<_N$ and $<_H$ of N and H, respectively, such that the reverse lexicographic ordering of G with respect to $<_N$ and $<_H$ is an ordering.*

Proof. Let $<_G$ be an ordering of G. Since N and H are subgroups of G (or more precisely, there exists isomorphic copies of N and H inside G), we can let $<_N$ and $<_H$ be the restriction of $<_G$ to N and H, respectively. Then φ_h is order-preserving with respect to $<_N$ for all $h \in H$. Thus by Proposition 2.1.3, the reverse lexicographic ordering of G with respect to $<_N$ and $<_H$ is an ordering. □

Notice in the above proof that if G is orderable then there exists some ordering of N with respect to which φ is order-preserving. This in conjunction with Propositions 2.1.3 and 2.1.4 gives us the following.

Theorem 2.1.5. *Let $G = N \rtimes_\varphi H$ be a semidirect product of orderable groups. Then the following are equivalent:*

(a) *G is orderable.*

(b) *G has a reverse lexicographic ordering.*

(c) *φ is order-preserving with respect to some order of N.*

Proof. The implication (a) $\Rightarrow$ (b) is the content of Proposition 2.1.4 and, of course, (b) $\Rightarrow$ (a). The implication (c) $\Rightarrow$ (a) follows from Proposition 2.1.3. To prove (a) $\Rightarrow$ (c), let $<_G$ be an ordering of G and let $<_N$ be the restriction of $<_G$ to N. Then since the action of φ is conjugation inside G, it readily follows that φ is order-preserving with respect to $<_N$. $\square$

In part (b) of the above theorem, G only needs to have a reverse lexicographic ordering with respect to some order of N and H. It does not necessarily need to have a reverse lexicographic ordering with respect to every order of N and H. It is interesting to note that in Theorem 2.1.5 when trying to determine if G is orderable, the specific order of H is not important but what order we choose for N is.

We should preface everything we have said so far by adding that not every semidirect product of two orderable groups is orderable. In lieu of Theorem 2.1.5, if we want an example of a semidirect product that is not orderable, we want to cleverly choose φ so that it does not preserve any order of N. The following is one such example.

Example 2.1.6 (A non-orderable semidirect product)**.** We showed in Example 1.2.11 that the fundamental group of the Klein bottle is not orderable. Recall

$$\pi_1(\text{Klein bottle}) \cong \mathbb{Z} \rtimes \mathbb{Z} \cong \langle x \rangle \rtimes \langle y \rangle \cong \langle x, y \mid yxy^{-1} = x^{-1} \rangle.$$

Note the action of y on x is inversion which is always an automorphism of an abelian group. We already saw one argument for why this group is not orderable. Another way to see this is to note that

$$(xy)^2 = xyxy = xx^{-1}yy = y^2$$

but $xy \neq y$ and so we do not have unique roots. Thus we see one way to make a semidirect product be not orderable is to force the conjugation action induced inside it to be inversion which can never be order-preserving.

To finish this section, we mention a surprising result related to finite extensions of groups shown by Neumann and Shepperd [NS57]. The proof makes use of a deep result from group theory. The most difficult part is showing that the prescribed positive cone in the proof is a semigroup. We state the result in a slightly different language. See [NS57] for details and a proof.

Theorem 2.1.7 ([NS57, Theorem 3.1])**.** *Let $(N, <_N)$ be an ordered normal subgroup of a group G and assume N has finite index in G. Suppose G is torsion-free and the conjugation action of G upon N preserves $<_N$. Then G is orderable and can be given an ordering that extends $<_N$.*

2.2 Semidirect products and computability

In this section, we look at the semidirect product and make connections to computability. Given a semidirect product $N \rtimes_\varphi H$ with $\varphi\colon H \to \mathrm{Aut}(N)$, we can represent the map φ as a function from $H \times N \to N$. Let $f\colon H \times N \to N$ be defined as $f(h, n) = \varphi_h(n)$ for all $h \in H$ and $n \in N$. Henceforth, whenever N and H are computable groups and we say φ is uniformly computable, we will mean that f is computable. So in particular, if φ is uniformly computable, then φ_h is computable for all $h \in H$.

Proposition 2.2.1. *Let $(N, \cdot_N)$ and $(H, \cdot_H)$ be computable groups, and let $\varphi\colon H \to \mathrm{Aut}(N)$*

be a group homomorphism that is uniformly computable. Then the semidirect product $N \rtimes_\varphi H$ *has a computable presentation.*

Proof. We can let the domain of G be the computable set $N \times H$. We can define a computable function $\cdot_G$ as follows

$$(n, h) \cdot_G (n', h') = (n \cdot_N \varphi_h(n'), h \cdot_H h').$$

Then $(G, \cdot_G)$ is a computable presentation of $N \rtimes_\varphi H$. □

For two computable groups N and H the *standard presentation* of $N \rtimes_\varphi H$ will be a group $(G, \cdot_G)$ with domain $G = N \times H$ such that $G \cong N \rtimes_\varphi H$. (Note we do not assume that in the standard presentation the group operation $\cdot_G$ is a computable function.)

Proposition 2.2.2. *Let* $(N, \cdot_N)$ *and* $(H, \cdot_H)$ *be computable groups, and let* $\varphi\colon H \to \mathrm{Aut}(N)$ *be a group homomorphism. Let* $(G, \cdot_G)$ *be the standard presentation of* $N \rtimes_\varphi H$ *with* $\cdot_G$ *computable. Then* φ *is uniformly computable.*

Proof. Observe that from the standard presentation of $N \rtimes_\varphi H$ we can recover the maps φ_h. For any $h \in H$ and $n \in N$, we have

$$(1, h) \cdot_G (n, 1) \cdot_G (1, h^{-1}) = (\varphi_h(n), 1).$$

Thus we can define $\varphi_h(n)$ to be the projection along the first component of the above. This procedure is also clearly uniform so therefore φ is uniformly computable. □

If the homomorphism φ is not uniformly computable then it is possible to have the standard presentation of a semidirect product with a noncomputable group operation.

Theorem 2.2.3. *There exists computable groups* N *and* H*, and a sequence of computable group homomorphisms* $\varphi_h\colon N \to N$ *for each* $h \in H$ *such that* $\varphi : H \to \mathrm{Aut}(N)$*, defined via* $\varphi(h) = \varphi_h$*, is a group homomorphism and for the standard presentation* $(G, \cdot_G)$ *of* $N \rtimes_\varphi H$*, we have* $\mathbf{0'} \equiv_T \cdot_G$*.*

Proof. Let N and H denote the group $(\bigoplus_\omega \mathbb{Z}, +)$. We define a group homomorphism $\varphi\colon H \to \operatorname{Aut}(N)$ by specifying the images of the generators $\{e_n\}_{n\in\omega}$ where $e_n = 0^n1$. If $n \in \mathbf{0}'$, then $\varphi(e_n)$ is the automorphism of N that swaps the components in position $2n$ and $2n+1$. If $n \notin \mathbf{0}'$, then $\varphi(e_n)$ is the identity automorphism. Let $\varphi_h = \varphi(h)$ for all $h \in H$. Note that $\varphi_h\colon N \to N$ is a computable function for all $h \in H$.

Consider the group $N \rtimes_\varphi H$. Let $(G, \cdot_G)$ be the standard presentation of $N \rtimes_\varphi H$. Notice that we can represent the elements of G as a pair of finite strings of integers with no leading zeros. We claim that $\mathbf{0}' \equiv_T \cdot_G$. To see that $\mathbf{0}' \leq_T \cdot_G$, given some $n \in \omega$, we ask what is $(0, e_n) \cdot_G (e_{2n}, 0)$? If the answer is (e_{2n}, e_n), then $n \notin \mathbf{0}'$. Otherwise if the answer is (e_{2n+1}, e_n), then $n \in \mathbf{0}'$. For $\cdot_G \leq_T \mathbf{0}'$, to find $(n, h) \cdot_G (n', h')$, decompose h into a finite string of integers. Then from the finitely many generator elements that appear in the string, using $\mathbf{0}'$ determine how h acts to modify the string being represented by n'. From there we can find the product of (n, h) and (n', h'). Hence, we have that $\mathbf{0}' \equiv_T \cdot_G$. □

Next, we show that the group $N \rtimes_\varphi H$ defined in the above proof has a computable copy $A \rtimes_\psi B$ with A and B computable, and ψ uniformly computable. Moreover, A and B have computable basis as torsion-free abelian groups.

Theorem 2.2.4. *Let N and H denote the group $(\bigoplus_\omega \mathbb{Z}, +)$. Let φ be as defined in the proof of Theorem 2.2.3. Then $N \rtimes_\varphi H$ has a computable presentation.*

Proof. We construct a computable group $(G, \cdot_G)$ that is classically isomorphic to $N \rtimes_\varphi H$, i.e., we give a computable presentation of $N \rtimes_\varphi H$. Let $K = \{x \in \omega \mid \varphi_x(x)\downarrow\}$ be the halting set. Fix an enumeration $\{K_s\}_{s\in\omega}$ of K such that $|K_{s+1} \setminus K_s| = 1$ for all $s \in \omega$. Assume $K_0 = \emptyset$. Let $\langle\cdot,\cdot\rangle$ denote the pairing function. Let $\pi_1(\langle x, y\rangle) = x$ and $\pi_2(\langle x, y\rangle) = y$ be the projection functions along the first and second components, respectively. We construct our group in stages G_s such that $G = \bigcup_{s\in\omega} G_s$. The domain of our group will be ω and to each $n \in G$, we will assign a pair of finite strings of integers $\sigma_n = (\alpha_n, \beta_n)$. We will define a total computable function $d(s)$ such that $G_s = \{0, \ldots, d(s)\}$ for all $s \in \omega$. We will also define a sequence of finite sets D_n such that $D_n \subseteq D_{n+1}$ for all n, and if $x \in D_n$ with $x = \langle i, j \rangle$, then the action of the group element $(0, 0^i 1)$ on the pair of strings $(\alpha, 0)$ will be to swap the $2j$ and $2j+1$ components of α. We have the following requirements:

- (Group generators) B_n : The generators $(0^n 1, 0)$ and $(0, 0^n 1)$ of the group are represented in G.

- (Group closure) C_n : If $n = \langle p, q \rangle$, then $p \cdot_G q$ is defined.

- (Group inverses) I_p : There exists q such that $p \cdot_G q = 0$. (The zero of ω will be the identity of G.)

Definition 2.2.5. (1) We say B_n *requires attention* at stage $s+1$ if there exists no $p, q \leq d(s)$ such that $\sigma_p = (0^n 1, 0)$, and $\sigma_q = (0, 0^n 1)$.

(2) We say C_n *requires attention* at stage $s+1$ if $n = \langle p, q \rangle$ with $p, q \leq d(s)$ and there is no $r \leq d(s)$ such that $p \cdot_G q = r$.

(3) We say I_p *requires attention* at stage $s+1$ if $p \leq d(s)$ and there is no $q \leq d(s)$ such that $p \cdot_G q = 0$.

We fix a priority ordering on our requirements as follows

$$B_0 < C_0 < I_0 < B_1 < C_1 < I_1 < \cdots .$$

Construction:

Stage 0: Let $D_0 = \emptyset$. Set $d(0) = 0$ and set σ_0 to be the empty string.

Stage $s+1$: Let $n \in K_{s+1} \setminus K_s$ and set $D_{s+1} = D_s \cup \{\langle n, k \rangle\}$, where $k > 2s$ is a large enough number not yet used in the construction. Find the highest priority requirement that requires attention.

- If B_n, let $d(s+1) = d(s) + 2$. Define $\sigma_{d(s)+1} = (0^n 1, 0)$ and $\sigma_{d(s)+2} = (0, 0^n 1)$.

- If C_n where $n = \langle p, q \rangle$, let $d(s+1) = d(s)+1$. Suppose $\sigma_p = (\alpha_p, \beta_p)$ and $\sigma_q = (\alpha_q, \beta_q)$. We will define $\sigma_{d(s+1)} = (\alpha_{d(s+1)}, \beta_{d(s+1)})$. Let $\beta_{d(s+1)} = \beta_p + \beta_q$ (here we mean summation componentwise). Let $D' = \{\pi_2(x) \mid x \in D_{s+1} \text{ and } \beta_p(\pi_1(x)) \text{ is odd.}\}$. Define the string α'_q as follows:
$$\alpha'_q(i) = \begin{cases} \alpha_q(2x+1) & \text{if } i = 2x \text{ for some } x \in D', \\ \alpha_q(2x) & \text{if } i = 2x+1 \text{ for some } x \in D', \\ \alpha_q(i) & \text{else.} \end{cases}$$
Let $\alpha_{d(s+1)} = \alpha_p + \alpha'_q$ and set $\sigma_{d(s+1)} = (\alpha_{d(s+1)}, \beta_{d(s+1)})$. Note that we have set $p \cdot_G q = d(s+1)$.

- If I_p, let $d(s+1) = d(s)+1$. Suppose $\sigma_p = (\alpha_p, \beta_p)$. We will define $\sigma_{d(s+1)} = (\alpha_{d(s+1)}, \beta_{d(s+1)})$. Let $D' = \{\pi_2(x) \mid x \in D_{s+1} \text{ and } \beta_p(\pi_1(x)) \text{ is odd.}\}$. Define α'_p as follows:
$$\alpha'_p(i) = \begin{cases} \alpha_p(2x+1) & \text{if } i = 2x \text{ for some } x \in D', \\ \alpha_p(2x) & \text{if } i = 2x+1 \text{ for some } x \in D', \\ \alpha_p(i) & \text{else.} \end{cases}$$
Set $\sigma_{d(s+1)} = (-\alpha'_p, -\beta_p)$. Note that we have set $p \cdot_G d(s+1) = 0$.

If no requirement needs attention, continue to the next stage. This completes the construction. $\dashv$

By inspection, $d(s)$ is a monotonically increasing function, so the domain of our group will be ω. Each requirement needs attention at most once and since all the requirements will eventually be met, we will have that $(G, \cdot_G)$ is a group. Moreover, by the subsequent discussion, it will follow that $G \cong N \rtimes_\varphi H$.

Suppose $A = B = (\bigoplus_\omega \mathbb{Z}, +)$ with generators $\{a_i\}_{i\in\omega}$ and $\{b_j\}_{j\in\omega}$, respectively. (Here $a_i = 0^i 1$ and $b_j = 0^j 1$.) Suppose we have a group homomorphism $\pi \colon B \to \mathrm{Aut}(A)$ such that we can partition the generators of B into two infinite subsets B_0 and B_1 as follows. All the elements in B_0 will map to the identity, and for each $b \in B_1$, $\pi(b)$ swaps two adjacent components of A. Moreover, if $b, b' \in B_1$ with $b \neq b'$, then the pair of components they act on are different. Also, there are infinitely many components of A that are never acted on by any $b \in B_1$. We claim $A \rtimes_\pi B \cong N \rtimes_\varphi H$.

We define two automorphisms $f\colon N \to A$ and $g\colon H \to B$. Suppose $K = \{n_0 < n_1 < \dots\}$ and $\omega \setminus K = \{m_0 < m_1 < \dots\}$. Let $B_0 = \{b_{x_0}, b_{x_1}, \dots\}$ and $B_1 = \{b_{y_0}, b_{y_1}, \dots\}$. If $n_i \in K$, then define $g(e_{n_i}) = b_{y_i}$, $f(e_{2n_i}) = a_{2j}$ and $f(e_{2n_i+1}) = a_{2j+1}$, where $2j$ and $2j+1$ are the two components swapped by $\pi(b_{y_i})$. If $m_i \in \omega \setminus K$, define $g(e_{m_i}) = b_{x_i}$, $f(e_{2m_i}) = a_{2k}$ and $f(e_{2m_i+1}) = a_{2k+1}$, where components $2k$ and $2k+1$ are never swapped by any $b \in B_1$. Define $\psi : B \to \mathrm{Aut}(A)$ by $\psi(b) = f \circ \varphi(g^{-1}(b)) \circ f^{-1}$. We claim $\psi = \pi$.

If $b \in B_0$, then $\pi(b)$ is identity. Also by definition of g, $\varphi(g^{-1}(b))$ is identity. Hence, $\psi(b)$ is also identity. If $b \in B_1$ and say $b = b_{y_i}$. Then $\pi(b_{y_i})$ swaps some components $2j$ and $2j+1$ of A. So then $\varphi(g^{-1}(b_{y_i}))$ swaps components $2n_i$ and $2n_i+1$. By definition of f, we have $\psi(b) = \pi(b)$. Finally, it can be shown that $N \rtimes_\varphi H \cong A \rtimes_\psi B$ under the mapping that sends (n, h) to $(f(n), g(h))$. This completes the proof of Theorem 2.2.4. $\square$

It is well-known that if $G = N \rtimes_\varphi H$ is a semidirect product, then there exists subgroups $\hat{N}$ and $\hat{H}$ of G such that

(1) $\hat{N} \cong N$ and $\hat{H} \cong H$,

(2) $\hat{N} \trianglelefteq G$,

(3) $\hat{N} \cap \hat{H}$ is trivial, and

(4) $G = \hat{N}\hat{H}$.

This is sometimes referred to as the recognition theorem for semidirect products. We can use this recognition theorem to give a sufficient condition for when a semidirect product is computably categorical.

Theorem 2.2.6. *Let $G = N \rtimes_\varphi H$ be a semidirect product that has a computable presentation. Suppose for every computable copy $\mathcal{A}$ of G, there exists computable subgroups $N_\mathcal{A}$ and $H_\mathcal{A}$ of $\mathcal{A}$ such that*

(i) $N_{\mathcal{A}} \cong N$ *and* $H_{\mathcal{A}} \cong H$,

(ii) $N_{\mathcal{A}} \trianglelefteq \mathcal{A}$,

(iii) $N_{\mathcal{A}} \cap H_{\mathcal{A}}$ *is trivial, and*

(iv) $\mathcal{A} = N_{\mathcal{A}} H_{\mathcal{A}}$.

Furthermore, suppose for any two computable copies $\mathcal{A}$ *and* $\mathcal{B}$ *of* G *there exists computable isomorphisms* $f \colon N_{\mathcal{A}} \to N_{\mathcal{B}}$ *and* $g \colon H_{\mathcal{A}} \to H_{\mathcal{B}}$. *Then* G *is computably categorical.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two computable copies of G. We want to show that $\mathcal{A}$ and $\mathcal{B}$ are computably isomorphic. Fix the subgroups $N_{\mathcal{A}}$ and $H_{\mathcal{A}}$ of $\mathcal{A}$ that satisfy the conditions of the hypothesis. Similarly, fix subgroups $N_{\mathcal{B}}$ and $H_{\mathcal{B}}$ of $\mathcal{B}$. Given any $a \in \mathcal{A}$, we can fix a unique pair of elements $x \in N_{\mathcal{A}}$ and $y \in H_{\mathcal{A}}$ such that $a = x \cdot_{\mathcal{A}} y$. Then we can map a to $b = f(x) \cdot_{\mathcal{B}} g(y)$ in $\mathcal{B}$. Since f and g are computable isomorphisms, and elements of $N_{\mathcal{B}}$ and $H_{\mathcal{B}}$ uniquely decompose $\mathcal{B}$, our mapping will be a computable isomorphism. $\square$

Chapter 3

Space of orders and Cantor-Bendixson rank

3.1 Space of orders with Cantor-Bendixson rank 2

In this section we provide a complete and detailed proof that the group constructed in [But71] is orderable and has exactly countably many orders. We will then prove that its space of orders has Cantor-Bendixson rank 2. We begin by describing the family of groups constructed in [But71].

Let A denote the subgroup of the additive group of rational numbers consisting of the dyadic rationals, that is,

$$A = \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{Z} \right\}.$$

Let X be a subset of A defined by $X = \{x \in A \mid 0 \leq x < 1\}$. For each $z \in \mathbb{Z}$ and $x \in X$, let $H_{z,x}$ and $K_{z,x}$ be copies of the group $(\mathbb{Q}, +)$, the rational numbers under addition. Define

$$H_z = \bigoplus_{x \in X} H_{z,x}$$

and

$$K_z = \bigoplus_{x \in X} K_{z,x}$$

to be restricted direct products of copies of $\mathbb{Q}$ for all $z \in \mathbb{Z}$. Let

$$H = \bigoplus_{z\in\mathbb{Z}} H_z = \bigoplus_{z\in\mathbb{Z}} \bigoplus_{x\in X} H_{z,x},$$

$$K = \bigoplus_{z\in\mathbb{Z}} K_z = \bigoplus_{z\in\mathbb{Z}} \bigoplus_{x\in X} K_{z,x},$$

and

$$P = H \times K.$$

Next, we want to construct a semidirect product of the groups P and A. To do this, let us first fix some notation. Let $h^r_{z,x} \in H_{z,x}$ and $k^r_{z,x} \in K_{z,x}$ denote the number $r \in \mathbb{Q}$. We will write arbitrary elements of the groups A and $\mathbb{Z}$ as λ^α and ζ^β, respectively, where $\alpha \in A$ and $\beta \in \mathbb{Z}$. Fix two distinct prime numbers p and q. To construct our semidirect product we define an action of each element of A on the elements $h^r_{z,x}$ and $k^r_{z,x}$. The action is defined as

$$\lambda^{-\alpha} h^r_{z,x} \lambda^\alpha = h^{rp^n}_{z,x+\alpha 2^z - n} \tag{3.1}$$

where $n \leq x + \alpha 2^z < n+1$ for $n \in \mathbb{Z}$. Analogously,

$$\lambda^{-\alpha} k^r_{z,x} \lambda^\alpha = k^{rq^n}_{z,x+\alpha 2^z - n} \tag{3.2}$$

where $n \leq x + \alpha 2^z < n+1$ for $n \in \mathbb{Z}$. It can be checked that these actions define a group homomorphism from A into $\mathrm{Aut}(P)$, thus we can construct the semidirect product $M = P \rtimes A$. Now, we define an action on the group M by $\mathbb{Z}$ via

$$\zeta^{-\beta} h^r_{z,x} \zeta^\beta = h^r_{z+\beta,x}, \tag{3.3}$$

$$\zeta^{-\beta} k^r_{z,x} \zeta^\beta = k^r_{z+\beta,x}, \tag{3.4}$$

and

$$\zeta^{-\beta} \lambda^\alpha \zeta^\beta = \lambda^{\frac{\alpha}{2^\beta}}. \tag{3.5}$$

Using these group actions we can construct the semidirect product

$$G(p,q) = M \rtimes \mathbb{Z} = (P \rtimes A) \rtimes \mathbb{Z}.$$

Observe that for each pair of distinct prime numbers we can construct the group $G(p, q)$, so in fact, we get a family of groups. We want to prove two facts about $G(p, q)$: it has exactly countably many different orders and its space of orders has Cantor-Bendixson rank 2. We begin by first showing that $\mathbb{X}(G(p, q))$ is countable. We will need the following results. These results are mentioned in [But71] without proof; we provide proofs for them.

Proposition 3.1.1. *Let $(G, <)$ be an ordered group. Suppose $x, y_i \in G$ where $i \in \mathbb{Z}$.*

(a) *If $x^{-1}y_i^{k_i}x = y_i^{l_i}$ with $k_i, l_i \in \mathbb{Z}$ and $k_i \neq l_i$, then $|y_i| \ll |x|$ and $|y_i| \sim |y_j|$ implies $k_i l_j = k_j l_i$.*

(b) *If $x^{-1}y_i x = y_{i+1}$ and $|y_i| \sim |y_{i+1}|$ does not hold, then either*

$$\cdots \ll |y_{i-1}| \ll |y_i| \ll |y_{i+1}| \ll \cdots \ll |x|$$

or

$$\cdots \ll |y_{i+1}| \ll |y_i| \ll |y_{i-1}| \ll \cdots \ll |x|.$$

Proof. We first prove (a). Let $x, y \in G$. Suppose $x^{-1}y^k x = y^l$ for some $k, l \in \mathbb{Z}$ with $k \neq l$. We can assume $k, l \neq 0$. Observe that

$$[x^{-1}, y^k] = x^{-1}y^k x y^{-k} = y^l y^{-k} = y^{l-k}.$$

So $[x^{-1}, y^k] \sim |y|$ since $l - k \neq 0$. By Proposition 1.6.15, we see that $|y| \ll \max(|x|, |y^k|)$. But then it must be the case that $\max(|x|, |y^k|) = |x|$. For otherwise it would mean that $|y| \ll |y^k|$ and this is impossible. Thus $|y| \ll |x|$.

Next, let $x, y_i, y_j \in G$. Suppose $x^{-1}y_i^{k_i}x = y_i^{l_i}$ and $x^{-1}y_j^{k_j}x = y_j^{l_j}$ for $k_i, k_j, l_i, l_j \in \mathbb{Z}$ with $k_i \neq l_i$ and $k_j \neq l_j$. Assume $k_i l_j \neq k_j l_i$. We have to show that $|y_i| \not\sim |y_j|$. Suppose for contradiction that $|y_i| \sim |y_j|$. Without loss of generality, we can assume that y_i and y_j are positive elements and $y_i < y_j$. Fix an integer $n \geq 1$ such that $y_i^n \leq y_j < y_i^{n+1}$. Then by raising this relation to $(k_i l_j)^m$ power we arrive at

$$y_i^{n(k_i l_j)^m} \leq y_j^{(k_i l_j)^m} < y_i^{(n+1)(k_i l_j)^m}$$

which is true for all integers $m \geq 1$. We also have

$$y_i^{n(k_ik_j)^m} \leq y_j^{(k_ik_j)^m} < y_i^{(n+1)(k_ik_j)^m}$$

for all $m \geq 1$. Conjugate the above relation by x to get

$$y_i^{n(k_jl_i)^m} \leq y_j^{(k_il_j)^m} < y_i^{(n+1)(k_jl_i)^m}.$$

To get a contradiction, it suffices to show that either

$$y_i^{(n+1)(k_jl_i)^m} < y_i^{n(k_il_j)^m}$$

or

$$y_i^{(n+1)(k_il_j)^m} < y_i^{n(k_jl_i)^m}$$

for some m. Equivalently, show that for some m either

$$(n+1)(k_jl_i)^m < n(k_il_j)^m$$

or

$$(n+1)(k_il_j)^m < n(k_jl_i)^m$$

holds. We have two cases to consider. For the first case, suppose $k_jl_i < k_il_j$. Then

$$\lim_{m\to\infty} \frac{(n+1)(k_jl_i)^m}{n(k_il_j)^m} = 0.$$

Therefore, for sufficiently large m, we have

$$\frac{(n+1)(k_jl_i)^m}{n(k_il_j)^m} < 1.$$

So $(n+1)(k_jl_i)^m < n(k_il_j)^m$ and we have arrived at a contradiction. Identically, in the second case, if $k_il_j < k_jl_i$, we will arrive at the contradiction that $(n+1)(k_il_j)^m < n(k_jl_i)^m$ for large enough m. Hence, we must have that $|y_i| \not\sim |y_j|$, as desired. This completes the proof of (a).

To prove (b), assume $x^{-1}y_ix = y_{i+1}$ and $|y_i| \not\sim |y_{i+1}|$ for all integers i. Without loss of generality, we may assume that each y_i and x are positive. Notice by our initial assumptions it follows that $x \neq y_i$ for all i. Suppose $y_0 < y_1$. We will show that

$$\cdots \ll y_{i-1} \ll y_i \ll y_{i+1} \ll \cdots \ll x.$$

If we conjugate $y_0 < y_1$ by x and x^{-1}, we have $y_{-1} < y_0$ and $y_1 < y_2$. That is, $y_{-1} < y_0 < y_1 < y_2$. By continuing along like this, we will get

$$\cdots < y_{-2} < y_{-1} < y_0 < y_1 < y_2 < \cdots .$$

Since $y_i \not\sim y_{i+1}$ for all i, this implies

$$\cdots \ll y_{-2} \ll y_{-1} \ll y_0 \ll y_1 \ll y_2 \ll \cdots ,$$

which follows from Proposition 1.5.9(i).

To finally complete the proof, we need to show that $y_i \ll x$ for all i. If it was the case that $x \sim y_j$ for some j, then there would exist some $n \geq 1$ such that $x < y_j^n$ and $y_j < x^n$. Conjugating both of the previous two inequalities by x^{-1} would give us that $x < y_{j+1}^n$ and $y_{j+1} < x^n$, which together say that $x \sim y_{j+1}$. But then $x \sim y_j$ and $x \sim y_{j+1}$ would imply $y_j \sim y_{j+1}$, a contradiction to our initial hypothesis. So then for each i, either $x \ll y_i$ or $y_i \ll x$.

Assume for a contradiction, there exists some j such that $x < y_j$ (which is equivalent to $x \ll y_j$). Then $x^{-1}y_jx < x^{-1}y_jy_j$ and in turn $y_{j+1} < x^{-1}y_j^2$. Also, $x^{-1} < y_j$ because x is positive, and this gives $x^{-1}y_j^2 < y_j^3$. Putting everything together, we have $y_{j+1} < y_j^3$. But then since we already know that $y_j < y_{j+1}$, this would mean that $y_j \sim y_{j+1}$, which is a contradiction. Therefore $y_i < x$ for all i, or equivalently, $y_i \ll x$ for all i. This completes the proof of (b). □

Our first step is showing that the group $G(p,q)$ is orderable.

Proposition 3.1.2. *The group $G(p,q) = M \rtimes \mathbb{Z} = (P \rtimes A) \rtimes \mathbb{Z}$ is orderable.*

Proof. We want to argue that the group $G(p,q)$ is orderable. We will do this by first showing that the semidirect product $M = P \rtimes A$ is orderable and then using that to show $G(p,q)$ is orderable. Consider the following order $<_{H_z}$ on the group H_z. Given an element $u \in H_z$, we can express it as $h_{z,x_1}^{r_1} \cdots h_{z,x_n}^{r_n}$ with $x_i \in X$ and $r_i \in \mathbb{Q}$. Define

$$1 <_{H_z} u \text{ if and only if } 0 <_{\mathbb{R}} r_1 p^{x_1} + \cdots + r_n p^{x_n}$$

where $<_{\mathbb{R}}$ denotes the usual order of $\mathbb{R}$ and p is the prime fixed from earlier. (See Appendix A.2 for a proof of the fact that the set $\{p^x \mid x \in X\}$ is a linearly independent subset of $\mathbb{R}$ regarded as a $\mathbb{Q}$-vector space.) The ordering $<_{H_z}$ is Archimedean and makes H_z an Archimedean ordered group. We can next define an ordering on the group H. For a nonidentity element $h = h_{z_1} \cdots h_{z_m} \in H$ with $h_{z_i} \in H_{z_i}$ and $z_1 < \cdots < z_m$, define

$$1 <_H h \text{ if and only if } 1 <_{H_{z_1}} h_{z_1}.$$

We can similarly order the groups K_z and K. (Note when ordering K_z we use the prime q instead.) With an ordering on H and K fixed, we can order $P = H \times K$ lexicographically. That is, if $hk \in P$, then

$$1 <_P hk \text{ if and only if either } 1 <_H h, \text{ or else } h = 1 \text{ and } 1 <_K k.$$

Since A is a subgroup of $\mathbb{Q}$ and the rationals can only be ordered in one of two ways as a group, we can simply fix one of these two orders for A and denote it by $<_A$. Likewise, fix an order $<_{\mathbb{Z}}$ on the group $\mathbb{Z}$. It can be verified that the group actions described in (3.1) and (3.2) are order preserving with respect to the orderings $<_H$ and $<_K$ on the groups H and K. So in turn, the conjugation action of A upon P preserves the given ordering $<_P$ of P. By way of example, if $1 <_P h_{z_1,x_1}^{r_1} k_{z_2,x_2}^{r_2}$, then $0 <_{\mathbb{R}} r_1 p^{x_1}$. If we act on $h_{z_1,x_1}^{r_1} k_{z_2,x_2}^{r_2}$ by $\lambda \in A$ we have

$$\lambda^{-1} h_{z_1,x_1}^{r_1} k_{z_2,x_2}^{r_2} \lambda = h_{z_1,x_1+2^{z_1}-n_1}^{r_1 p^{n_1}} k_{z_2,x_2+2^{z_2}-n_2}^{r_2 q^{n_2}}.$$

But then $h_{z_1,x_1+2^{z_1}-n_1}^{r_1 p^{n_1}} k_{z_2,x_2+2^{z_2}-n_2}^{r_2 q^{n_2}}$ is positive since $r_1 p^{n_1} p^{x_1+2^{z_1}-n_1} = r_1 p^{x_1+2^{z_1}}$ is positive with respect to $<_{\mathbb{R}}$. Therefore by Proposition 2.1.3, the group $M = P \rtimes A$ is an orderable group.

Let $<_M$ be an ordering on M defined as follows. Let $\rho = \alpha\beta \in M$ with $\alpha \in P$ and $\beta \in A$. Define

$$1 <_M \rho \text{ if and only if } 1 <_A \beta, \text{ or } \beta = 1 \text{ and } 1 <_P \alpha.$$

Note that $<_M$ is simply a lexicographic order on M. It can be checked that the conjugation actions described in (3.3), (3.4) and (3.5) preserve the order $<_M$ of M. Therefore we can finally define an order $<_G$ on $G(p,q) = M \rtimes \mathbb{Z}$. Suppose $\rho = \alpha\beta \in G(p,q)$ where $\alpha \in M$ and $\beta \in \mathbb{Z}$. Then

$$1 <_G \rho \text{ if and only if } 1 <_{\mathbb{Z}} \beta, \text{ or } \beta = 1 \text{ and } 1 <_M \alpha.$$

The order $<_G$ is an invariant order on $G(p,q)$. Hence, we can finally conclude that $G(p,q)$ is orderable. $\square$

Our next step is showing that the subgroups H_z and K_z inherit unique orders up to duals from $G(p,q)$.

Lemma 3.1.3. *In any order of $G(p,q)$, the order of each subgroup H_z and K_z is Archimedean and unique up to duals.*

Proof. Fix an ordering $<_G$ of $G(p,q)$ and fix $z \in \mathbb{Z}$. Suppose $x_1, x_2 \in X$ with $x_1 < x_2$. Then $0 < x_2 - x_1 < 1$ and so we may write $x_2 - x_1 = \frac{m}{2^n}$ with $m < 2^n$ and $m, n \in \mathbb{Z}^+$. Let $\alpha = \frac{m}{2^{n+z}} \in A$. Observe that

$$\lambda^{-2^n\alpha} h_{z,x_1} \lambda^{2^n\alpha} = h_{z,x_1}^{p^m} \tag{3.6}$$

and

$$\lambda^{-\alpha} h_{z,x_1} \lambda^{\alpha} = h_{z,x_2}. \tag{3.7}$$

Suppose on the contrary that H_z is not Archimedean. Let $\ll$ denote the induced linear order on the Archimedean classes under $<_G$. Without loss of generality, assume $h_{z,x_1} \ll h_{z,x_2}$. Then if we conjugate both sides by the element λ^α from (3.7) we get

$$h_{z,x_2} \ll \lambda^{-2\alpha} h_{z,x_1} \lambda^{2\alpha}.$$

Since the relation $\ll$ is transitive, we can conclude that

$$h_{z,x_1} \ll \lambda^{-2\alpha} h_{z,x_1} \lambda^{2\alpha}.$$

If we continue conjugating by λ^{α}, we will arrive at the relation $h_{z,x_1} \ll \lambda^{-2^n\alpha} h_{z,x_1} \lambda^{2^n\alpha}$. But according to (3.6), in fact, $h_{z,x_1} \ll h_{z,x_1}^{p^m}$. But this is a contradiction since $\mathbb{Q}$ only has two orders and both of those orders are Archimedean. Thus it must be the case that h_{z,x_1} and h_{z,x_2} are Archimedean equivalent. Since x_1 and x_2 were arbitrary elements of X, it follows that H_z is an Archimedean ordered group with respect to any order it inherits from $G(p,q)$.

Next, we want to show that any order of $G(p,q)$ restricted to H_z is unique up to duals. Let $<_{H_z}$ be the restriction of $<_G$ to H_z. By above $(H_z, <_{H_z})$ is Archimedean so it is order-isomorphic to a subgroup of the real numbers under their usual ordering. Let $\varphi\colon H_z \to \mathbb{R}$ be the isomorphism in question, and let $L = \varphi(H_z)$. Suppose $h_{z,0}$ is positive under $<_{H_z}$. We can further assume that $\varphi(h_{z,0}) = 1 \in \mathbb{R}$. Fix $0 < x \in X$. Then if $x = \frac{m}{2^n}$, set $\alpha = \frac{m}{2^{n+z}}$. By (3.6) and (3.7)

$$\lambda^{-2^n\alpha} h_{z,0} \lambda^{2^n\alpha} = h_{z,0}^{p^m} \tag{3.8}$$

and

$$\lambda^{-\alpha} h_{z,0} \lambda^{\alpha} = h_{z,x}. \tag{3.9}$$

Now (3.9) is an order-preserving automorphism of H_z, so this gives an automorphism of L which, according to Proposition 1.2.17, is determined by multiplication by a positive real number, say $r \in \mathbb{R}$. Now (3.8) says $r^{2^n} = p^m$ and this implies that $r = p^{\frac{m}{2^n}} = p^x$. By (3.9), $\varphi(h_{z,x}) = p^x$. Therefore for all $y \in X$ and $s \in \mathbb{Q}$, we have $\varphi(h_{z,y}^s) = sp^y$ (with p^y always taken to be a positive real number). That is all to say, the choice of a sign for $h_{z,0}$ completely determines $\varphi(H_z)$. The image of H_z in $\mathbb{R}$ will always be the same. So to compare elements in H_z, we can instead compare elements in $\varphi(H_z)$. This means there is exactly one order possible on H_z if we choose that $h_{z,0}$ is positive, while its dual occurs if we impose $h_{z,0}$ is negative. This proves that the order of H_z is unique up to duals. Of course, similar results are true for K_z and this completes the proof. $\square$

The next two lemmas describe the relationships among the Archimedean classes of $G(p,q)$ and how these classes are ordered.

Lemma 3.1.4. *In any order of $G(p,q)$, either*

$$\cdots \ll h_{z-1,0} \ll h_{z,0} \ll h_{z+1,0} \ll \cdots \ll \lambda \ll \zeta$$

or

$$\cdots \ll h_{z+1,0} \ll h_{z,0} \ll h_{z-1,0} \ll \cdots \ll \lambda \ll \zeta$$

holds, and either

$$\cdots \ll k_{z-1,0} \ll k_{z,0} \ll k_{z+1,0} \ll \cdots \ll \lambda \ll \zeta$$

or

$$\cdots \ll k_{z+1,0} \ll k_{z,0} \ll k_{z-1,0} \ll \cdots \ll \lambda \ll \zeta$$

is true.

Proof. We first want to show that $h_{z,0} \not\sim h_{z+1,0}$ for all $z \in \mathbb{Z}$. Suppose $z \geq 0$. Then $\lambda^{-1}h_{z,0}\lambda = h_{z,0}^{p^{2^z}}$ and $\lambda^{-1}h_{z+1,0}\lambda = h_{z+1,0}^{p^{2^{z+1}}}$. Since $p^{2^z} \neq p^{2^{z+1}}$, by Proposition 3.1.1(a) it follows $h_{z,0} \not\sim h_{z+1,0}$. Now, suppose $z < 0$ and let $z = -n$. Then $\lambda^{-2^{n+1}}h_{z,0}\lambda^{2^{n+1}} = h_{z,0}^{p^2}$ and $\lambda^{-2^{n+1}}h_{z-1,0}\lambda^{2^{n+1}} = h_{z-1,0}^{p}$. Again, by Proposition 3.1.1(a), we have $h_{z,0} \not\sim h_{z-1,0}$. We are just left to show that $h_{-1,0} \not\sim h_{0,0}$. Observe that $\lambda^{-2}h_{-1,0}\lambda^2 = h_{-1,0}^{p}$ and $\lambda^{-2}h_{0,0}\lambda^2 = h_{0,0}^{p^2}$. So $h_{-1,0} \not\sim h_{0,0}$. Therefore, $h_{z,0} \not\sim h_{z+1,0}$ for all z. Using this fact along with relation $\zeta^{-1}h_{z,0}\zeta = h_{z+1,0}$ we can apply Proposition 3.1.1(b) to get that either

$$\cdots \ll h_{z-1,0} \ll h_{z,0} \ll h_{z+1,0} \ll \cdots \ll \zeta$$

or

$$\cdots \ll h_{z+1,0} \ll h_{z,0} \ll h_{z-1,0} \ll \cdots \ll \zeta.$$

Next, by the relation $\lambda^{-2^{-z}}h_{z,0}\lambda^{2^{-z}} = h_{z,0}^{p}$ and Proposition 3.1.1(a), we can conclude that

$h_{z,0} \ll \lambda^{2^{-z}}$. But since $\lambda \sim \lambda^{2^{-z}}$, in fact, $h_{z,0} \ll \lambda$ for all z. We can also conclude that $\lambda \ll \zeta$ via $\zeta\lambda\zeta^{-1} = \lambda^2$. Putting everything together, it is the case that either

$$\cdots \ll h_{z-1,0} \ll h_{z,0} \ll h_{z+1,0} \ll \cdots \ll \lambda \ll \zeta$$

or

$$\cdots \ll h_{z+1,0} \ll h_{z,0} \ll h_{z-1,0} \ll \cdots \ll \lambda \ll \zeta.$$

If we replace $h_{z,0}$ by $k_{z,0}$ in the above argument we will get an analogous result that either

$$\cdots \ll k_{z-1,0} \ll k_{z,0} \ll k_{z+1,0} \ll \cdots \ll \lambda \ll \zeta$$

or

$$\cdots \ll k_{z+1,0} \ll k_{z,0} \ll k_{z-1,0} \ll \cdots \ll \lambda \ll \zeta. \qquad \square$$

Lemma 3.1.5. *For all $i, j \in \mathbb{Z}$, the elements $h_{i,0}$ and $k_{j,0}$ are not Archimedean equivalent.*

Proof. Suppose $i = j$. Note that $\lambda^{-2^{-i}} h_{i,0} \lambda^{2^{-i}} = h_{i,0}^p$ and $\lambda^{-2^{-j}} k_{j,0} \lambda^{2^{-j}} = k_{j,0}^q$. Since $p \neq q$, we must have $h_{i,0} \not\sim k_{j,0}$ by Proposition 3.1.1(a). Now suppose $i \neq j$. We can fix an integer $m \geq 1$ such that either $m = i - j$ or $m = j - i$. Without loss of generality, say $m = i - j$. Then $\lambda^{-2^{-i+m}} h_{i,0} \lambda^{2^{-i+m}} = h_{i,0}^{p^{2^m}}$ and $\lambda^{-2^{-j}} k_{j,0} \lambda^{2^{-j}} = k_{j,0}^q$. Since $-i + m = -j$ and $p^{2^m} \neq q$, it follows that $h_{i,0} \not\sim k_{j,0}$ again from Proposition 3.1.1(a). $\square$

Using the above lemmas, we can deduce the following result.

Lemma 3.1.6. *The Archimedean classes of $G(p, q)$ with respect to any order are*

$$\{[1], [\lambda], [\zeta]\} \cup \{[h_{z,0}] \mid z \in \mathbb{Z}\} \cup \{[k_{z,0}] \mid z \in \mathbb{Z}\}.$$

Proof. This follows since we can express each element of $G(p, q)$ in a unique way. Suppose $g \in G(p, q)$ is a nonidentity element. Then we can express g uniquely in the form $hk\lambda^a\zeta^b$ where $h \in H, k \in K, a \in A$ and $b \in \mathbb{Z}$. Furthermore, we can express h uniquely as $h_{y_1} \cdots h_{y_n}$ with $h_{y_i} \in H_{y_i}$ and $y_1 < \cdots < y_n$, and k uniquely as $k_{z_1} \cdots k_{z_m}$ with $k_{z_i} \in K_{z_i}$ and $z_1 < \cdots < z_m$. Now by Proposition 1.5.10(iv), the Archimedean class of g will be whatever is the largest

Archimedean class among the classes determined by the elements $h_{y_i}, k_{z_j}, \lambda^a$ and ζ^b. (Note that $\lambda^a \sim \lambda$ and $\zeta^b \sim \zeta$ whenever $a, b \neq 0$.) $\square$

We state a definition before moving on to proving our main results for this section.

Definition 3.1.7. Let A and B be two subgroups of an ordered group.

(1) We will write $A \ll B$ to denote $a \ll b$ for all $a \in A \backslash \{1\}$ and $b \in B \backslash \{1\}$.

(2) We will say that A and B are *mixed* if $A \not\ll B$ and $B \not\ll A$.

We are now ready to prove our first main result for this section.

Theorem 3.1.8. *The group $G(p, q)$ has exactly countably many distinct orders. In other words, the space of orders $\mathbb{X}(G(p, q))$ is countable.*

Proof. To fix some notation, if L is a subgroup of $G(p, q)$, we will write $\mathbb{X}_G(L)$ to indicate the set of orders L inherits from $G(p, q)$. In other words, $\mathbb{X}_G(L)$ contains the orders of L that are compatible with some order on $G(p, q)$.

Let $g \in G(p, q)$ be an arbitrary element. Then g can be written as $\rho\lambda^a\zeta^b$ where $\rho \in P, a \in A$ and $b \in \mathbb{Z}$. By Lemma 3.1.4, since $\rho \ll \lambda^a \ll \zeta^b$, then we have that g is positive if and only if

$$\zeta^b \text{ is positive; or } b = 0 \text{ and } \lambda^a \text{ is positive; or } a = b = 0 \text{ and } \rho \text{ is positive.}$$

Notice in particular that this means that the orders on $G(p, q)$ are lexicographical type orders. Therefore the number of orders of $G(p, q)$ are

$$|\mathbb{X}_G(\mathbb{Z})| \times |\mathbb{X}_G(A)| \times |\mathbb{X}_G(P)|.$$

Note that above we are only counting the number of orders each respective subgroup inherits from $G(p, q)$. We have that $|\mathbb{X}_G(\mathbb{Z})| = |\mathbb{X}_G(A)| = 2$ since both $\mathbb{Z}$ and A are rank 1 Abelian groups. Thus to determine the size of $\mathbb{X}(G(p, q))$, it suffices to determine the size of $\mathbb{X}_G(P)$. We claim that $\mathbb{X}_G(P)$ is a countable set and from this fact it will follow that $\mathbb{X}(G(p, q))$ is countable.

We make some observations about the orders of H and K. By Lemma 3.1.3, the signs of the elements $h_{z,0}$ and $k_{z,0}$ determines the order of H_z and K_z, respectively. Moreover, by the relations

$$\zeta^{-z} h_{0,0} \zeta^z = h_{z,0}$$

and

$$\zeta^{-z} k_{0,0} \zeta^z = k_{z,0}$$

for $z \in \mathbb{Z}$, the signs of $h_{0,0}$ and $k_{0,0}$ determines the order of each H_z and K_z. If $h = h_{z_1} \cdots h_{z_n} \in H$ is a nonidentity element with $h_{z_i} \in H_{z_i}$ and $z_1 < \cdots < z_n$, then by Lemma 3.1.4, either

$$h_{z_1} \ll \cdots \ll h_{z_n} \tag{3.10}$$

or

$$h_{z_n} \ll \cdots \ll h_{z_1}. \tag{3.11}$$

(We can assume all h_{z_i} are nonzero.) So any order on H is either the lexicographical or the reverse lexicographical order. Say if (3.10) is true, then h is positive if and only if h_{z_n} is positive. And if (3.11) is true, then h is positive if and only if h_{z_1} is positive. In each of the two cases above, the choice of an ordering for H_0 determines the order of each H_z. Since H_0 has only two orders by Lemma 3.1.3, we have that H has exactly $2 \times 2 = 4$ orders. In particular, $|\mathbb{X}_G(H)| = 4$. Similar result also holds for K with $|\mathbb{X}_G(K)| = 4$.

Having determined how the subgroups H and K are ordered inside $G(p,q)$, we now turn to figuring out all the possible ways the subgroup $P = H \times K$ can be ordered inside $G(p,q)$. We have three cases to consider:

(1) $H \ll K$;

(2) $K \ll H$;

(3) H and K are mixed.

In cases (1) and (2), the orders on P are lexicographical orders. In case (1), if $\rho = hk \in P$, then ρ is positive if and only if either k is positive, or else $k = 1$ and h is positive. Since H and K each have only 4 possible orders, this gives $4 \times 4 = 16$ total orders on P. Similarly, in case (2), we will have exactly 16 possible orders on P. Thus all together the first two cases give 32 possible different orders on P.

In case (3), there exists integers i, j, n, m such that

$$k_{j,0} \ll h_{i,0} \tag{3.12}$$

and

$$h_{n,0} \ll k_{m,0}. \tag{3.13}$$

Conjugate (3.12) by ζ^{m-j} to get $k_{m,0} \ll h_{i+m-j,0}$ and this implies $h_{n,0} \ll k_{m,0} \ll h_{i+m-j,0}$. By Lemma 3.1.4, we can fix an integer u such that either $h_{u,0} \ll k_{m,0} \ll h_{u+1,0}$ or $h_{u,0} \ll k_{m,0} \ll h_{u-1,0}$ depending on how the Archimedean classes of H are ordered. If we conjugate these by ζ^{-u} we get that there exists an integer v such that either

$$h_{0,0} \ll k_{v,0} \ll h_{1,0}$$

or

$$h_{0,0} \ll k_{v,0} \ll h_{-1,0}.$$

If we conjugate the above two relations by various integral powers of ζ we will see that either

$$\cdots \ll h_{0,0} \ll k_{v,0} \ll h_{1,0} \ll k_{v+1,0} \ll h_{2,0} \ll \cdots \tag{3.14}$$

or

$$\cdots \ll h_{0,0} \ll k_{v,0} \ll h_{-1,0} \ll k_{v-1,0} \ll h_{-2,0} \ll \cdots \tag{3.15}$$

holds. For each integer v, either one of (3.14) and (3.15) is possible. Meaning, for every $v \in \mathbb{Z}$, there exists an invariant order in $\mathbb{X}_G(P)$ such that either one of (3.14) and (3.15) is true.

As a way of example, fix an integer v and suppose we are in the case where (3.14) holds. If $\rho \in P$ is a nonzero element, then we can express it as $\rho = \rho_1 \cdots \rho_n$ where either $\rho_i \in H_s$ for some s or $\rho_i \in K_t$ for some t, and $\rho_1 \ll \cdots \ll \rho_n$. Assume all ρ_i are nonzero. Then ρ is positive if and only if ρ_n is positive. This is simply a lexicographical ordering. The ordering is determined by a choice of signs for $h_{0,0}$ and $k_{0,0}$. This gives four different choices, so we get four total orders on P with a fixed v and fixing either one of (3.14) or (3.15).

Therefore for each integer v, we get $2 \times 4 = 8$ different lexicographical orderings on P. Hence, in case (3), we have $|\mathbb{Z}| \times 8 = \aleph_0$ many orderings on P. Putting all three cases together, it follows that $|\mathbb{X}_G(P)| = \aleph_0$, that is, P has exactly countably many orderings it inherits from $G(p,q)$. Thus

$$|\mathbb{X}(G(p,q))| = |\mathbb{X}_G(\mathbb{Z})| \times |\mathbb{X}_G(A)| \times |\mathbb{X}_G(P)| = 2 \times 2 \times \aleph_0 = \aleph_0.$$

This proves that $G(p,q)$ has exactly countably many different orders. □

Our next goal is proving that $\mathbb{X}(G(p,q))$ has Cantor-Bendixson rank 2. We will need one more lemma to help us prove this result.

Lemma 3.1.9. *Let $<$ be an order on $G(p,q)$ such that either*

$$h_{0,0} \ll h_{1,0} \textit{ and } k_{1,0} \ll k_{0,0},$$

or

$$h_{1,0} \ll h_{0,0} \textit{ and } k_{0,0} \ll k_{1,0}.$$

Then either $H \ll K$ or $K \ll H$. In other words, H and K are not mixed.

Proof. Suppose $h_{0,0} \ll h_{1,0}$ and $k_{1,0} \ll k_{0,0}$. Assume neither $H \ll K$ or $K \ll H$ is true. So then H and K are mixed. As shown in the proof of Theorem 3.1.8, since $h_{0,0} \ll h_{1,0}$, there exists some $v \in \mathbb{Z}$ such that $h_{0,0} \ll k_{v,0} \ll h_{1,0}$. Conjugating by ζ gives $h_{1,0} \ll k_{v+1,0} \ll h_{2,0}$, which implies $k_{v,0} \ll k_{v+1,0}$. Conjugating by ζ^{-v}, it follows that $k_{0,0} \ll k_{1,0}$, contradicting our assumption that $k_{1,0} \ll k_{0,0}$. Thus either $H \ll K$ or $K \ll H$. The other case when

$h_{1,0} \ll h_{0,0}$ and $k_{0,0} \ll k_{1,0}$ is proven similarly. $\square$

Recall that an ordering of G is isolated in $\mathbb{X}(G)$ if it is the only ordering satisfying some finite set of inequalities.

Theorem 3.1.10. *The space of orders $\mathbb{X}(G(p,q))$ has Cantor-Bendixson rank 2.*

Proof. An order on $G(p,q)$ is determined by making a choice of signs for the elements $h_{0,0}, k_{0,0}, \lambda$ and ζ, and choosing how the groups H and K are ordered relative to each other. There are three different possibilities to consider when ordering the group $P = H \times K$. Either

(1) $H \ll K$;

(2) $K \ll H$;

(3) H and K are mixed.

First, we show that all the orderings that arise from case (3) are isolated. In case (3), once we choose an integer v as in the proof of Theorem 3.1.8, there are eight possible orderings on P and each of the orderings will satisfy exactly one of the following inequalities (depending on the signs of $h_{0,0}$ and $k_{0,0}$):

(i) $h_{0,0} < k_{v,0} < h_{1,0}$

(ii) $h_{0,0}^{-1} < k_{v,0} < h_{1,0}^{-1}$

(iii) $h_{0,0} < k_{v,0}^{-1} < h_{1,0}$

(iv) $h_{0,0}^{-1} < k_{v,0}^{-1} < h_{1,0}^{-1}$

(v) $h_{0,0} < k_{v,0} < h_{-1,0}$

(vi) $h_{0,0}^{-1} < k_{v,0} < h_{-1,0}^{-1}$

(vii) $h_{0,0} < k_{v,0}^{-1} < h_{-1,0}$

(viii) $h_{0,0}^{-1} < k_{v,0}^{-1} < h_{-1,0}^{-1}$.

Furthermore, taking into account the signs of the elements λ and ζ, each of the eight different possibilities above gives us four different orders on the group $G(p,q)$. Each of these orders will be isolated since, for example, say $h_{0,0}, k_{0,0}, \lambda$ and ζ are all positive. Then the string of inequalities

$$1 < h_{0,0} < k_{v,0} < h_{1,0} < \lambda < \zeta \tag{3.16}$$

witnesses that the order is isolated. There is no other order possible on $G(p, q)$ with $h_{0,0}, k_{0,0}, \lambda$ and ζ all positive that can also satisfy the inequality (3.16). Thus, in case (3) all the orderings are isolated.

We now turn to the orderings arisen from cases (1) and (2). We will only discuss case (1) since both cases are handled similarly. In case (1), there are exactly 16 different orderings possible for P. This is so because the only ordering possible on P in case (1) is the lexicographic ordering, and the groups H and K each have only four possible orderings. This gives us 16 total orders on P. We show half of these orders are isolated and the other half are limit points in $\mathbb{X}(G(p, q))$.

Fix a finite set of positive elements $g_1, \ldots, g_n \in G(p, q)$. First, fix an ordering $<$ of $G(p, q)$ from one of these 16 possible orders where it is the case $h_{0,0} \ll h_{1,0}$ and $k_{0,0} \ll k_{1,0}$. We want to find another ordering $\prec$ of type (3) under which $g_1, \ldots, g_n$ are all positive. We can write each g_i as $g_i = h_{y_1} \cdots h_{y_s} k_{z_1} \cdots k_{z_t} \lambda^a \zeta^b$ with $h_i \in H_i, k_j \in K_j, a \in A$ and $b \in \mathbb{Z}$. By Lemma 3.1.6 under any ordering on $G(p, q)$, the following are the set of equivalences classes under the Archimedean equivalent relation: $\{[1], [h_{z,0}], [k_{z,0}], [\lambda], [\zeta]\}$. Choose $u \in \mathbb{Z}$ large enough so that no element from H_u appears in any g_i, and also choose $v \in \mathbb{Z}$ small enough so that no element from K_v appears in any g_i. We can now define an ordering $\prec$ on $G(p, q)$ as follows. The elements $h_{0,0}, k_{0,0}, \lambda$ and ζ will have the same signs as they did under $<$ and $\prec$ will satisfy the following Archimedean chain

$$h_{u,0} \ll k_{v,0} \ll h_{u+1,0} \ll \lambda \ll \zeta.$$

It can be checked that in the ordering $\prec$ the elements $g_1, \ldots, g_n$ will all be positive. This is so because under $\prec$ for each h_{y_i} and k_{z_j} that appears in some g_i, we will have that $h_{y_i} \ll k_{z_j}$. So the Archimedean classes that appear in $g_1, \ldots, g_n$ are ordered in the exact same way as they were under $<$. Thus $g_1, \ldots, g_n$ will remain positive under $\prec$. Likewise, if our fixed ordering $<$ satisfies $h_{1,0} \ll h_{0,0}$ and $k_{1,0} \ll k_{0,0}$. Then we can straightforwardly modify the above argument to find another ordering in which $g_1, \ldots, g_n$ are all positive.

This only leaves the cases where $<$ satisfies either $h_{0,0} \ll h_{1,0}$ and $k_{1,0} \ll k_{0,0}$, or $h_{1,0} \ll h_{0,0}$ and $k_{0,0} \ll k_{1,0}$. In these cases, the orderings will be isolated. For example, suppose $<$ satisfies $h_{0,0} \ll h_{1,0}$ and $k_{1,0} \ll k_{0,0}$, and the signs of the elements $h_{0,0}, k_{0,0}, \lambda, \zeta$ are all positive. Then the inequality

$$1 < h_{0,0} < h_{1,0} < k_{1,0} < k_{0,0} < \lambda < \zeta \tag{3.17}$$

witnesses that the ordering is isolated. To see this, let U be the open neighborhood determined by (3.17). First, notice that an ordering from case (1) can only be a limit point of set of orders from case (3) since cases (1) and (2) only give rise to finitely many orders and any open neighborhood of a limit point in $\mathbb{X}(G(p,q))$ must contain infinitely many distinct points. Therefore, if $<$ is to be a limit point, then U must contain infinitely many orders of type (3). On the other hand, by Lemma 3.1.9, any order in U cannot have H and K mixed. In particular, U cannot contain any order of type (3). So we see that it is not possible for $<$ to be a limit point. Thus an order where the Archimedean classes of H and K are going in opposite directions must be an isolated order.

We can now see that $\mathbb{X}(G(p,q))$ has infinitely many isolated points and only finitely many limit points. Therefore $\mathbb{X}(G(p,q))'$ is a finite set and so $\mathbb{X}(G(p,q))^{(2)} = \emptyset$. Hence $\mathbb{X}(G(p,q))$ has Cantor-Bendixson rank 2. $\square$

We now discuss an alternative way to view the above proof. We have already seen that we can view elements of $\mathbb{X}(G)$ as paths through a binary branching tree. As we progress along the tree, at each node we choose whether an element is positive or negative. So then a path through the tree precisely describes a positive cone of an order.

For the group $G(p,q)$, we can build a tree T such that the paths through T correspond to the orderings of the group $G(p,q)$. We start at the root node that represents the empty string. At level one, we decide whether the element ζ is positive or negative. At level two, we decide whether λ is positive and negative. Similarly, at levels three and four, we decide the signs of the elements $h_{0,0}$ and $k_{0,0}$, respectively. At level five, decide the sign of $|h_{0,0}|^{-1}|h_{1,0}|$. This determines whether $|h_{0,0}| < |h_{1,0}|$ or $|h_{1,0}| < |h_{0,0}|$ will be hold in the ordering. At level six, choose the sign of $|k_{0,0}|^{-1}|k_{1,0}|$. So far we have built the first six levels of the tree. The choices on level five and six determine the order of the Archimedean classes for groups H and K. That is, once we have decided whether or not $|h_{0,0}|^{-1}|h_{1,0}|$ is positive or negative, we have determined for H either

$$\cdots \ll h_{-1,0} \ll h_{0,0} \ll h_{1,0} \ll \cdots$$

or

$$\cdots \ll h_{1,0} \ll h_{0,0} \ll h_{-1,0} \ll \cdots$$

is true. We have a similar result for K as well.

Consider the nodes at level six that say that the chain of the Archimedean classes of H and K go in opposite directions. All the paths that pass through these nodes will be isolated. This is so because once the chain of the Archimedean classes are going in opposite directions, it is no longer possible to mix the Archimedean classes of H and K.

Next, consider the nodes at level six that say that the chains go in the same direction. At subsequent levels choose the signs of $|h_{0,0}|^{-1}|k_{v,0}|$, $|k_{v,0}|^{-1}|h_{1,0}|$ and $|k_{v,0}|^{-1}|h_{-1,0}|$ for all integers v. At this point, we are waiting to see if the Archimedean classes of H and K mix together or not. If we see that either

$$|h_{0,0}| < |k_{v,0}| < |h_{1,0}|$$

or

$$|h_{0,0}| < |k_{v,0}| < |h_{-1,0}|$$

for some v, then the orders above these nodes will be isolated. Since once the classes are mixed together, they will be the only orderings that satisfy an inequality of the form

$$|h_{0,0}| < |k_{v,0}| < |h_{1,0}| < |\lambda| < |\zeta|$$

or

$$|h_{0,0}| < |k_{v,0}| < |h_{-1,0}| < |\lambda| < |\zeta|$$

depending on the order of the Archimedean classes of H and K. If we never see the Archimedean classes mixing together, then these paths will be limit points.

We now want to argue that $[T]$ has Cantor-Bendixson rank 2. As mentioned above, once we reach level six, any order whose Archimedean chain for H and K go in opposite directions will be isolated. In addition, any order where the Archimedean chain go in the same direction but such that either $|h_{0,0}| < |k_{v,0}| < |h_{1,0}|$ or $|h_{0,0}| < |k_{v,0}| < |h_{-1,0}|$ holds will be isolated. Note that after the first derivative we are only left with finitely many paths or orders, and these are precisely the ones where the Archimedean classes of H and K did not mix together and the classes go in the same direction. These finitely many paths will be isolated. Therefore, taking the derivative one more time will leave no paths on the tree. Hence $[T]$ has Cantor-Bendixson rank 2.

3.2 Space of orders with Cantor-Bendixson rank n

In this section we want to prove that for every integer $n \geq 2$ there exists an orderable group such that its space of orders has Cantor-Bendixson rank n. We will build upon the group $G(p, q)$ from Section 3.1 to construct our orderable groups in question. Once again, let

$$A = \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{Z} \right\}$$

be the additive group of dyadic rationals. Let X be a subset of A defined by

$$X = \{x \in A \mid 0 \leq x < 1\}.$$

Fix an integer $n \geq 2$. For each integer $1 \leq i \leq n$, we will define the group H^i. (Here H^1 and H^2 will play the roles of H and K, respectively, from Section 3.1.) For all $z \in \mathbb{Z}$ and $x \in X$, let $H^i_{z,x}$ be copies of the group $(\mathbb{Q}, +)$. For each $z \in \mathbb{Z}$, define

$$H^i_z = \bigoplus_{x \in X} H^i_{z,x}.$$

Let

$$H^i = \bigoplus_{z\in\mathbb{Z}} H^i_z = \bigoplus_{z\in\mathbb{Z}} \bigoplus_{x\in X} H^i_{z,x}$$

and

$$P_n = H^1 \times H^2 \times \cdots \times H^n.$$

Our goal is to construct a family of groups similar to how $G(p,q)$ was constructed in Section 3.1. Fix a collection of distinct primes $p_1, \ldots, p_n$. First we want to construct a semidirect product $M_n = P_n \rtimes A$. We will write $h^{i,r}_{z,x} \in H^i_{z,x}$ to denote the number $r \in \mathbb{Q}$. Let λ^α for $\alpha \in A$ denote an arbitrary element of A and let ζ^β for $\beta \in \mathbb{Z}$ denote an arbitrary element of $\mathbb{Z}$. We define a group action on P_n by A similar to the group actions in (3.1) and (3.2). Our group action will be

$$\lambda^{-\alpha} h^{i,r}_{z,x} \lambda^\alpha = h^{i,rp_i^m}_{z,x+\alpha 2^z - m}$$

where $m \leq x + \alpha 2^z < m+1$ for $m \in \mathbb{Z}$. Of course, this action is only defined on the basic components of P_n but we can extend this action to the whole group P_n componentwise. Under these group actions, we can construct the semidirect product $M_n = P_n \rtimes A$. Next we define a group action on M_n by $\mathbb{Z}$ via

$$\zeta^{-\beta} h^{i,r}_{z,x} \zeta^\beta = h^{i,r}_{z+\beta,x}$$

and

$$\zeta^{-\beta} \lambda^\alpha \zeta^\beta = \lambda^{\frac{\alpha}{2^\beta}}.$$

With the actions described above we can finally construct the semidirect product

$$G_n = M_n \rtimes \mathbb{Z} = (P_n \rtimes A) \rtimes \mathbb{Z}.$$

Once again, we get a family of groups by choosing different collections of primes.

The next set of results are true for the group G_n with almost identical proofs as those given in Section 3.1 for the group $G(p,q)$.

Proposition 3.2.1. *The group $G_n = M_n \rtimes \mathbb{Z} = (P_n \rtimes A) \rtimes \mathbb{Z}$ is orderable.*

Proof. This result is proven in an analogous fashion as Proposition 3.1.2. We give an outline of the proof without going into details. We start by defining an ordering on the each of the subgroups H_z^i as done in beginning of the proof of Proposition 3.1.2. We can extend this to define an ordering of each H^i and then an ordering of P_n. Verify that the ordering of P_n is preserved under the group actions and then define an ordering on $M_n = P_n \rtimes A$. Lastly, verify the ordering of M_n is preserved by the group actions and conclude that $G_n = M_n \rtimes \mathbb{Z}$ is orderable. $\square$

We point out that in the proofs of Lemmas 3.1.3 and 3.1.4 the results were only proven for the subgroup H (or in the notation of this section, for the subgroup H^1 of G_2) and it was clear that similar arguments would lead to the same conclusions about the subgroup K. We can prove the next two lemmas using identical arguments.

Lemma 3.2.2. *In any order of G_n, the order of each subgroup H_z^i is Archimedean and unique up to duals.*

Proof. We describe the unique Archimedean order on H_z^i up to dual. Let $h \in H_z^i$. We can write $h = h_{z,x_1}^{i,r_1} \cdots h_{z,x_m}^{i,r_m}$ with $x_j \in X$ and $r_j \in \mathbb{Q}$. Define

$$h \text{ is positive if and only if } 0 <_{\mathbb{R}} r_1 p_i^{x_1} + \cdots + r_m p_i^{x_m}$$

where $<_{\mathbb{R}}$ denotes the usual ordering of $\mathbb{R}$. See proof of Lemma 3.1.3 for details. $\square$

Lemma 3.2.3. *In any order of G_n, for all $1 \le i \le n$, either*

$$\cdots \ll h_{z-1,0}^{i,1} \ll h_{z,0}^{i,1} \ll h_{z+1,0}^{i,1} \ll \cdots \ll \lambda \ll \zeta$$

or

$$\cdots \ll h_{z+1,0}^{i,1} \ll h_{z,0}^{i,1} \ll h_{z-1,0}^{i,1} \ll \cdots \ll \lambda \ll \zeta$$

holds.

Proof. See Lemma 3.1.4. $\square$

Lemma 3.2.4. *For all $i \neq j$, the elements $h^{i,1}_{u,0}$ and $h^{j,1}_{v,0}$ are not Archimedean equivalent for any $u, v \in \mathbb{Z}$.*

Proof. This result is proven similar to Lemma 3.1.5. □

Lemma 3.2.5. *The Archimedean classes of G_n with respect to any order are*

$$\{[1], [\lambda], [\zeta]\} \cup \{[h^{1,1}_{z,0}] \mid z \in \mathbb{Z}\} \cup \cdots \cup \{[h^{n,1}_{z,0}] \mid z \in \mathbb{Z}\}.$$

Proof. Follows from Lemmas 3.2.2, 3.2.3 and 3.2.4. □

3.2.1 Invariants of orderings of G_n

In this section we show a useful way to be able to describe all the orderings of G_n. Under any fixed order of G_n, by Lemma 3.2.3, we know that the Archimedean classes of each H^i can only ordered in exactly two ways. With this in mind we make the following definition.

Definition 3.2.6. Fix an ordering $<$ of G_n.

(1) We will say that the direction of H^i is *positive* if it is the case that

$$\cdots \ll h^{i,1}_{z-1,0} \ll h^{i,1}_{z,0} \ll h^{i,1}_{z+1,0} \ll \cdots .$$

(2) We will say that the direction of H^i is *negative* if it is the case that

$$\cdots \ll h^{i,1}_{z+1,0} \ll h^{i,1}_{z,0} \ll h^{i,1}_{z-1,0} \ll \cdots .$$

We now introduce an auxiliary relation that will help us understand the orderings of G_n. Fix an ordering $<$ of G_n. Let $\overline{n}$ denote the set $\{1, \ldots, n\}$. Define $\sim_<$ on $\overline{n}$ by $i \sim_< j$ if and only if either $i = j$ or else H^i and H^j are mixed. (Recall Definition 3.1.7.)

Lemma 3.2.7. *If $i \sim_< j$, then H^i and H^j have the same direction. That is, the direction of both is either positive or negative.*

Proof. Suppose $i \neq j$. Then H^i and H^j are mixed and there exists integers s, t, u, v such that

$$h^{i,1}_{u,0} \ll h^{j,1}_{v,0}$$

and

$$h^{j,1}_{t,0} \ll h^{i,1}_{s,0}.$$

Conjugate the above relation by ζ^{v-t} to get $h^{j,1}_{v,0} \ll h^{i,1}_{s+v-t,0}$ and this implies $h^{i,1}_{u,0} \ll h^{j,1}_{v,0} \ll h^{i,1}_{s+v-t,0}$. By Lemma 3.2.3, we can fix an integer m such that either $h^{i,1}_{m,0} \ll h^{j,1}_{v,0} \ll h^{i,1}_{m+1,0}$ or $h^{i,1}_{m,0} \ll h^{j,1}_{v,0} \ll h^{i,1}_{m-1,0}$ depending on how the Archimedean classes of H^i are ordered. If we conjugate these by ζ^{-m} we get that there exists an integer l such that either

$$h^{i,1}_{0,0} \ll h^{j,1}_{l,0} \ll h^{i,1}_{1,0}$$

or

$$h^{i,1}_{0,0} \ll h^{j,1}_{l,0} \ll h^{i,1}_{-1,0}.$$

By conjugating the above two relations by various integral powers of ζ we will see that either

$$\cdots \ll h^{i,1}_{0,0} \ll h^{j,1}_{l,0} \ll h^{i,1}_{1,0} \ll h^{j,1}_{l+1,0} \ll h^{i,1}_{2,0} \ll \cdots$$

or

$$\cdots \ll h^{i,1}_{0,0} \ll h^{j,1}_{l,0} \ll h^{i,1}_{-1,0} \ll h^{j,1}_{l-1,0} \ll h^{i,1}_{-2,0} \ll \cdots$$

holds. Thus we see that the Archimedean classes of H^i and H^j have the same direction. $\square$

Lemma 3.2.8. *The relation $\sim_<$ is an equivalence relation.*

Proof. It is easy to see that $\sim_<$ is both reflexive and symmetric. We just need to show that $\sim_<$ is transitive. Suppose $i \sim_< j$ and $j \sim_< k$. Suppose H^i, H^j and H^k all have positive direction. We will only prove this case, the other case when all have negative direction is handled similarly. From the proof of Lemma 3.2.7 we know there exists integers u and v such that

$$h_{0,0}^{i,1} \ll h_{u,0}^{j,1} \ll h_{1,0}^{i,1} \ll h_{u+1,0}^{j,1} \ll h_{2,0}^{i,1}$$

and

$$h_{u,0}^{j,1} \ll h_{v,0}^{k,1} \ll h_{u+1,0}^{j,1}.$$

It follows $h_{0,0}^{i,1} \ll h_{u,1}^{j,1} \ll h_{v,0}^{k,1}$ and $h_{v,0}^{k,1} \ll h_{u+1,0}^{j,1} \ll h_{2,0}^{i,1}$. Thus H^i and H^k are mixed, and $i \sim_< k$. □

We will refer to the direction of an equivalence class $[i]$ as being *positive* or *negative*, by which we will mean that the direction of H^i is positive or negative, respectively.

Lemma 3.2.9. *If $i \sim_< k, j \sim_< l$ and $i \not\sim_< j$, then*

$$H^i \ll H^j \text{ if and only if } H^k \ll H^l.$$

Proof. Assume $H^i \ll H^j$. Since $\sim_<$ is an equivalence relation, it follows that $k \not\sim_< l$. Then either $H^k \ll H^l$ or $H^l \ll H^k$. By way of contradiction, suppose $H^l \ll H^k$. Since $j \sim_< l$ and $i \sim_< k$, there exists integers u and v such that $h_{0,0}^{j,1} \ll h_{u,0}^{l,1}$ and $h_{0,0}^{k,1} \ll h_{v,0}^{i,1}$. By assumption $H^l \ll H^k$ and so $h_{0,0}^{j,1} \ll h_{u,0}^{l,1} \ll h_{0,0}^{k,1} \ll h_{v,0}^{i,1}$. But this contradicts the initial assumption $H^i \ll H^j$. Therefore it must be that $H^k \ll H^l$. A similar argument shows that $H^k \ll H^l$ implies $H^i \ll H^j$. □

Observe that Lemma 3.2.9 shows that the $\ll$ relation induces a strict total order $\lesssim$ on the set of equivalence classes of $\overline{n}$ under $\sim_<$. Given two equivalence classes $[i]$ and $[j]$, we can define $[i] \lesssim [j]$ if and only if $[i] = [j]$ or $H^i \ll H^j$.

Lemma 3.2.10. *Let C be an equivalence class under $\sim_<$ with $|C| = k$ and $k \geq 2$. Let i_0 be the least integer of C.*

(a) *Suppose the direction of H^i is positive for all $i \in C$. Then there is an enumeration*

$i_0, i_1, \ldots, i_{k-1}$ of C and integers $u_1, \ldots, u_{k-1}$ such that

$$h^{i_0,1}_{0,0} \ll h^{i_1,1}_{u_1,0} \ll \cdots \ll h^{i_{k-1},1}_{u_{k-1},0} \ll h^{i_0,1}_{1,0}.$$

(b) *Suppose the direction of H^i is negative for all $i \in C$. Then there is an enumeration $i_0, i_1, \ldots, i_{k-1}$ of C and integers $u_1, \ldots, u_{k-1}$ such that*

$$h^{i_0,1}_{0,0} \ll h^{i_1,1}_{u_1,0} \ll \cdots \ll h^{i_{k-1},1}_{u_{k-1},0} \ll h^{i_0,1}_{-1,0}.$$

Proof. We prove only (a). From the proof of Lemma 3.2.7 we know there exists integers $u_1, \ldots, u_{k-1}$ such that

$$h^{i_0,1}_{0,0} \ll h^{i_1,1}_{u_1,0} \ll h^{i_0,1}_{1,0}, \ldots, h^{i_0,1}_{0,0} \ll h^{i_{k-1},1}_{u_{k-1},0} \ll h^{i_0,1}_{1,0}.$$

The elements $h^{i_1,1}_{u_1,0}, \ldots, h^{i_{k-1},1}_{u_{k-1},0}$ are linearly ordered with respect to $\ll$. So be reindexing as necessary, we can conclude that

$$h^{i_0,1}_{0,0} \ll h^{i_1,1}_{u_1,0} \ll \cdots \ll h^{i_{k-1},1}_{u_{k-1},0} \ll h^{i_0,1}_{1,0}. \qquad \square$$

We next want to describe a collection of invariants that uniquely describe an ordering of G_n. Recall, we had fixed an ordering $<$ of G_n. We can define a *positivity string* $\gamma \in 2^{n+2}$ that will encode the signs of the elements $h^{1,1}_{0,0}, \ldots, h^{n,1}_{0,0}, \lambda$ and ζ under $<$. The bits in positions 0 to $n-1$ will encode the signs of $h^{1,1}_{0,0}, \ldots, h^{n,1}_{0,0}$, respectively. The bit in position n will encode the sign of λ and the bit in position $n+1$ will encode the sign of ζ. A bit of 0 will correspond to the element being negative and a bit of 1 will correspond to the element being positive.

Fix the equivalence relation $\sim_<$ on $\overline{n}$ as defined above and suppose we have m many equivalence classes. We have the induced ordering relation $\lesssim$ on the equivalence classes of $\overline{n}$ under $\sim_<$. When we say the i-th equivalence class, we will mean the i-th equivalence class with respect to the $\lesssim$ ordering. (So for example, the 0-th equivalence class will refer to the least class under the $\lesssim$ order.) Define a *direction string* $\delta \in 2^m$ where $\delta(i) = 0$ if the direction of the i-th equivalence class is negative, and $\delta(i) = 1$ if the direction of the i-th equivalence class is positive. Furthermore, by Lemma 3.2.10, for each equivalence class with at least two elements, we can assign to it an enumeration of its elements and a finite set of integers that tell us how the Archimedean classes of various H^i are mixed. For example, let C be a class with k many elements, we can fix a string of pairs of the form $\langle i_1, u_1\rangle, \ldots, \langle i_{k-1}, u_{k-1}\rangle$ such that the Archimedean classes of $H^{i_0}, H^{i_1}, \ldots H^{i_{k-1}}$ are mixed as determined by this string.

To summarize, we have the following list of invariants we can assign to each order $<$ of G_n:

(3.I) A positivity string $\gamma \in 2^{n+2}$ that will encode the signs of the elements $h^{1,1}_{0,0}, \ldots, h^{n,1}_{0,0}, \lambda$ and ζ under $<$.

(3.II) An equivalence relation $\sim_<$ on $\overline{n}$ with m many equivalence classes.

(3.III) The induced relation $\lesssim$ on the equivalence classes of $\overline{n}$ under $\sim_<$.

(3.IV) A direction string $\delta \in 2^m$ where $\delta(i)$ encodes if the i-th equivalence class is positive or negative.

(3.V) A string of pairs that encodes how the various H^i in each equivalence class are mixed under the Archimedean less than relation.

Conversely, we can start by fixing these invariants and from there define a unique ordering of G_n.

Proposition 3.2.11. *Suppose we have fixed the invariants* (3.I)–(3.V) *as describe above. Then we can define a unique ordering of G_n.*

Proof. We show how we can use these invariants to define a unique ordering $\prec$ of G_n. For (3.I), we pick a string $\gamma \in 2^{n+2}$ that will determine the signs of the elements $h^{1,1}_{0,0}, \ldots, h^{n,1}_{0,0}, \lambda$ and ζ under $\prec$. For (3.II), we define some equivalence relation, denoted by $\sim_\prec$, on the set $\overline{n}$. Or equivalently, we define some partition of the set $\overline{n}$ and define $\sim_\prec$ to be the equivalence relation corresponding to this partition. This equivalence relation will determine the Archimedean classes of which H^i are mixed. For (3.III), fix some total ordering $\lesssim$ of the equivalence classes of $\sim_\prec$. For (3.IV), suppose we have m many equivalence classes under $\sim_\prec$, pick a string in $\delta \in 2^m$ that will determine for each equivalence class whether it is positive or negative. Lastly, for (3.V), for any equivalence class of $\sim_\prec$ with at least two elements, we fix an enumeration of its elements and pick one less than the size of the class many integers. The enumerations and the choice of integers will determine how are the Archimedean classes of the H^i are mixed.

Next, having fixed our collection of invariants, we now explain how to define an ordering $\prec$ of G_n. Let $g \in G_n$ be an arbitrary nonidentity element. We show how to determine whether g is positive or negative under $\prec$. We can write $g = \rho\lambda^a\zeta^b$ for some unique $\rho \in P_n$, $\lambda^a \in A$ and $\zeta^b \in \mathbb{Z}$. We first look at $\gamma(n+1)$ to determine the sign of ζ. Suppose $\gamma(n+1) = 1$. If $b > 0$, then g is positive, and if $b < 0$, then g is negative. Next suppose $\gamma(n+1) = 0$. If $b > 0$, then g is negative, and if $b < 0$, then g is positive. If $b = 0$, then we look at the bit in $\gamma(n)$. In an analogous fashion, if $\gamma(n) = 1$ and $a > 0$, then g is positive, and if $a < 0$, then g is negative. If $\gamma(n) = 0$ and $a > 0$, then g is negative and if $a < 0$, then g is positive. If $a = b = 0$, then $g = \rho$. In this case, we can express g as $g = h^{l_1} \cdots h^{l_s}$ where $h^{l_i} \in H^{l_i}$. Furthermore, for each l_i, we can express h^{l_i} as $h^{l_i}_{z_1} \cdots h^{l_i}_{z_t}$ where $h^{l_i}_{z_j} \in H^{l_i}_{z_j}$.

Using the $\lesssim$ ordering of the equivalences classes, we can order the classes $[l_1], \ldots, [l_s]$. Without loss of generality, assume $[l_1] \lesssim \cdots \lesssim [l_s]$. (Of course, it is possible some of these equivalence classes are equal.) Define C to the largest equivalence class, i.e., $C = [l_s]$. Recall, we had assumed we have m many equivalence classes under $\sim_\prec$ and so $\delta(m-1)$ determines whether the largest equivalence class is positive or negative. First, suppose $C = \{[l_s]\} = \{[l]\}$, that is, C consists of a single element. In this case, we express h^l as $h^l_{z_1} \ldots h^l_{z_t}$ with $h_{z_j} \in H^l_{z_j}$ and $z_1 < \cdots < z_t$. Now, if $\delta(m-1) = 0$, then g is positive if $h^l_{z_1}$ is positive, and g is negative is $h^l_{z_1}$ is negative. If $\delta(m-1) = 1$, then g is positive if $h^l_{z_t}$ is positive, and g is negative if $h^l_{z_t}$ is negative. To determine the sign of $h^l_{z_1}$ or $h^l_{z_t}$, we can use the positivity string γ and Lemma 3.2.2.

Next, suppose $C = \{[k_0], \ldots, [k_r]\}$. Assume $k_0, \ldots, k_r$ is the enumeration we have assigned to C and $u_1, \ldots, u_r$ are the integers we have picked. Assume $\delta(m-1) = 1$. Now, faithfully in our ordering $\prec$ that we are defining, the Archimedean less than relation "should" behave as following

$$\cdots \ll h_{0,0}^{k_0,1} \ll h_{u_1,0}^{k_1,1} \ll \cdots \ll h_{u_r,0}^{k_r,1} \ll h_{1,0}^{k_0,1} \ll h_{u_1+1,0}^{k_1,1} \ll \cdots \ll h_{u_r+1,0}^{k_r,1} \ll h_{2,0}^{k_0,1} \ll \cdots .$$

So keeping this in mind, we search $g = h^{l_1} \cdots h^{l_s}$ to find an element of the form $h_z^{k_i} \in H_z^{k_i}$ that will be the "largest" with respect to the Archimedean less than relation given above. Once again, g is positive if $h_z^{k_i}$ is positive, and g is negative if $h_z^{k_i}$ is negative. And we can determine the sign of $h_z^{k_i}$ using the positivity string γ and Lemma 3.2.2. The case when $\delta(m-1) = 0$ can be handled similarly. The only difference in that case will be that the Archimedean less than relation described above will be going in the negative direction. $\square$

We mention that for the ordering $\prec$ described in the above proof, we can start with this ordering and then define the invariants (3.I)–(3.V) from it. In this case, we will find that we get exactly the same set of invariants. Therefore fixing our set of invariants is same as fixing an ordering of G_n. In the next result, we show that this way of describing the orders of G_n will actually completely describe any possible ordering of G_n.

Theorem 3.2.12. *The group G_n has exactly countably many distinct orders. In other words, the space of orders $\mathbb{X}(G_n)$ is countable.*

Proof. Let $g \in G_n$ be an arbitrary element. Then g can be written as $\rho\lambda^a\zeta^b$ where $\rho \in P_n$, $\lambda^a \in A$ and $\zeta^b \in \mathbb{Z}$. By Lemma 3.2.3, since $\rho \ll \lambda^a \ll \zeta^b$, then we have that g is positive if and only if

$$\zeta^b \text{ is positive; or } b = 0 \text{ and } \lambda^a \text{ is positive; or } a = b = 0 \text{ and } \rho \text{ is positive.}$$

Therefore all the orders on G_n are lexicographical type orders. Hence we just need to count all the ways the subgroups P_n, A and $\mathbb{Z}$ can be ordered inside G_n. We have that A and $\mathbb{Z}$ are both rank one Abelian groups, so they can only be ordered in one of two ways. We are left with figuring out all the possible ways to order P_n. This is where the invariants we described earlier will be useful.

For $P_n = H^1 \times \cdots \times H^n$ we need to figure out all the possible ways the Archimedean classes of $H^1, \ldots, H^n$ can be ordered relative to each other. But this is precisely what the invariants (3.I)–(3.V) are describing. They are describing which Archimedean classes should be mixed and how exactly are they mixed, and as we showed in Proposition 3.2.11, the invariants give us enough information to describe an ordering of G_n. Thus, we simply need to count all the different collections of invariants we can fix.

For (3.I) and (3.IV), we are picking a finite binary string of some fixed length. There are only finitely many such strings. For (3.II) and (3.III), there are only finitely many different equivalence classes we can define on the set $\overline{n}$ and only finitely many different ways to order any collection of finitely many equivalence classes. For (3.V), there are only finitely many different ways to enumerate any particular equivalence class, but we also have to pick some finite set of integers for each equivalence class. And here we see that we can actually make countably many different choices when picking our finite sets of integers. Hence we have countably many different possibilities when fixing our invariants. In turn, we get that P_n can ordered in countably many different ways. Therefore G_n has exactly countably many distinct orders. $\square$

3.2.2 Limit points of $\mathbb{X}(G_n)$

The goal of this section is to prove that the Cantor-Bendixson rank of $\mathbb{X}(G_n)$ is n. Let us first discuss the limit points of $\mathbb{X}(G_n)$. Let $< \in \mathbb{X}(G_n)$ be a limit point. Suppose H^i, H^j and H^k are mixed for some $i, j, k \in \overline{n}$. Suppose $<$ satisfies the following string of inequalities

$$1 < h^{i,1}_{0,0} < h^{j,1}_{u,0} < h^{k,1}_{v,0} < h^{i,1}_{1,0}$$

where $u, v \in \mathbb{Z}$. Notice in particular the above inequalities specify how exactly are the Archimedean classes of H^i, H^j and H^k are mixed. Another way to look at it is, the inequalities are specifying part of the information needed to describe invariant (3.V). These string of inequalities also determine an open neighborhood of $<$ in $\mathbb{X}(G_n)$ and any order in this open neighborhood must also satisfy these inequalities. So any order in this neighborhood must also have H^i, H^j and H^k mixed in exactly the same way. Furthermore, if we look at the equivalence classes under $\sim_<$, for any equivalence class with more than one element we can write a similar string of inequalities specifying how the Archimedean classes are mixed and then we can put together all of these strings of inequalities as a single string of inequalities.

As an example, consider the group G_6 and fix $< \in \mathbb{X}(G_6)$ to be a limit point. Suppose $\{1,2,3\},\{4\},\{5,6\}$ are the equivalence classes under $\sim_<$. Assume the direction of all these classes is positive and they are ordered as $[1] \lesssim [4] \lesssim [5]$. Assume $h^{1,1}_{0,0},\ldots,h^{6,1}_{0,0}$, λ and ζ are all positive. Then we can write

$$1 < h^{1,1}_{0,0} < h^{2,1}_{u,0} < h^{3,1}_{v,0} < h^{1,1}_{1,0} < h^{5,1}_{0,0} < h^{6,1}_{w,0} < h^{5,1}_{1,0}$$

for some $u,v,w \in \mathbb{Z}$. We can also add in the elements $h^{4,1}_{0,0}$, λ and ζ to the above string

$$1 < h^{1,1}_{0,0} < h^{2,1}_{u,0} < h^{3,1}_{v,0} < h^{1,1}_{1,0} < h^{4,1}_{0,0} < h^{4,1}_{1,0} < h^{5,1}_{0,0} < h^{6,1}_{w,0} < h^{5,1}_{1,0} < \lambda < \zeta. \tag{3.18}$$

If $<$ is a limit point then every open neighborhood of it must contain points from $\mathbb{X}(G_6)$, in fact, it must contain infinitely many different points from $\mathbb{X}(G_6)$. In particular, in the neighborhood U determined by (3.18), we need to be able find infinitely many orders that satisfy (3.18). Let us consider what invariants (3.18) would specify. It fixes (3.I), (3.IV) and (3.V). So for any order in U these three invariants cannot change. The only ones we can vary are (3.II) and (3.III). Notice if we in addition keep (3.II) fixed, then we have only finitely many different choices to make for (3.III) and this would only give us finitely many different orders at most. Thus there must necessarily exist infinitely many orders in U such that the invariant (3.II) is different for them compared to $<$. In addition, for any order $\prec \in U$, when specifying the equivalence relation $\sim_\prec$, its equivalence classes must be coarser compared to the equivalence classes of $\sim_<$. That is, we can add elements to an equivalence class but we cannot split apart an equivalence class or make it more finer. This is so because (3.18) is saying H^1, H^2, H^3 are mixed and H^5, H^6 are mixed. Thus we can make a choice about mixing H^4 with one of these, but we cannot "unmix" any Archimedean classes that are already mixed.

We can summarize the above discussion as saying that an ordering can only be a limit point of a subset that contains infinitely many orders more mixed than it (mixed here again referring to the mixing of the Archimedean classes of different H^i). We finally arrive at our main result for this section.

Theorem 3.2.13. *The Cantor-Bendixson rank of $\mathbb{X}(G_n)$ is n.*

Proof. We want to show that $\mathrm{CB}(\mathbb{X}(G_n)) = n$. In fact, we will argue that n is least positive integer such that $\mathbb{X}(G_n)^{(n)} = \emptyset$. We first show that $\mathrm{CB}(\mathbb{X}(G_n)) \leq n$. As discussed above, an ordering can only be a limit point if there exists infinitely many orders more mixed than it in $\mathbb{X}(G_n)$. Thus any order that has all of the H^i mixed together must be an isolated ordering because since all of the H^i are already mixed, there aren't any collection of orders of which it can be a limit point of. Therefore $\mathbb{X}(G_n)^{(1)}$ can only contain orders that have at most $n-1$ many of $H^1, \dots, H^n$ mixed and no orders where all of the H^i are mixed. Similarly, $\mathbb{X}(G_n)^{(2)}$ can only contain orders that have at most $n-2$ many of $H^1, \dots, H^n$ mixed. Continuing in this way, $\mathbb{X}(G_n)^{(n-1)}$ can only contains orders that have none of the H^i mixed. But now $\mathbb{X}(G_n)^{(n-1)}$ must be a finite set since G_n has only finitely many orders with no H^i mixed. Hence $\mathbb{X}(G_n)^{(n)} = \emptyset$ and $\mathrm{CB}(\mathbb{X}(G_n)) \leq n$.

Next, we will show that there exists a point in $\mathbb{X}(G_n)$ with Cantor-Bendixson rank at least $n-1$ and this will imply that $\mathrm{CB}(\mathbb{X}(G_n)) = n$. Note since we already know that $\mathbb{X}(G_n)^{(n)} = \emptyset$, then for every $P \in \mathbb{X}(G_n)$, the Cantor-Bendixson rank of P is a finite number. So every point of $\mathbb{X}(G_n)$ has finite rank. Thus if there exists some $P \in \mathbb{X}(G_n)$ with $\mathrm{CB}(P) \geq n-1$, then $\mathbb{X}(G_n)^{(n-1)} \neq \emptyset$. But since $\mathbb{X}(G_n)^{(n)} = \emptyset$, we will have that $\mathrm{CB}(\mathbb{X}(G_n)) = n$, as desired.

Let $\mathcal{O}_k \subseteq \mathbb{X}(G_n)$ for $2 \leq k \leq n$ be defined as follows. An ordering $< \in \mathcal{O}_k$ if and only if ζ, λ and $h^{i,1}_{0,0}$ for all i are positive under $<$; the direction of each H^i is positive, i.e., the

Archimedean classes of each H^i are ordered as

$$\cdots \ll h^{i,1}_{-1,0} \ll h^{i,1}_{0,0} \ll h^{i,1}_{1,0} \ll \cdots ;$$

the induced order $\ll$ on the Archimedean classes satisfies

$$H^j \ll H^{k+1} \ll \cdots \ll H^n$$

with $1 \leq j \leq k$; and $<$ satisfies an inequality of the form

$$h^{1,1}_{0,0} < h^{2,1}_{u_1,0} < \cdots < h^{k,1}_{u_{k-1},0} < h^{1,1}_{1,0}$$

for some $u_1, \ldots, u_{k-1} \in \mathbb{Z}$. Observe that all of the orders in $\mathcal{O}_k$ have $H^1, \ldots, H^k$ mixed. Let $\mathcal{O}_1 \subseteq \mathbb{X}(G_n)$ be defined as follows. An ordering $< \in \mathcal{O}_1$ if and only if ζ, λ and $h^{i,1}_{0,0}$ for all i are positive under $<$; the direction of each H^i is positive; and the induced order $\ll$ on the Archimedean classes satisfies

$$H^1 \ll H^2 \ll \cdots \ll H^n.$$

Now if $\prec \in \mathcal{O}_n$, then $\prec$ is an isolated ordering and $\mathrm{CB}(\prec) = 0$. Next, each $\prec \in \mathcal{O}_{n-1}$ is a limit point of the set $\mathcal{O}_n$. As a limit point of a set of rank 0 points, we have $\mathrm{CB}(\prec) \geq 1$ for all $\prec \in \mathcal{O}_{n-1}$. Similarly, we have that every point of $\mathcal{O}_{n-2}$ is a limit point of the set $\mathcal{O}_{n-1}$, and so $\mathrm{CB}(\prec) \geq 2$ for all $\prec \in \mathcal{O}_{n-2}$. Continuing along, we see that every $\prec \in \mathcal{O}_k$ for $1 \leq k \leq n-1$ is a limit point of the set $\mathcal{O}_{k+1}$, and it also follows that $\mathrm{CB}(\prec) \geq n-k$. In particular, we have that every point in $\mathcal{O}_1$ has Cantor-Bendixson rank at least $n-1$. Thus there exists a point in $\mathbb{X}(G_n)$ with Cantor-Bendixson rank at least $n-1$ and this completes the proof. □

3.3 G_n computably categorical

In this section, we want to show that the group $G(p,q) = G_2$ is computably categorical. A similar argument will show that G_n is computably categorical for all $n \geq 2$. First we show that $G(p,q)$ has a computable presentation. Observe that each element of $G(p,q)$ has a normal form, that is, we can fix a unique representation for each element of the group. Let $g \in G(p,q)$ be a nonidentity element. Then g can be uniquely expressed as

$$h_{i_1,x_1}^{p_1} \cdots h_{i_s,x_s}^{p_s} \cdot k_{j_1,y_1}^{q_1} \cdots k_{j_t,y_t}^{q_t} \cdot \lambda^a \cdot \zeta^b$$

where $i_1 \leq \cdots \leq i_s, j_1 \leq \cdots \leq j_t, x_n < x_{n+1}$ if $i_n = i_{n+1}$, and $y_n < y_{n+1}$ if $j_n = j_{n+1}$. Note that the sets $X, A, \mathbb{Q}$ and $\mathbb{Z}$ are all computable via suitable coding of them into $\mathbb{N}$. Let $D = \mathbb{Z} \times X \times \mathbb{Q}^*$ where $\mathbb{Q}^* = \mathbb{Q} \backslash \{0\}$. Then D is a computable set. If $\sigma \in D$ with $\sigma = \langle z, x, r \rangle$, then we will think of σ representing the element $h_{z,x}^r \in H_{z,x}$. Let $C \subseteq D^{<\omega}$ be defined as $\tau = \langle \sigma_1, \ldots, \sigma_n \rangle \in C$ if and only if $\sigma_i \neq \sigma_j$; there exists at most one σ_i such that $\sigma_i(0) = z$ and $\sigma_i(1) = x$ for all $z \in \mathbb{Z}, x \in X$; $\sigma_1(0) \leq \cdots \leq \sigma_n(0)$; and $\sigma_1(1) < \cdots < \sigma_n(1)$. The set C is also computable. An element $\tau \in C$ will represent some arbitrary element of H of the form $h_{z_1,x_1}^{r_n} \cdots h_{z_n,x_n}^{r_n}$.

Now using C we can effectively represent elements of the group H. Given some $h \in H$ either h is the identity or we can fix a $\tau \in C$ that uniquely represents h. We can also use C to effectively represent elements of K, since H and K are identical as sets. The domain of $G(p,q)$ as a set can be written as $H \times K \times A \times \mathbb{Z}$. Since we can code each component of this direct product into $\mathbb{N}$, it follows we have an effective way of representing elements of $G(p,q)$ using each element's unique normal form. Thus, we can code the domain of $G(p,q)$ into $\mathbb{N}$ effectively.

Next observe that the group actions that were used to build $G(p,q)$ are uniformly computable. (Here it is important that we have an effective way of representing the elements of $G(p,q)$.) So then the group multiplication is a computable function. Hence, $G(p,q)$ has a computable presentation. We denote this computable presentation by $\mathcal{G}$. Having shown $G(p,q)$ has a computable presentation we now show it is computably categorical.

Theorem 3.3.1. *The group $G(p,q)$ is computably categorical.*

Proof. Fix two copies M and N of $G(p,q)$. Assume the domains of M and N are ω. Let

$n_0, n_1, n_2, n_3, n_4 \in N$ correspond to elements $1_G, h_{0,0}, k_{0,0}, \lambda$ and ζ, respectively. Similarly, let $m_0, m_1, m_2, m_3, m_4 \in M$ correspond to elements $1_G, h_{0,0}, k_{0,0}, \lambda$ and ζ, respectively. We can non-uniformly map the elements $1_G, h_{0,0}, k_{0,0}, \lambda$ and ζ in the two copies M and N.

Given $n_1, \ldots, n_4 \in N$, we can determine what elements in N correspond to λ^a and ζ^b for $a \in A$ and $b \in \mathbb{Z}$. If $a = i/2^j$, then the unique solution to the equation $2^j \cdot y = j \cdot n_3$ corresponds to λ^a. If $b \geq 0$ then take integral powers of n_4 to get $b \cdot n_4$ representing ζ^b. If $b \leq 0$, we can find n_4' that corresponds to ζ^{-1} and then take its integral powers. Knowing how to search for elements λ^a and ζ^b, we can also search for elements of the form $h^r_{z,x}$ and $k^r_{z,x}$. If $x \in X$ and $z \in \mathbb{Z}$, then $\lambda^{-x} h_{0,0} \lambda^x = h_{0,x}$ and $\zeta^{-z} h_{0,x} \zeta^z = h_{z,x}$. Thus we can determine the elements corresponding to $h_{z,x}$ and $k_{z,x}$. Now, if $n \in N$ represents $h_{z,x}$ and $r = i/j$, then the unique solution to $j \cdot y = i \cdot n$ represents the element $h^r_{z,x}$. Therefore, we can computably search through N and enumerate over all possible normal forms of elements of $G(p,q)$ in some fixed order. So in particular, we can fix a computable isomorphism between N and $\mathcal{G}$. Similarly, using the same idea, we can fix a computable isomorphism between M and $\mathcal{G}$. Hence we can build a computable isomorphism between M and N and this proves that $G(p,q)$ is computably categorical. $\square$

We can adapt the proof of the previous theorem along with the discussion preceding it to conclude the following theorem.

Theorem 3.3.2. *The group G_n is computably categorical for all $n \geq 2$.*

Observe that all the orders on G_n are lexicographical type orders and for each order we only need to specify some finite amount of information to be able to describe it. Since the group is computably categorical and we have the computable presentation $\mathcal{G}$ described above, in any given presentation we can effectively fix a unique normal form for each element of G_n and from this normal form we can compute the set of positive elements under any fixed order. Therefore all the orders on G_n are computable in any given computable presentation.

Chapter 4

Group with no computable Archimedean orders

In this chapter we construct an orderable computable group that has no computable Archimedean orders but has at least one computable non-Archimedean order.

Theorem 4.0.1. *There exists a computable torsion-free abelian group G that is classically isomorphic to $\bigoplus_\omega \mathbb{Z}$ such that G has no computable Archimedean orders but G does have a computable non-Archimedean order.*

Proof. Let H denote the group $(\bigoplus_\omega \mathbb{Z}, +)$. We will construct a computable group $(G, \cdot_G)$ that will be isomorphic to H. We will construct our group in stages $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ with $G = \bigcup_{s \in \omega} G_s$. We will define a total computable monotonic function $d(s)$ such that at each stage s, the domain of our group will be $G_s = \{0, \ldots, d(s)\}$. At each stage s, we will also define a map $F_s \colon G_s \to \bigoplus_\omega \mathbb{Z}$ such that for all $g \in G_s$, $F_s(g) = \sigma_{g,s}$ will be a finite string of integers with no trailing zeros. In the end, we will ensure that $\lim_s \sigma_{g,s}$ exists, and $F \colon G \to H$ defined via $F(g) = \lim_s \sigma_{g,s} = \sigma_g$ will be a Δ^0_2 isomorphism between G and an infinitely generated subgroup of H.

We will define $p \cdot_G q = r$ if $F_s(p) + F_s(q) = F_s(r)$ for some s. We will build a computable non-Archimedean order $\prec$ of G as follows. We will set $p \prec q$ if $F_s(p) <_{\text{lex}} F_s(q)$ for some s, where $<_{\text{lex}}$ is the usual lexicographic order on $\bigoplus_\omega \mathbb{Z}$. We will ensure that our construction has the following two properties:

(1) If $F_s(p) + F_s(q) = F_s(r)$, then $F_t(p) + F_t(q) = F_t(r)$ for all $t \geq s$. (Therefore in the limit $F(p) + F(q) = F(r)$.)

(2) If $F_s(p) <_{\text{lex}} F_s(q)$, then $F_t(p) <_{\text{lex}} F_t(q)$ for all $t \geq s$.

We shall meet the following set of requirements:

- (Group generators) B_n : The generator element 0^n1 of H is represented in G.
- (Group closure) C_n : If $n = \langle p, q \rangle$, then $p \cdot_G q$ is defined.
- (Group inverses) I_n : There exists p such that $n \cdot_G p = 0$. (The zero of ω will be the identity of G.)
- (Non-Archimedean orders) N_e : φ_e does not define an Archimedean order on G.

We fix a priority ordering on our requirements as follows:

$$B_0 < C_0 < I_0 < N_0 < B_1 < C_1 < I_1 < N_1 < \cdots .$$

Our strategy for meeting the N_e requirements will be as so. For each N_e, we will fix two elements $p, q \in G$ and we will wait to see if φ_e ever says that p and q are Archimedean equivalent. If so, then N_e will act and ensure that φ_e is not an ordering of G. More specifically, we will attempt to satisfy each N_e in three steps as following. First, we will wait until the elements $0^{2e}1$ and $0^{2e+1}1$ are represented in G_s. Suppose $\sigma_{p,s} = 0^{2e}1$ and $\sigma_{q,s} = 0^{2e+1}1$ for some $p, q \in G_s$. Second, we wait until $\varphi_{e,s}(0,p)\downarrow$ and $\varphi_{e,s}(0,q)\downarrow$. Anticipating φ_e to be an ordering, we will write $x <_e y$ if $\varphi_e(x,y)\downarrow \neq 0$ and $y <_e x$ if $\varphi_e(x,y)\downarrow = 0$. Assume p and q are both $<_e$-positive. For the third step, we will wait until we see that for some positive integers m and n, the elements p^n, q^m and q^{m+1} are defined in G_s and φ_e says that $q^m <_e p^n <_e q^{m+1}$. At this point, we will satisfy N_e by making sure φ_e is not an ordering of G. The exact details of the diagonalization will be mentioned in the group construction.

Definition 4.0.2. We say

(1) B_n *requires attention* at stage $s+1$ if there exists no $p \leq d(s)$ such that $\sigma_{p,s} = 0^n 1$.

(2) C_n *requires attention* at stage $s+1$ if $n = \langle p, q \rangle$ with $p, q \leq d(s)$ and there is no $r \leq d(s)$ such that $\sigma_{p,s} + \sigma_{q,s} = \sigma_{r,s}$.

(3) I_n *requires attention* at stage $s+1$ if $n \leq d(s)$ and there is no $p \leq d(s)$ such that $\sigma_{n,s} + \sigma_{p,s} = \lambda$ (the empty string).

We will say that a requirement N_e is *primed* at stage $s+1$ if there are $p, q \leq d(s)$ such that $\sigma_{p,s} = 0^{2e}1$ and $\sigma_{q,s} = 0^{2e+1}1$ with $\varphi_{e,s}(0,p)\downarrow$ and $\varphi_{e,s}(0,q)\downarrow$. We will say that N_e *requires attention* at stage $s+1$ if N_e is primed and there exist $1 \leq m, n \leq s+1$ such that one of the following holds true depending on the "signs" of p and q:

Case (i): If $0 <_e p$ and $0 <_e q$, then the elements p^n, q^m and q^{m+1} are defined in G_s (i.e. there exists $i, j, k \leq d(s+1)$ such that $\sigma_{i,s} = 0^{2e}n$, $\sigma_{j,s} = 0^{2e+1}m$ and $\sigma_{k,s} = 0^{2e+1}(m+1)$), and $\varphi_{e,s}$ is compatible with the inequality

$$q^m <_e p^n <_e q^{m+1}.$$

Case (ii): If $0 <_e p$ and $q <_e 0$, then the elements p^n, q^{-m} and q^{-m-1} are defined in G_s, and $\varphi_{e,s}$ is compatible with the inequality

$$q^{-m} <_e p^n <_e q^{-m-1}.$$

Case (iii): If $p <_e 0$ and $0 <_e q$, then the elements p^{-n}, q^m and q^{m+1} are defined in G_s, and $\varphi_{e,s}$ is compatible with the inequality

$$q^m <_e p^{-n} <_e q^{m+1}.$$

Case (iv): If $p <_e 0$ and $q <_e 0$, then the elements p^{-n}, q^{-m} and q^{-m-1} are defined in G_s, and $\varphi_{e,s}$ is compatible with the inequality

$$q^{-m} <_e p^{-n} <_e q^{-m-1}.$$

Construction:

Stage 0: Let $d(0) = 0$ and let $\sigma_{0,0}$ be the empty string.

Stage $s+1$: Find the highest priority requirement that requires attention.

- If B_n, let $d(s+1) = d(s)+1$. For all $p \leq d(s+1)$, define

$$\sigma_{p,s+1} = \begin{cases} \sigma_{p,s} & \text{if } p \leq d(s), \\ 0^n 1 & \text{if } p = d(s)+1. \end{cases}$$

- If C_n with $n = \langle p, q \rangle$, let $d(s+1) = d(s)+1$. For all $r \leq d(s+1)$, define

$$\sigma_{r,s+1} = \begin{cases} \sigma_{r,s} & \text{if } r \leq d(s), \\ \sigma_{p,s} + \sigma_{q,s} & \text{if } r = d(s)+1. \end{cases}$$

- If I_n, let $d(s+1) = d(s)+1$. For all $p \leq d(s+1)$, define

$$\sigma_{p,s+1} = \begin{cases} \sigma_{p,s} & \text{if } p \leq d(s), \\ -\sigma_{n,s} & \text{if } p = d(s)+1. \end{cases}$$

- If N_e, let $d(s+1) = d(s)$. Let m and n be such that they satisfy the respective inequality as witnessed by $\varphi_{e,s}$. Let

$$l = 2\max\{|\sigma_{r,s}(2e+1)| : r \leq d(s+1)\} + m + n + 1.$$

We remark that we have chosen our l sufficiently large so that p^l and q^l are not yet defined in G_s. For all $r \le d(s+1)$, define

$$\sigma_{r,s+1}(i) = \begin{cases} \sigma_{r,s}(i) & \text{if } i \neq 2e, 2e+1, \\ l\sigma_{r,s}(2e) + \sigma_{r,s}(2e+1) & \text{if } i = 2e, \\ 0 & \text{if } i = 2e+1. \end{cases}$$

Our action ensures that $\sigma_{p,s+1} = 0^{2e}l$ and $\sigma_{q,s+1} = 0^{2e}1$. So in particular, we have set that $p = q^l$.

After a requirement has received attention and acted, we will declare it satisfied for rest of the construction. This completes the construction. $\dashv$

Verification:

The following lemmas verify the required properties of the construction.

Lemma 4.0.3.

(i) *The function $d(s)$ is a monotonically increasing function, that is,* $\lim_s d(s) = \infty$.

(ii) *Each requirement requires attention at most finitely often.*

(iii) *The requirements B_n, C_n and I_n are all met.*

Proof. The first two statements are clear from our construction. For the last statement, each of the requirements B_n, C_n and I_n will act at most once and be satisfied permanently. $\dashv$

Lemma 4.0.4. *For all $p \in \omega$,* $\lim_s \sigma_{p,s} = \sigma_p$ *exists and is a finite string.*

Proof. We have $\sigma_{p,s} \neq \sigma_{p,s+1}$ only when some N_e requirement received attention and acted to diagonalize for φ_e. Also note that whenever an N_e requirement acts, it only modifies the numbers in components $2e$ and $2e+1$. Since each $\sigma_{p,s}$ is a finite string and the length of $\sigma_{p,s}$ decreases as $s \to \infty$, there are only finitely many N_e requirements that when they act can modify $\sigma_{p,s}$. Since each N_e acts at most once, we can fix a large enough stage t such that any N_e that requires attention and acts after stage t will not modify any component of $\sigma_{p,s}$. $\dashv$

For the next set of lemmas, the only case of worry is when act to satisfy some N_e because we modify our string representations only when some N_e acts. Thus we only need to check the stages when some N_e receives attention and acts.

Lemma 4.0.5. *For all $p, q \leq d(s)$, if $p \neq q$, then $\sigma_{p,s} \neq \sigma_{q,s}$.*

Proof. Fix a stage s and assume N_e acts at stage $s+1$. Suppose $p, q \leq d(s)$ with $p \neq q$ and $\sigma_{p,s} \neq \sigma_{q,s}$. Assume towards a contraction, $\sigma_{p,s+1} = \sigma_{q,s+1}$. For $i \neq 2e, 2e+1$, we get $\sigma_{p,s+1}(i) = \sigma_{q,s+1}(i)$ implies $\sigma_{p,s}(i) = \sigma_{q,s}(i)$. Next, we consider the numbers in component $2e$ after N_e acts. By assumption, we have

$$\sigma_{p,s+1}(2e) = l\sigma_{p,s}(2e) + \sigma_{p,s}(2e+1) = l\sigma_{q,s}(2e) + \sigma_{q,s}(2e+1) = \sigma_{q,s+1}(2e).$$

Let $x = \sigma_{p,s}(2e+1) - \sigma_{q,s}(2e+1)$ and $y = \sigma_{q,s}(2e) - \sigma_{p,s}(2e)$. Then $x = ly$. By our construction, $1 < l$. If $x = 0$, then $y = 0$ as well, but this is a contradiction since $\sigma_{p,s} \neq \sigma_{q,s}$. So assume $x \neq 0$. Then $y \neq 0$ and $|x| \leq 2\max\{|\sigma_{r,s}(2e+1)| : r \leq d(s+1)\} < l$. On the other hand, since $1 \leq |y|$, this means that $l \leq |ly|$. Thus $|x| < l \leq |ly|$ and it is not possible that $x = ly$, contrary to our assumption. Hence, we can conclude that $\sigma_{p,s+1} \neq \sigma_{q,s+1}$. The statement now follows by induction. ⊣

Lemma 4.0.6. *For all $p, q, r \leq d(s)$, if $\sigma_{p,s} + \sigma_{q,s} = \sigma_{r,s}$, then $\sigma_{p,t} + \sigma_{q,t} = \sigma_{r,t}$ for all $t \geq s$.*

Proof. Suppose N_e acts at stage $s+1$. If $i \neq 2e$, then it is clear from the construction that $\sigma_{p,s+1}(i) + \sigma_{q,s+1}(i) = \sigma_{r,s+1}(i)$. Next, we have that

$$\sigma_{p,s+1}(2e) + \sigma_{q,s+1}(2e) = l\sigma_{p,s}(2e) + \sigma_{p,s}(2e+1) + l\sigma_{q,s}(2e) + \sigma_{q,s}(2e+1).$$

It follows by assumption that

$$\sigma_{p,s}(2e) + \sigma_{q,s}(2e) = \sigma_{r,s}(2e)$$

and

$$\sigma_{p,s}(2e+1) + \sigma_{q,s}(2e+1) = \sigma_{r,s}(2e+1).$$

Therefore,

$$l\sigma_{p,s}(2e) + \sigma_{p,s}(2e+1) + l\sigma_{q,s}(2e) + \sigma_{q,s}(2e+1) = l\sigma_{r,s}(2e) + \sigma_{r,s}(2e+1) = \sigma_{r,s+1}(2e).$$

By induction, we get that $\sigma_{p,t} + \sigma_{q,t} = \sigma_{r,t}$ for all $t \geq s$. ⊣

Lemma 4.0.7. *For all $p, q \leq d(s)$, if $\sigma_{p,s} <_{lex} \sigma_{q,s}$, then $\sigma_{p,t} <_{lex} \sigma_{q,t}$ for all $t \geq s$.*

Proof. We need to check that if some N_e acts at stage $s+1$ we do not change our lexicographic ordering. Suppose $\sigma_{p,s} <_{\text{lex}} \sigma_{q,s}$ and fix the least i such that $\sigma_{p,s}(i) < \sigma_{q,s}(i)$. Let $a = \sigma_{p,s}(2e), b = \sigma_{p,s}(2e+1), c = \sigma_{q,s}(2e)$ and $d = \sigma_{q,s}(2e+1)$. We have four possibilities.

(1) Suppose $i < 2e$. Then N_e will not modify the ith component and $\sigma_{p,s+1} <_{\text{lex}} \sigma_{q,s+1}$.

(2) Suppose $i = 2e$. Then $a < c$ by assumption and $b - d \leq |b| + |d| < l$ by our choice of l. Observe that $b - d < l$ implies $b < l + d$ and, in turn,

$$la + b < l(a+1) + d \leq lc + d.$$

Thus $\sigma_{p,s+1}(2e) = la + b < lc + d = \sigma_{q,s+1}(2e)$ and $\sigma_{p,s+1} <_{\text{lex}} \sigma_{q,s+1}$.

(3) Suppose $i = 2e+1$. Then $a = c$ and $b < d$. So $la = lc$ and $la + b < lc + d$. It follows $\sigma_{p,s+1}(2e) = la + b < lc + d = \sigma_{q,s+1}(2e)$ and $\sigma_{p,s+1} <_{\text{lex}} \sigma_{q,s+1}$.

(4) Suppose $i > 2e+1$. Then as in the first case, N_e will not modify the ith component and $\sigma_{p,s+1} <_{\text{lex}} \sigma_{q,s+1}$. ⊣

Define a map $F\colon G \to H$ by $F(p) = \lim_s \sigma_{p,s} = \sigma_p$ for all $p \in G$. This map is well-defined by Lemma 4.0.4 and injective by Lemma 4.0.5. Define $\cdot_G$ via: $p \cdot_G q = r$ if and only if $\sigma_{p,s} + \sigma_{q,s} = \sigma_{r,s}$ for some s. By Lemma 4.0.6, this operation does not depend on s and so $\cdot_G$ is computable. Moreover, $p \cdot_G q = r$ implies $F(p) + F(q) = F(r)$. Hence, F is an injective group homomorphism between G and H. Therefore, G is an infinitely generated torsion-free abelian group.

We define a computable non-Archimedean ordering of G by $p \prec g$ if and only if $\sigma_{p,s} <_{\text{lex}} \sigma_{q,s}$ for some s. It is a non-Archimedean ordering because it is a lexicographic ordering and Lemma 4.0.7 shows that $\prec$ is a computable ordering.

Lemma 4.0.8. *Every N_e requirement is met and therefore $(G, \cdot_G)$ has no computable Archimedean orders.*

Proof. Consider some fixed N_e. Suppose φ_e is a computable Archimedean order on G. Then φ_e is total and there must exist a stage s such that N_e will require attention at stage $s+1$. Assume Case (i) applies. Then there exist integers m, n such that $q^m <_e p^n <_e q^{m+1}$. But now the action of N_e will ensure that $q^l = p$ and since $m+1 < l$, we cannot have that $q^m <_e q^{ln} <_e q^{m+1}$.

Next, assume Case (ii) applies. Then $0 <_e p$, $q <_e 0$, and there exist integers m, n such that $q^{-m} <_e p^n <_e q^{-m-1}$. Now, our action will again force that $q^l = p$ but this time we observe that $p \cdot_G q^{-l} = 0$, that is, a product of positive elements is the identity. Thus φ_e is not an order invariant under $\cdot_G$.

The other two cases can be handled similarly, Case (iii) is analogous to Case (ii) and Case (iv) is analogous to Case (i). Hence, in any case, we see that φ_e cannot be an Archimedean ordering of G and N_e is met. $\dashv$

This conclude the proof of Theorem 4.0.1. $\square$

Appendix A

Miscellaneous proofs

A.1 Divisible closure of an abelian group

In this section we show that every torsion-free abelian group has a divisible closure and the divisible closure is a vector space over $\mathbb{Q}$. We will write groups additively. Informally, we say a group is divisible if we can "divide" by integers. More precisely, a nontrivial abelian group A is called *divisible* if for every $g \in A$ and every nonzero integer n there exists an element $h \in A$ such that $nh = g$.

Theorem A.1.1. *If A is an abelian group, then there exists a divisible group D such that*

(i) *there is an injective homomorphism $\varphi\colon A \to D$, and*

(ii) *for all $g \in D$ there is a positive integer n with $ng \in \varphi(A)$.*

Moreover, if A is torsion-free, then D is also torsion-free.

Proof. Let $\mathbb{Z}^+$ denote the set of positive integers. Consider the set $A \times \mathbb{Z}^+$. Define an equivalence relation on $A \times \mathbb{Z}^+$ by $(g, m) \sim (h, n)$ if and only if $ng = mh$ for all $g, h \in A$ and $m, n \in \mathbb{Z}^+$. Let D denote the set of equivalence classes of $A \times \mathbb{Z}^+$ modulo $\sim$ and write g/n to denote the equivalence class of an element (g, n). We define addition on D by $g/m + h/n = (ng + mh)/mn$. It is routine to verify that this operation is well-defined and satisfies the group axioms. For example, the identity of D is $0/n$. It follows from these verifications that D is a divisible group. Lastly, the map $\varphi\colon A \to D$ given by $\varphi(g) = g/1$ is an injective homomorphism and satisfies property (ii) in the theorem statement. $\square$

The group D in Theorem A.1.1 is called a *divisible closure of* A. Next, we want to show that the divisible closure of a torsion-free abelian group is a vector space over $\mathbb{Q}$.

Theorem A.1.2. *Every torsion-free divisible group D is a vector space over $\mathbb{Q}$.*

Proof. One short proof of this fact follows from considering the tensor product $D \otimes_{\mathbb{Z}} \mathbb{Q}$. Since D is torsion-free, it will follow that the natural map $D \to D \otimes_{\mathbb{Z}} \mathbb{Q}$ defined via $g \mapsto g \otimes 1$ is injective and $D \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-vector space.

Here is a more illuminating proof. First we claim that if D is a torsion-free divisible group, then the equation $nx = g$ has a unique solution for any $g \in D$ and any nonzero integer n. Suppose for contradiction there exists two distinct elements x and y of D such that $nx = ny = g$. But then $nx = ny$ implies $nx - ny = n(x - y) = 0$. That is $x - y$ is a nonzero torsion element of D, a contradiction.

We next show how to define scalar multiplication on D. Let $g \in D$ and let $r = m/n \in \mathbb{Q}$. Define $r \cdot g$ to be the unique solution to the equation $nx = mg$. It can be verified that this multiplication satisfies the usual properties of scalar multiplication in a vector space. $\square$

A.2 Linear independence

In this section we regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Fix a prime number p. For all integers $n \geq 1$, let

$$B_n = \left\{ p^{\frac{m}{2^n}} \mid m \in \mathbb{Z} \text{ and } 0 \leq m < 2^n \right\}.$$

Define

$$B = \bigcup_{1 \leq n} B_n = \left\{ p^{\frac{m}{2^n}} \mid m, n \in \mathbb{Z}, 1 \leq n \text{ and } 0 \leq m < 2^n \right\}.$$

We want to show that B is a linearly independent subset of $\mathbb{R}$. Observe that B is linearly independent if and only if B_n is linearly independent for all $n \geq 1$. Thus it suffices to show that B_n is a linearly independent subset of $\mathbb{R}$ for all $n \geq 1$. Fix $n \geq 1$ and let $\alpha = p^{\frac{1}{2^n}}$. Observe that α is a root of the polynomial $x^{2^n} - p \in \mathbb{Q}[x]$ and $B_n = \{1, \alpha, \ldots, \alpha^{2^n-1}\}$.

Lemma A.2.1. *The polynomial $x^{2^n} - p \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.*

Proof. This follows by applying Eisenstein's criterion with the prime p. We can express $x^{2^n} - p$ as

$$a_{2^n}x^{2^n} + a_{2^n-1}x^{2^n-1} + \cdots + a_1x + a_0$$

where $a_0 = -p, a_{2^n} = 1$ and $a_i = 0$ for $0 < i < 2^n$. Apply Eisenstein's criterion with the prime p to conclude that $x^{2^n} - p$ is irreducible over $\mathbb{Q}$. □

Lemma A.2.2. *The real number α is not the root of a polynomial in $\mathbb{Q}[x]$ of degree less than 2^n.*

Proof. Suppose α is the root of a polynomial $f(x) \in \mathbb{Q}[x]$ and the degree of $f(x)$ is less than 2^n. Without loss of generality, let $f(x)$ be the monic polynomial of least degree that has α as a root. In other words, let $f(x)$ be the minimal polynomial for α over $\mathbb{Q}$. Then $f(x)$ divides every polynomial in $\mathbb{Q}[x]$ which has α as a root. In particular, $f(x)$ divides $x^{2^n} - p$, but this contradicts the previous lemma. □

Proposition A.2.3. *The set B_n is linearly independent.*

Proof. For a contradiction, suppose that B_n is linearly dependent. Let $a_0, a_1, \ldots, a_{2^n-1} \in \mathbb{Q}$ such that not all a_i are zero and $a_0 + a_1\alpha + \cdots + a_{2^n-1}\alpha^{2^n-1} = 0$. Let $f(x) \in \mathbb{Q}[x]$ be the polynomial $a_0 + a_1x + \cdots + a_{2^n-1}x^{2^n-1}$. Note that $f(x)$ is a polynomial of degree less than 2^n and α is a root of $f(x)$. This contradicts the above lemma and completes the proof. □

Corollary A.2.4. *The set B is a linearly independent subset of $\mathbb{R}$.*

Proof. Note that B is linearly independent if and only if B_n is linearly independent for all $n \geq 1$. □

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