# Computability Theory on Polish Metric Spaces

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## ABSTRACT

Computability theoretic aspects of Polish metric spaces are studied by adapting notions and methods of computable structure theory. In this dissertation, we mainly investigate index sets and classification problems for computably presentable Polish metric spaces. We find the complexity of a number of index sets, isomorphism problems and embedding problems for computably presentable metric spaces. We also provide several computable structure theory results related to some classical Polish metric spaces such as the Urysohn space $\mathbb{U}$, the Cantor space $2^{\mathbb{N}}$, the Baire space $\mathbb{N}^{\mathbb{N}}$, and spaces of continuous functions.

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## APPROVAL PAGE

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# Computability Theory on Polish Metric Spaces

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# Contents

# Chapter 1

# Introduction

Computable analysis is a study of mathematical analysis from the perspective of computability theory. For a background in computable analysis, see, e.g. [1], [14] and [21]. In computability theory, we mainly study computational properties of sets and functions on the natural numbers, and by using codings, we can study other countable objects such as finite strings of natural numbers, integers, or rational numbers. However, most objects we study in analysis are uncountable objects, for example, we study the space $\mathbb{R}$ of real numbers, the spaces $C(X)$ of real-valued continuous functions on $X$, $L^p$-spaces, etc. Hence, in order to study algorithmic properties of these uncountable objects, we need some ways to represent them by countable objects. In 2013, Melnikov [12] proposed a way to adapt notions and methods of computable structure theory to Polish metric spaces. This paper introduces a notion of computable metric space, and contains several results and open problems about computable metric spaces that give us the motivation for this dissertation.

In this dissertation, we study computability theoretic aspects of Polish metric spaces. We focuses on index sets and classification problems for computable metric spaces. This dissertation is organized as follows. In Chapter 1, we provide some background and terminology in computability theory and computable metric spaces. In Chapter 2, we find the complexity of a number of basic index sets of computable metric spaces. In Chapter 3, we investigate the complexity of isomorphism problems and embedding problems for computable metric spaces in general and for finite metric spaces. We also consider embedding problems for some infinite metric spaces. In Chapter 4, we find the complexity of the index sets for perfect computable metric spaces and for discrete computable metric spaces. In Chapter 5, we study the Urysohn space $\mathbb{U}$ and bounded Urysohn spaces $\mathbb{U}_{\leq r}$ about their characterizations, computable presentations and index sets. In Chapter 6, we consider some embedding problems of the Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$. Finally, in Chapter 7, we gives a few results on the computable categoricity of the space $C(2^{\mathbb{N}})$ of continuous functions on the Cantor space.

## 1.1 Computability Theory

In this section, we provide some background and terminology in computablity theory. For more details, see, e.g. [16], [17] and [20].

Let $\mathbb{N}$ denote the set of all natural numbers, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

**Definition 1.1.1.**

- A *partial function* on $\mathbb{N}$ is a function $f : \mathbb{N} \to \mathbb{N}$ with $dom(f) \subseteq \mathbb{N}$.

If $x \in dom(f)$, we say that $f(x)$ *converges*, or *is defined*, and write $f(x)\downarrow$.

If $x \notin dom(f)$, we say that $f(x)$ *diverges*, or *is undefined*, and write $f(x)\uparrow$.

- A *total function* on $\mathbb{N}$ is a partial function $f : \mathbb{N} \to \mathbb{N}$ with $dom(f) = \mathbb{N}$.

**Definition 1.1.2.** For partial functions $f$ and $g$, we write $f = g$ if $dom(f) = dom(g)$ and $f(x) = g(x)$ for all $x \in dom(f)$.

**Definition 1.1.3.** A partial function $f : \mathbb{N} \to \mathbb{N}$ is *partial computable* if there is an effective procedure (or a computer program or a Turing machine) which takes a natural number $n$ as an input, and then either the procedure eventually halts with an output $f(n)$ or it never halts. We say $f$ is *computable* if $f$ is partial computable and $f$ is total. (For details on Turing machines and a more formal definition of partial computable functions, see, e.g. [16], [17] and [20].)

**Definition 1.1.4.** A set $A \subseteq \mathbb{N}$ is *computable* if its characteristic function $\chi_A$ is computable.

Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the standard (bijective) coding function (or pairing function) defined by

$$\langle x, y \rangle = \frac{(x+y)(x+y+1)}{2} + x.$$

By iteration, we can encode all tuples in $\mathbb{N}^k$ where $k \geq 1$ as natural numbers. For example, a tuple $(x, y, z) \in \mathbb{N}^3$ is coded by $\langle x, y, z \rangle := \langle \langle x, y \rangle, z \rangle$. So we can think of a function $f : \mathbb{N}^3 \to \mathbb{N}$ as the function from $\mathbb{N}$ into $\mathbb{N}$ that maps $\langle x, y, z \rangle$ to $f(x, y, z)$. Since integers and rationals can be represented by tuples of natural numbers, we can also encode them as natural numbers.

We can encode each partial computable function (or Turing machine) as a natural number $e$. This number $e$ is called an *index* of the function. Throughout this dissertation, we fix an effective enumeration of all partial computable functions:

$$\varphi_0, \varphi_1, \varphi_2, \ldots .$$

**Definition 1.1.5.** A set $A$ is *computably enumerable*, written *c.e.*, if $A = dom(\varphi_e)$ for some $e \in \mathbb{N}$. Equivalently, $A$ is c.e. if and only if $A = range(\varphi_e)$ for some $e \in \mathbb{N}$, that is, we can effectively enumerate all elements of $A$. For each $e \in \mathbb{N}$, let $W_e$ denote the $e$-th c.e. set, that is, $W_e := dom(\varphi_e)$.

A Turing machine can be equipped with an external database, called an **oracle**. During its computation, the Turing machine can ask the oracle finitely many questions to get extra information. For example, a Turing machine with a set $A \subseteq \mathbb{N}$ as the oracle can ask $A$ finitely many questions of the form "is $n$ in $A$?".

We can effectively list all Turing machines with oracle $A$ as

$$\Phi_0^A, \Phi_1^A, \Phi_2^A, \ldots .$$

We usually write $\Phi_e$ instead of $\Phi_e^{\emptyset}$. Then a function $f$ is partial computable if and only if $f = \Phi_e$ for some $e \in \omega$.

We can think of a Turing machine $\Phi_e$ as a functional that takes an oracle set $A$ as an input and gives the partial function $\Phi_e^A$ as the output. We called this $\Phi_e$ a *Turing functional*.

If $\Phi_e^A$ on input $x$ halts within $s$ steps of computation and gives an output $y$, then we write $\Phi_{e,s}^A(x) \downarrow= y$. If it does not halt within $s$ steps, we write $\Phi_{e,s}^A(x) \uparrow$.

We can *relativize* the notions for computable functions and computable sets to an oracle $A$ as follows.

**Definition 1.1.6.** Let $A \subseteq \mathbb{N}$.

- A partial function $f : \mathbb{N} \to \mathbb{N}$ is *partial $A$-computable* (or *partial computable relative to $A$* or *Turing computable in $A$*), written $f \leq_T A$, if $f = \Phi_e^A$ for some $e \in \mathbb{N}$. If $f$ is also total, then we say $f$ is *$A$-computable*.
- A set $B$ is *$A$-computable* (or *computable relative to $A$* or *Turing reducible to $A$*), written $B \leq_T A$, if $\chi_B$ is $A$-computable.

Other notions can be relativized in the same way.

**Definition 1.1.7.**

- We say $A$ and $B$ are *Turing equivalent*, written $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$. Note that $\leq_T$ is reflexive and transitive. So $\equiv_T$ is an equivalence relation.
- The *Turing degree* of $A$ is the equivalence class $deg(A) := \{B : B \equiv_T A\}$.

**Definition 1.1.8.** The *Halting set*, denoted by $0'$, is the set $\{e \in \mathbb{N} : \varphi_e(e) \downarrow\}$. For each $A \subseteq \mathbb{N}$, the *halting set relative to $A$*, denoted by $A'$, is the set $\{e \in \mathbb{N} : \Phi_e^A(e) \downarrow\}$. For each $n \in \mathbb{N}$, we define $0^{(n)}$ inductively by

- $0^{(0)} := \emptyset$,
- $0^{(n+1)} := (0^{(n)})'$.

**Definition 1.1.9.** A set $A$ is *many-one reducible* to a set $B$, written $A \leq_m B$, if there is a computable function $f$ such that for all $n \in \mathbb{N}$,

$$n \in A \Longleftrightarrow f(n) \in B.$$

**Definition 1.1.10.** Let $R(x; X)$ be a relation where $x$ is a number variable (ranging over natural numbers) and $X$ is a set variable (ranging over subsets of $\mathbb{N}$). A relation $R(x; X)$ is *computable* if there is an $e \in \mathbb{N}$ such that for all $x \in \mathbb{N}$ and $X \subseteq \mathbb{N}$,

$$\Phi_e^X(x) = \begin{cases} 1 & \text{if } R(x; X) \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

**Definition 1.1.11.** Let $n \geq 1$. We define *complexity classes* $\Sigma_n^0$, $\Pi_n^0$, $\Delta_n^0$, $d$-$\Sigma_n^0$, $\Sigma_1^1$, $\Pi_1^1$, and $\Delta_1^1$ as follows.

- $A \subseteq \mathbb{N}$ is $\Sigma_n^0$ if there is a computable relation $R(x, y_1, \ldots, y_n)$ such that for all $x, y_1, \ldots, y_n \in \mathbb{N}$,

  $$x \in A \Longleftrightarrow \exists y_1 \forall y_2 \exists y_3 \ldots Q y_n R(x, y_1, \ldots, y_n),$$

  where $Q$ is $\exists$ if $n$ is odd, and $Q$ is $\forall$ if $n$ is even.

- $A \subseteq \mathbb{N}$ is $\Pi_n^0$ if $\mathbb{N} \backslash A$ is $\Sigma_n^0$, that is, there is a computable relation $R(x, y_1, \ldots, y_n)$ such that for all $x, y_1, \ldots, y_n \in \mathbb{N}$,

  $$x \in A \Longleftrightarrow \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n R(x, y_1, \ldots, y_n),$$

  where $Q$ is $\forall$ if $n$ is odd, and $Q$ is $\exists$ if $n$ is even.

- $A$ is $\Delta^0_n$ if $A$ is both $\Sigma^0_n$ and $\Pi^0_n$.
- $A$ is *d*-$\Sigma^0_n$ if $A = B \setminus C$ for some $\Sigma^0_n$ sets $B$ and $C$.
- $A$ is *arithmetical* if $A$ is $\Sigma^0_n$ for some $n \in \mathbb{N}$.
- $A$ is $\Sigma^1_1$ if there is an arithmetical relation $R(x; X)$, where $X$ is a set variable, such that for all $x \in \mathbb{N}$,

$$x \in A \iff (\exists X \subseteq \mathbb{N}) R(x; X).$$

- $A$ is $\Pi^1_1$ if $\mathbb{N} \setminus A$ is $\Sigma^1_1$.
- $A$ is $\Delta^1_1$ if $A$ is both $\Sigma^1_1$ and $\Pi^1_1$.

**Definition 1.1.12.** Let $\Gamma$ be a complexity class.

- $A$ is $\Gamma$-*hard* if $B \leq_m A$ for all $\Gamma$ sets $B$.
- $A$ is $\Gamma$-*complete* if $A$ is $\Gamma$ and $A$ is $\Gamma$-hard.

**Definition 1.1.13.** A set $A \subseteq \mathbb{N}$ is an *index set* if for every $e, e' \in \mathbb{N}$, if $e \in A$ and $\varphi_e = \varphi_{e'}$, then $e' \in A$.

**Example 1.1.14.** The following sets are index sets.

- $0^{(n)}$ is $\Sigma^0_n$-complete for all $n \geq 1$.
- $Tot := \{e \in \mathbb{N} : \varphi_e \text{ is total}\}$ is $\Pi^0_2$-complete.
- $Inf := \{e \in \mathbb{N} : dom(\varphi_e) \text{ is infinite}\}$ is $\Pi^0_2$-complete.
- $Fin := \{e \in \mathbb{N} : dom(\varphi_e) \text{ is finite}\}$ is $\Sigma^0_2$-complete.

Let $I$ be a set and $A \subseteq I$. In some cases, given that we know $e \in I$, the problem of deciding if $e \in A$ might be simpler than detecting whether $e$ belongs to the set $I$. In these cases, we usually consider the complexity of index sets "within" the set $I$. This leads to the following definition.

**Definition 1.1.15.** Let $\Gamma$ be a complexity class, $I$ be a set and $A \subseteq I$. We say that

(1) $A$ is $\Gamma$ *within* $I$ if there exists a $B \in \Gamma$ such that $A = B \cap I$.

(2) $A$ is $\Gamma$-*hard within* $I$ if for every $B \in \Gamma$, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) \in I$ and $(n \in B \iff f(n) \in A)$.

(3) $A$ is $\Gamma$-*complete within* $I$ if $A$ is $\Gamma$ within $I$ and $A$ is $\Gamma$-hard within $I$.

**Remark 1.1.16.**

(1) *$A$ is $\Gamma \implies A$ is $\Gamma$ within $I$.*

(2) *$A$ is $\Gamma$-hard within $I \implies A$ is $\Gamma$-hard.*

The following theorem will be used throughout this dissertation when we work with index sets.

**Theorem 1.1.17.** **($s$-$m$-$n$ Theorem)** *For every $n, m \geq 1$, there is a computable injective function $S_n^m : \mathbb{N}^{n+1} \to \mathbb{N}$ such that for all $e \in \mathbb{N}$, $\overline{x} \in \mathbb{N}^n$ and $\overline{y} \in \mathbb{N}^m$,*

$$\Phi_{S_n^m(e,\overline{x})}(\overline{y}) = \Phi_e(\overline{x}, \overline{y}).$$

Next, we gives some standard notations for strings of natural numbers.

Let $2^{<\mathbb{N}}$ denote the set of all finite binary strings, and let $\mathbb{N}^{<\mathbb{N}}$ denote the set of all strings of natural numbers. We can think of a finite string of natural numbers as a function from a finite initial segment of $\mathbb{N}$ into $\mathbb{N}$. For example, if $\sigma$ is the string $(1, 0, 4)$, then we view $\sigma$ as the function $\sigma : \{0, 1, 2\} \to \mathbb{N}$ where $\sigma(0) = 1, \sigma(1) = 0$ and $\sigma(4) = 4$. We usually use $\sigma, \tau, \rho, \ldots$ to range over finite strings of natural numbers.

- Let $\lambda$ denote the *empty string*, that is, $\lambda = \emptyset$.
- Let $|\sigma|$ denote the length of the string $\sigma$, that is, $|\sigma| := |dom(\sigma)|$. For example, $|(1, 0, 1)| = 3$.
- For each $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$, we let $\sigma^\frown\tau$ denote the *concatenation* of $\sigma$ and $\tau$, that is, the string obtained from joining $\tau$ at the end of $\sigma$. For example, $(0, 1, 5)^\frown(2, 3) = (0, 1, 5, 2, 3)$. For $i \in \mathbb{N}$, we simply write $\sigma i$ or $\sigma^\frown i$ instead of $\sigma^\frown(i)$.
- $\sigma$ is an *initial segment* of $\tau$, written $\sigma \subseteq \tau$, if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for all $i < |\sigma|$. $\sigma$ is a *proper initial segment* of $\tau$, written $\sigma \subsetneq \tau$, if $\sigma \subseteq \tau$ and $\sigma \neq \tau$.
- We identify a natural number $n$ with the set $\{0, 1, \ldots, n - 1\}$.
- We identify a set $A \subseteq \mathbb{N}$ with its characteristic function $\chi_A$, that is, $A(n) = 1$ if $n \in A$, and $A(n) = 0$ if $n \notin A$. So we can think of $A$ as an infinite binary string.
- For $A \subseteq \mathbb{N}$, we write $\sigma \subseteq A$ if $\sigma$ is an initial segment of $A$, that is, $\sigma(i) = A(i)$ for all $i < |\sigma|$. Similarly, for $f : \mathbb{N} \to \mathbb{N}$, we write $\sigma \subseteq f$ if $\sigma$ is an initial segment of $f$.

For $A, B \subseteq \mathbb{N}$, we let $B^A$ denote the set of all functions from $A$ into $B$. So we can think of $2^{\mathbb{N}}$ as the set of all infinite binary strings, $\mathbb{N}^{\mathbb{N}}$ as the set of all infinite strings of natural numbers, and $2^n$, where $n \in \mathbb{N}$, as the set of all finite binary strings of length $n$. We can also think of $2^{\mathbb{N}}$ as the power set $\mathcal{P}(\mathbb{N}) := \{X : X \subseteq \mathbb{N}\}$.

**Definition 1.1.18.**

- A *tree* is a subset $T$ of $\mathbb{N}^{<\mathbb{N}}$ that is closed under initial segments, that is, if $\sigma \in T$ and $\tau \subseteq \sigma$, then $\tau \in T$. So every nonempty tree $T$ contains the empty string $\lambda$. We call $\lambda$ the *root* of $T$.
- A *binary tree* is a tree $T$ such that $T \subseteq 2^{<\mathbb{N}}$.
- An *(infinite) path* through a tree $T$ is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f \upharpoonright n \in T$. So we can think of a path through a binary tree as a set of natural numbers. We let $[T]$ denote the set of all infinite paths through $T$.

Note that, by coding finite strings as natural numbers, we can think of a tree $T$ as a set of natural numbers.

The following fact can be used to show $\Sigma_1^1$(or $\Pi_1^1$)-hardness of an index set. For the definition of primitive recursive trees, see, e.g. [16], [17] and [20].

**Fact 1.1.19** (see [15])**.**

- *There is a computable sequence of all primitive recursive trees.*
- *If $(T_e)_{e\in\mathbb{N}}$ is a computable sequence of all primitive recursive trees, then the set $\{e \in \mathbb{N} : T_e$ has an infinite path$\}$ is $\Sigma_1^1$-complete.*

**Definition 1.1.20.**

- A *computable sequence of rationals* is a sequence $(r_n)_{n\in\mathbb{N}}$ of rationals such that there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) = r_n$, that is, $f(n)$ is the code of the rational $r_n$.
- A real number $r$ is *computable* if there is a computable sequence $(r_n)_{n\in\mathbb{N}}$ of

rationals such that for all $n \in \mathbb{N}$,

$$|r - r_n| \leq 2^{-n}.$$

That is, $r_n$ is a rational approximation within $2^{-n}$ of $r$.

- A real number $r$ is *left-c.e.* if the set $\{q \in \mathbb{Q} : q < r\}$ is c.e. Equivalently, $r$ is *left-c.e.* if and only if there is a computable increasing (or strictly increasing) sequence $(r_n)_{n\in\mathbb{N}}$ of rationals such that $\lim_{n\to\infty} r_n = r$. We define *right-c.e.* reals similarly.

## 1.2 Computable Polish Metric Spaces

First, we review the definitions of pseudometric spaces and metric spaces.

**Definition 1.2.1.** A *pseudometric* on a set $X$ is a function $d : X \times X \to \mathbb{R}$ such that for every $x, y, z \in X$,

(1) $d(x, x) = 0$

(2) $d(x, y) = d(y, x)$ (symmetry)

(3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Note that (1)-(3) imply that $d(x, y) \geq 0$ for all $x, y \in X$.

A *pseudometric space* is a pair $(X, d)$ where $X$ is a set and $d$ is a pseudometric on $X$.

Note that, in a pseudometric space, it is possible that $d(x, y) = 0$ but $x \neq y$.

**Definition 1.2.2.** A *metric* on a set $X$ is a pseudometric $d$ on $X$ such that for every $x, y \in X$, $d(x, y) = 0 \Longrightarrow x = y$. A *metric space* is a pair $(X, d)$ where $X$ is a set and $d$ is a metric on $X$.

**Definition 1.2.3.** A metric space $(X, d)$ is called a *rational metric space* if $d(x, y) \in \mathbb{Q}$ for all $x, y \in X$.

For a metric space $(X, d)$, we let $diam(X)$ denote the *diameter* of $X$, that is,

$$diam(X) := \sup\{d(x, y) : x, y \in X\} \in [0, \infty].$$

For $Y \subseteq X$, we let $cl(Y)$ denote the closure of $Y$ in $X$, that is, $z \in cl(Y)$ if and only if there is a sequence $(y_n)_{n \in \mathbb{N}}$ in $Y$ that converges to $z$ in $X$.

**Example 1.2.4.** The following are metric spaces.

- The one-point metric space $\{x\}$.
- The space $(\mathbb{N}, d_{\mathbb{N}})$ where $d_{\mathbb{N}}(m, n) := |m - n|$ is the standard metric on $\mathbb{N}$.
- For any vector $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we define

  (1) *Euclidean metric*: $d_{euclid}(\vec{x}, \vec{y}) := \sqrt{\sum_{1 \leq i \leq n} (x_i - y_i)^2}$,

  (2) *Maximum metric*: $d_{max}(\vec{x}, \vec{y}) := \max_{1 \leq i \leq n} |x_i - y_i|$,

  (3) *Taxicab metric*: $d_{taxi}(\vec{x}, \vec{y}) := \sum_{1 \leq i \leq n} |x_i - y_i|$.

  Then $d_{euclid}, d_{max}$ and $d_{taxi}$ are metrics on $\mathbb{R}^n$.
- Let $G = (V, E)$ be a connected undirected (possibly weighted) graph, where $V$ is the set of all vertices in $G$, and $E$ is the set of all edges in $G$. For any

$u, v \in V$, we define the distance $d_G(u, v)$ to be the length of a shortest path from $u$ to $v$ in $G$. Then $d_G$ is a metric, and we call $d_G$ the *shortest path metric* on $G$.

**Definition 1.2.5.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.

(1) A function $f : X \to Y$ is called a *distance-preserving* function or an *isometric embedding*, written $f : X \hookrightarrow Y$, if for every $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

(2) An *isometry* is a bijective distance-preserving function.

(3) We say that $X$ *embeds isometrically* into $Y$, written $X \hookrightarrow Y$, if there is an isometric embedding $f : X \hookrightarrow Y$.

(4) We say that $X$ is *isometric* to $Y$, written $X \cong Y$, if there is an isometry $f : X \to Y$.

Note that distance-preserving functions are injective and continuous, and the relation $\cong$ is an equivalence relation.

**Definition 1.2.6.** A *Polish space* is a topological space that is homeomorphic to a complete separable metric space. A *Polish metric space* is a complete separable metric space.

To study computability theory on Polish metric spaces, we first need to find an effective way to represent these spaces. We will use terminology from [11].

**Definition 1.2.7.** A *computable presentation* of (or a *computable structure* on) a

Polish metric space $(M, d)$ is any dense sequence $(p_i)_{i\in\mathbb{N}}$ of points in $M$ such that the distance $d(p_i, p_j)$ is a computable real number uniformly in $i, j$. That is, there exists a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that for all $i, j, k \in \mathbb{N}$,

$$|f(\langle i, j, k\rangle) - d(p_i, p_j)| \leq 2^{-k}.$$

Equivalently, there is an algorithm such that if we input two indices $i$, $j$ and a positive rational $\varepsilon$, then it will output a rational number that approximates the distance between $p_i$ and $p_j$ with error less than $\varepsilon$.

**Definition 1.2.8.** A metric space is *computably presentable* if it has a computable presentation.

**Definition 1.2.9.** A *computable (Polish) metric space* is a pair $((M, d), (p_i)_{i\in\mathbb{N}})$ where $(M, d)$ is a Polish metric space and $(p_i)_{i\in\mathbb{N}}$ is a computable presentation of $(M, d)$. The points in the sequence $(p_i)_{i\in\mathbb{N}}$ are called the *rational points*.

Since the rational points are dense in $M$, every point $x \in M$ is a limit of a sequence $(p_i)_{i\in\mathbb{N}}$ of rational points, so we might use $(p_i)_{i\in\mathbb{N}}$ as an approximation of the point $x$. However, we do not know the rate of convergence of $(p_i)_{i\in\mathbb{N}}$. If $(p_i)_{i\in\mathbb{N}}$ converges to $x$ very slowly, then it would be a bad approximation. So, in order to get a good approximation, we want the sequence to converge fast enough, and this leads to the notion of Cauchy name.

**Definition 1.2.10.** Let $(p_i)_{i\in\mathbb{N}}$ be a computable presentation of a metric space $(M, d)$ and $x \in M$. A *Cauchy name* of $x$ in $(p_i)_{i\in\mathbb{N}}$ is a function $f : \mathbb{N} \to \mathbb{N}$ such that $(p_{f(k)})_{k\in\mathbb{N}}$

converges to $x$, and for all $k \in \mathbb{N}$ and $l > k$,

$$d(p_{f(k)}, p_{f(l)}) \leq 2^{-k}.$$

This implies that $d(p_{f(k)}, x) \leq 2^{-k}$ for all $k \in \mathbb{N}$.

We can think of a Cauchy name of $x$ as a Cauchy sequence $(p_{f(k)})_{k\in\mathbb{N}}$ of rational points that converges to $x$ rapidly in the sense that it satisfies the above inequalities. So we view a Cauchy name of $x$ as a good approximation of $x$. Note that, since $(p_i)_{i\in\mathbb{N}}$ is dense in $M$, every element $x$ in $M$ has a Cauchy name.

**Definition 1.2.11.** An element $x$ of $M$ is *computable* with respect to $(p_i)_{i\in\mathbb{N}}$ (written w.r.t. $(p_i)_{i\in\mathbb{N}}$) if it has a computable Cauchy name in $(p_i)_{i\in\mathbb{N}}$. Equivalently, $x$ is computable if and only if there is an algorithm such that, given a positive rational $\varepsilon$, it computes a rational point $p_i$ that is $\varepsilon$-close to $x$. So we can effectively approximate this element $x$ in this sense.

Note that every rational point $p_i$ has the constant sequence $(p_i, p_i, \dots)$ as a computable Cauchy name. So every rational point is computable.

Next, we give some examples of computable metric spaces. Recall that a computable metric space is a Polish metric space together with a dense sequence of points whose distances are uniformly computable reals.

**Example 1.2.12.**

- The space $(\mathbb{N}, d_{\mathbb{N}})$ with the usual metric $d_{\mathbb{N}}(n, m) := |n - m|$, where $(i)_{i\in\mathbb{N}}$ is a computable presentation.

- The space $\mathbb{R}$ with the Euclidean metric, where an effective list $(q_i)_{i\in\mathbb{N}}$ of all rationals is a computable presentation.

- The space $C[0,1]$ of real-valued continuous functions on $[0,1]$ with the supremum metric:

$$d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|,$$

  where an effective list $(p_i)_{i\in\mathbb{N}}$ of all rational polynomials is a computable presentation.

- The Cantor space $2^{\mathbb{N}}$ equipped with the metric

$$d(X,Y) := 2^{-\min\{n: X(n)\neq Y(n)\}},$$

  where an effective list $(\sigma_i^\frown 0^{\mathbb{N}})_{i\in\mathbb{N}}$ of all infinite binary strings that are eventually 0 is a computable presentation.

From the above examples, we know that $\mathbb{R}$ and $C[0,1]$ are computably presentable. In fact, many other classical metric spaces are computably presentable. For example, if $p \geq 1$ is a computable real, then every separable $L^p$ space is computably presentable.

There are also many metric spaces that are not computably presentable. For example, a two-point metric space $M = \{x, y\}$ is computably presentable if and only if $d(x,y)$ is a computable real. So we can study computable presentability of metric spaces. That is, for a metric space, we determine if it is computably presentable.

We already have the definition for an element $x$ of a Polish metric space to be computable. But what about a function between Polish metric spaces? What does it mean for a function between computable metric spaces to be computable? We usually think of a computable function $f$ as an algorithm such that, given an input $x$ it outputs the value of $f(x)$. We use this as the definition of computable functions $f : \mathbb{N} \to \mathbb{N}$. However, a point in a metric space usually requires an infinite amount of information to be specified, so we cannot use the exact value of a point as an input or an output for algorithm. What we can do is, instead of exact specifications of points, we use Cauchy names as approximations.

**Definition 1.2.13.** Let $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ be computable presentations of metric spaces $\mathcal{M}$ and $\mathcal{N}$, respectively. A map $F : \mathcal{M} \to \mathcal{N}$ is *computable* with respect to $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ (written w.r.t. $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$) if there is a Turing functional $\Phi$ such that, for every $x \in \mathcal{M}$ and Cauchy name $f$ of $x$ in $(p_i)_{i\in\mathbb{N}}$, the functional $\Phi$ with oracle $f$ is a Cauchy name of $F(x)$ in $(q_i)_{i\in\mathbb{N}}$.

It turns out that two computable presentations of a metric space might not have the same computational power even though they are presentations of the same metric space. This leads to the following definition.

**Definition 1.2.14.** Computable presentations $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ of a metric space $(M, d)$ are said to be *equivalent up to computable isometry* or *computably isometric*, if there exists a (surjective) self-isometry $U$ on $(M, d)$ that is computable with respect to $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$.

If two computable presentations are computably isometric, then they have the same computational power, that is, anything that can be done computably in one presentation can also be done in the other presentation. For example, on the space

$C[0,1]$, the list of all rational polynomials and the list of all rational piecewise linear functions are two computable presentations that are computably isometric.

**Definition 1.2.15.** A computably presentable metric space $(M, d)$ is *computably categorical* if every two computable presentations of $(M, d)$ are computably isometric. That is, it has a unique computable presentation up to computable isometry.

If a metric space is not computably categorical, we might want to know how much extra computational power we need to build an isometry between two computable presentations. So we relativize the definition as follows.

**Definition 1.2.16.** Let $\mathcal{M}$ be a Polish metric space and $\mathbf{d}$ be a Turing degree. We say $\mathcal{M}$ is $\mathbf{d}$-*categorical* if for any two computable presentations $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ of $\mathcal{M}$, there is an isometry $F : \mathcal{M} \to \mathcal{M}$ that is $\mathbf{d}$-computable with respect to $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$.

The *degree of categoricity* of $\mathcal{M}$ is the least Turing degree $\mathbf{d}$ such that $\mathcal{M}$ is $\mathbf{d}$-categorical. Note that the degree of categoricity may not exist.

The degree of categoricity of $\mathcal{M}$ tells us how much computational power we need (sufficient and necessary) to compute an isometry between any two computable presentations of $\mathcal{M}$. This leads to the study of computable categoricity and degree of categoricity of Polish metric spaces. For example, Pour-El and Richards [14] showed that every separable $L^2$ space is computably categorical. Moreover, McNicholl [9] showed that if $p \geq 1$ is computable and $p \neq 2$, then the degree of categoricity of the space $l^p$ is $\mathbf{0}'$.

## 1.3 Computable Metric Spaces with Operations

We use terminology from [11]. An operation $X$ on a metric space $(M,d)$ is a function which maps tuples of points to points (i.e. $X : M^k \to M$), or tuples of points to reals (i.e. $X : M^k \to \mathbb{R}$). A point $x$ in $M$ can be viewed as an operation $T_x : M \to M$ where $T_x(y) = x$ for all $y \in M$.

We view a direct power $M^k$ of $(M,d)$ as a metric space with the metric $d_{M^k}(x,y) := \sup_{i \leq k} d(\pi_i(x), \pi_i(y))$, where $\pi_i$ is the projection on the $i$-th component. Let $(p_i)_{i\in\mathbb{N}}$ be a computable presentation of $(M,d)$. The computable presentation $[(p_i)_{i\in\mathbb{N}}]^k$ of $(M^k, d_{M^k})$ is the effective list of $k$-tuples of rational points from $(p_i)_{i\in\mathbb{N}}$.

For convenience, if an operation $X : M^k \to M$ is computable w.r.t. $[(p_i)_{i\in\mathbb{N}}]^k$ and $(p_i)_{i\in\mathbb{N}}$, we simply say that $X$ is computable w.r.t. $(p_i)_{i\in\mathbb{N}}$. Similarly, if an operation $X : M^k \to \mathbb{R}$ is computable w.r.t. $[(p_i)_{i\in\mathbb{N}}]^k$ and $(q_i)_{i\in\mathbb{N}}$, where $(q_i)_{i\in\mathbb{N}}$ is the usual effective list of rationals, then we say that $X$ is computable w.r.t. $(p_i)_{i\in\mathbb{N}}$.

Since every Turing functional $\Phi_e$ can be effectively identified with its computable index $e$, we can speak of uniformly computable families of maps betweens computable metric spaces.

**Definition 1.3.1.** Let $(M, d, (X_j)_{j\in J})$ be a Polish metric space with distinguished operations $(X_j)_{j\in J}$, where $J$ is a computable set. A sequence $(p_i)_{i\in\mathbb{N}}$ is a *computable presentation of* (or a *computable structure* on) $(M, d, (X_j)_{j\in J})$ if $(M, d, (p_i)_{i\in\mathbb{N}})$ is a computable metric space and the operations $(X_j)_{j\in J}$ are computable w.r.t. $(p_i)_{i\in\mathbb{N}}$ uniformly in their respective indices $j \in J$. We say that the space $(M, d, (X_j)_{j\in J})$ is *computably presentable* if it has a computable presentation.

**Definition 1.3.2.** Let $T : M \to M$ be an operation. We say that $T$ *respects* an operation $X : M^k \to M$ if $T$ commutes with $X$ (i.e. $X \circ T = T \circ X$). We say that $T$ *respects* an operation $X : M^k \to \mathbb{R}$ if $T$ preserves the output of $X$ (i.e. $X \circ T = X$).

**Definition 1.3.3.** A computably presentable space $(M, d, (X_j)_{j \in J})$ is *computably categorical* if every two computable presentations $(p_i)_{i \in \mathbb{N}}$ and $(q_i)_{i \in \mathbb{N}}$ of $(M, d, (X_j)_{j \in J})$ are computably isometric via an isometry which respects $X_j$ for every $j \in J$.

**Definition 1.3.4.** We say that operations $(Y_i)_{i \in I}$ *effectively determine* operations $(X_j)_{j \in J}$ on a metric space $(M, d)$ if

(1) every isometry of $M$ that respects $(Y_i)_{i \in I}$ respects $(X_j)_{j \in J}$ as well,

(2) for any computable presentation $(p_i)_{i \in \mathbb{N}}$ of $(M, d)$, the uniform computablity of $(Y_i)_{i \in I}$ w.r.t $(p_i)_{i \in \mathbb{N}}$ implies the uniform computability of $(X_j)_{j \in J}$ w.r.t. $(p_i)_{i \in \mathbb{N}}$.

The following fact immediately follows from Definition 1.3.3 and Definition 1.3.4.

**Fact 1.3.5.** *If $(M, d, (X_j)_{j \in J}, (Y_i)_{i \in I})$ is computably categorical and $(Y_i)_{i \in I}$ effectively determine $(X_j)_{j \in J}$, then $(M, d, (Y_i)_{i \in I})$ is computably categorical.*

## 1.4 Computable Indices of Computable Metric Spaces

Recall that $(\varphi_e)_{e \in \mathbb{N}}$ is a fixed effective list of all partial computable functions. By coding, we can think of $(\varphi_e)_{e \in \mathbb{N}}$ as an effective list of all rational-valued partial computable functions.

**Definition 1.4.1.** If $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is a pseudometric on $\mathbb{N}$ and $\varphi$ is a rational-valued partial computable function such that $\varphi$ is total and $\varphi$ converges rapidly to $d$ in the sense that

(1) $\lim_{k\to\infty} \varphi(i,j,k) = d(i,j)$ for every $i, j \in \mathbb{N}$,

(2) $|\varphi(i,j,k) - \varphi(i,j,k+l)| \le 2^{-k}$ for every $i, j, k, l \in \mathbb{N}$,

then we say $d$ is the *pseudometric induced by* $\varphi$. If $\varphi_e$ induces a pseudometric, we let $d_e$ denote this pseudometric.

For a pseudometric $d$ on $\mathbb{N}$, we let $M_d$ denote the completion of the pseudometric space $(\mathbb{N}, d)$, where we identify every two points $i, j \in \mathbb{N}$ with $d(i,j) = 0$ as the same point in $M_d$, so that $M_d$ is a Polish metric space. We will write $M_e$ as a shorthand for $M_{d_e}$. So $M_e$ is the Polish metric space induced by $\varphi_e$. The natural number $e$ is called an *index* of $M_e$.

Note that $(M_e)_{e\in\mathbb{N}}$ is an effective list containing all computable metric spaces up to isometry. So a Polish metric space $M$ is computably presentable if and only if $M \cong M_e$ for some $e \in \mathbb{N}$.

We define an index set

$$PolSp := \{e \in \mathbb{N} : \varphi_e \text{ induces a pseudometric}\} = \{e \in \mathbb{N} : M_e \text{ is a Polish metric space}\}.$$

Then $PolSp$ is the index set of all computable Polish metric spaces.

Note that if $d$ is the pseudometric induced by $\varphi$, then for every $i, j, k \in \mathbb{N}$,

$$|d(i,j) - \varphi(i,j,k)| \le 2^{-k}.$$

So $\varphi(i,j,k)$ is a rational approximation within $2^{-k}$ of the real number $d(i,j)$.

Note that if $x(k)$ and $y(k)$ are rational approximations within $2^{-k}$ of real numbers $x$ and $y$, respectively, then

$$x = y \iff \forall k(|x(k) - y(k)| \leq 2^{-k+1})$$

$$x \neq y \iff \exists k(|x(k) - y(k)| > 2^{-k+1})$$

$$x \leq y \iff \forall k(x(k) \leq y(k) + 2^{-k+1})$$

$$x < y \iff \exists k(x(k) + 2^{-k+1} < y(k))$$

Hence, for computable reals $x$ and $y$, the statement "$x = y$" is $\Pi^0_1$, "$x \neq y$" is $\Sigma^0_1$, "$x \leq y$" is $\Pi^0_1$, and "$x < y$" is $\Sigma^0_1$. We can use this observation to find the complexity of conditions involving the pseudometric $d_e$ induced by $\varphi_e$.

# Chapter 2

# Basic Index Set Results

Let $\mathcal{K}$ be a class of Polish metric spaces. The complexity of the classification problem for computable members of $\mathcal{K}$ is measured using the following two index sets:

(1) The *characterization problem* (or the *index set*) of $\mathcal{K}$ is the set

$$\{e \in \mathbb{N} : M_e \in \mathcal{K}\}.$$

(2) The *isomorphism problem* of $\mathcal{K}$ is the set

$$\{(i, j) \in \mathbb{N}^2 : M_i, M_j \in \mathcal{K} \text{ and } M_i \cong M_j\}.$$

For a class $\mathcal{K}$, we can study the complexity of the index set of $\mathcal{K}$ and the isomorphism problem of $\mathcal{K}$ (in arithmetical hierarchy, hyperarithmetical, and analytical hierarchy). The complexity of the isomorphism problem is the complexity of the associated classification problem.

Suppose we have a set $A \subseteq \mathbb{N}$ that is $\Gamma$-hard where $\Gamma$ is a complexity class. Let $P$ be a property of Polish metric spaces that is preserved under isometry. To show that an index set of the form $I := \{e \in \mathbb{N} : M_e \text{ has property } P\}$ is $\Gamma$-hard within $PolSp$, by the *s-m-n* Theorem, it is enough to build a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$, $e \in A \iff X_e$ has property $P$.

To see why this is true, assume we have such a computable sequence $(X_e)_{e\in\mathbb{N}}$. Then, by the *s-m-n* Theorem, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $e \in \mathbb{N}$, $\varphi_{f(e)}$ induces $X_e$, and so $e \in PolSp$ and $M_{f(e)} \cong X_e$. Then for every $e \in \mathbb{N}$,

$$e \in A \iff X_e \text{ has property } P \iff M_{f(e)} \text{ has property } P \iff f(e) \in I.$$

So $A$ is many-one reducible to $I$. Therefore, since $A$ is $\Gamma$-hard and $X_e$ is a computable metric space for all $e \in \mathbb{N}$, we can conclude that $I$ is $\Gamma$-hard within $PolSp$.

First, we compute the complexity of some basic index sets.

**Theorem 2.0.1.** *$PolSp$ is $\Pi^0_2$-complete.*

*Proof.* Note that for any $e \in \mathbb{N}$, $\varphi_e$ induces a pseudometric if and only if the following conditions hold:

(1) $\varphi_e$ is total, (i.e. $(\forall i, j, k \in \mathbb{N})(\exists s \in \mathbb{N})(\varphi_{e,s}(i, j, k)\downarrow)$),

(2) $|\varphi_e(i, j, k) - \varphi_e(i, j, k + l)| \leq 2^{-k}$ for all $i, j, k, l \in \mathbb{N}$,

(2) guarantees that $\lim_{k\to\infty} \varphi_e(i,j,k)$ exists for all $i,j \in \mathbb{N}$, so we can define $d_e(i,j) := \lim_{k\to\infty} \varphi_e(i,j,k)$.

(3) $d_e(i,j) \geq 0$ for all $i,j \in \mathbb{N}$,

(4) $d_e(i,i) = 0$ for all $i \in \mathbb{N}$,

(5) $d_e(i,j) = d_e(j,i)$ for all $i,j \in \mathbb{N}$,

(6) $d_e(i,j) \leq d_e(i,k) + d_e(k,j)$ for all $i,j,k \in \mathbb{N}$.

Now, since $\varphi_e(i,j,t)$ is a rational approximation within $2^{-t}$ of $d_e(i,j)$, it is easy to see that the condition "$\varphi_e$ induces a pseudometric" is $\Pi^0_2$. That is, $PolSp$ is $\Pi^0_2$.

More precisely, condition (1) is a $\Pi^0_2$ statement and conditions (2)-(6) are $\Pi^0_1$ statements. For example, assuming $\varphi_e$ is total, condition (5) is equivalent to $(\forall i,j \in \mathbb{N})(\forall t \in \mathbb{N})(|\varphi_e(i,j,t) - \varphi_e(j,i,t)| \leq 2^{-t+1})$, and so (5) is a $\Pi^0_1$ statement (assuming $\varphi_e$ is total).

Next, we show that $PolSp$ is $\Pi^0_2$-hard. Recall that $Tot := \{e \in \mathbb{N} : \varphi_e \text{ is total}\}$ is $\Pi^0_2$-complete.

For each $e \in \mathbb{N}$, we construct a partial computable function $\psi_e$ uniformly in $e$ as follows.

By dovetailing, we compute $\varphi_e(\langle i,j\rangle)$ for all $i,j \in \mathbb{N}$. Whenever we see that $\varphi_e(\langle i,j\rangle)\downarrow$, we define $\psi_e(i,j,k) = 0$ for all $k \in \mathbb{N}$.

This ends the construction.

Since the construction of $\psi_e$ is effective uniformly in $e$, $\psi_e$ is partial computable uniformly in $e$. Thus, by the *s-m-n* Theorem, there is a computable function $f$ such that $\varphi_{f(e)} = \psi_e$ for all $e \in \mathbb{N}$.

If $e \in Tot$, then $\varphi_{f(e)}(i,j,k) = \psi_e(i,j,k) = 0$ for all $i,j,k \in \mathbb{N}$. So it is clear from the definition that $\varphi_{f(e)}$ induces a pseudometric, that is, $f(e) \in PolSp$. In fact, the pseudometric $d_e$ induced by $\varphi_{f(e)}$ is a metric such that $d_e(i,j) = 0$ for all $i,j \in \mathbb{N}$.

If $e \notin Tot$, then there are $i,j \in \mathbb{N}$ such that $\varphi_e(\langle i,j\rangle)\uparrow$. So we will never define $\psi_e(i,j,k)$ for all $k \in \mathbb{N}$, that is, $\psi_e(i,j,k)\uparrow$ for all $k \in \mathbb{N}$. Hence $\varphi_{f(e)} = \psi_e$ is not total. Therefore, $\varphi_{f(e)}$ does not induce a pseudometric, that is, $f(e) \notin PolSp$.

We conclude that for all $e \in \mathbb{N}$, $e \in Tot \iff f(e) \in PolSp$. So $Tot$ is many-one reducible to $PolSp$. Therefore, since $Tot$ is $\Pi^0_2$-hard, $PolSp$ is $\Pi^0_2$-hard. $\square$

In the proof for $\Pi^0_2$-hardness of $PolSp$, we only use the fact that $PolSp \subseteq Tot$ and $Tot$ is $\Pi^0_2$-hard. The proof can be modified to obtain the following stronger result:

**Theorem 2.0.2.** *For any nonempty class $\mathcal{K}$ of computable Polish metric spaces, the index set $\{e \in \mathbb{N} : M_e \in \mathcal{K}\}$ is $\Pi^0_2$-hard.*

*Proof.* Let $\mathcal{K}$ be a nonempty class of computable Polish metric spaces. Then there is an $e_0 \in \mathbb{N}$ such that $M_{e_0} \in \mathcal{K}$.

For each $e \in \mathbb{N}$, we construct a partial computable function $\psi_e$ uniformly in $e$ as follows.

By dovetailing, we compute $\varphi_e(\langle i,j\rangle)$ for all $i,j \in \mathbb{N}$. Whenever we see that $\varphi_e(\langle i,j\rangle)\downarrow$, we define $\psi_e(i,j,k) = \varphi_{e_0}(i,j,k)$ for all $k \in \mathbb{N}$.

This ends the construction.

Since the construction of $\psi_e$ is effective uniformly in $e$, $\psi_e$ is partial computable uniformly in $e$. Thus, by the *s-m-n* Theorem, there is a computable function $f$ such that $\varphi_{f(e)} = \psi_e$ for all $e \in \mathbb{N}$.

If $e \in Tot$, then $\varphi_e(\langle i,j\rangle)\downarrow$ for all $i,j \in \mathbb{N}$. So $\varphi_{f(e)}(i,j,k) = \psi_e(i,j,k) = \varphi_{e_0}(i,j,k)$ for all $i,j,k \in \mathbb{N}$, that is, $\varphi_{f(e)} = \varphi_{e_0}$. Thus, since $\varphi_{e_0}$ induces the computable metric space $M_{e_0}$, we have that $M_{f(e)} = M_{e_0} \in \mathcal{K}$.

If $e \notin Tot$, then there are $i,j \in \mathbb{N}$ such that $\varphi_e(\langle i,j\rangle)\uparrow$. So we will never define $\psi_e(i,j,k)$ for all $k \in \mathbb{N}$, that is, $\psi_e(i,j,k)\uparrow$ for all $k \in \mathbb{N}$. Hence $\varphi_{f(e)} = \psi_e$ is not total. Therefore, $M_{f(e)}$ is not a computable metric space, and so $M_{f(e)} \notin \mathcal{K}$.

We conclude that for all $e \in \mathbb{N}$, $e \in Tot \iff M_{f(e)} \in \mathcal{K}$. So $Tot$ is many-one reducible to $\{e : M_e \in \mathcal{K}\}$. Therefore, since $Tot$ is $\Pi^0_2$-hard, $\{e : M_e \in \mathcal{K}\}$ is $\Pi^0_2$-hard. □

By Theorem 2.0.1 and Theorem 2.0.2, we have the following remark.

**Remark 2.0.3.** *For any set $I \subseteq PolSp$ and complexity class $\Gamma$,*

(1) *$I$ is $\Pi^0_2 \iff I$ is $\Pi^0_2$ within PolSp.*

(2) *($\Gamma \subseteq \Pi^0_2$ and $I$ is $\Gamma$-complete within PolSp) $\implies I$ is $\Pi^0_2$-complete.*

**Theorem 2.0.4.** *The set $\{e \in \mathbb{N} : d_e$ is a metric$\}$ is $\Pi^0_2$-complete within PolSp, and so it is $\Pi^0_2$-complete.*

*Proof.* Note that for any $e \in PolSp$, $d_e$ is a pseudometric, and so

$$d_e \text{ is a metric } \iff (\forall i,j \in \mathbb{N})(d_e(i,j) = 0 \implies i = j)$$

$$\iff (\forall i, j \in \mathbb{N})(d_e(i,j) > 0 \vee i = j)$$

$$\iff (\forall i, j \in \mathbb{N})(\exists k \in \mathbb{N})(\varphi_e(i,j,k) > 2^{-k} \vee i = j)$$

Therefore, $\{e : d_e \text{ is a metric}\}$ is $\Pi_2^0$ within $PolSp$.

Next, we show that $\{e : d_e \text{ is a metric}\}$ is $\Pi_2^0$-hard within $PolSp$. For each $n \in \mathbb{N}$, let $a_n := 2n$ and $b_n := 2n+1$. Then

$$\mathbb{N} = \{a_n : n \in \mathbb{N}\} \sqcup \{b_n : n \in \mathbb{N}\}.$$

Let $A$ be a $\Pi_2^0$ set. Then there exists a computable relation $R_A$ such that for all $e \in \mathbb{N}$,

$$e \in A \iff \forall n \exists s R_A(e,n,s).$$

For each $e \in \mathbb{N}$, we define a partial computable function $\psi_e$ uniformly in $e$ as follows.

For all $n, m, k \in \mathbb{N}$, let

$$\psi_e(a_n, a_m, k) = \psi_e(b_n, b_m, k) = \begin{cases} 1 & \text{if } n \neq m \\ 0 & \text{if } n = m \end{cases}$$

For all $n, m, k \in \mathbb{N}$ with $m \neq n$, let

$$\psi_e(a_n, b_m, k) = \psi_e(b_m, a_n, k) = 1.$$

At stage $s$ where $s \in \mathbb{N}$: For each $n \leq s$, we do the following:

(1) If $R_A(e,n,s)$, then let $s_n := \min\{s' : R_A(e,n,s')\}$ and for all $k \in \mathbb{N}$, let

$$\psi_e(a_n,b_n,k) = \psi_e(b_n,a_n,k) = \begin{cases} 2^{-k} & \text{if } k < s_n \\ 2^{-s_n} & \text{if } k \geq s_n \end{cases}$$

(2) If $\neg R_A(e,n,s)$, then for all $k \leq s$ such that $\psi_e(a_n,b_n,k)$ and $\psi_e(b_n,a_n,k)$ have not been defined yet (to avoid conflicts with (1)), we let

$$\psi_e(a_n,b_n,k) = \psi_e(b_n,a_n,k) = 2^{-k}.$$

This ends the construction.

Note that if we do (1) for $(a_n,b_n)$ at some stage $s_0$, then we will never do (2) for $(a_n,b_n)$ at or after stage $s_0$.

By the *s-m-n* Theorem, there is a computable function $f$ such that $\varphi_{f(e)} = \psi_e$ for all $e \in \mathbb{N}$.

We claim that for all $e \in \mathbb{N}$, $\varphi_{f(e)}$ induces a pseudometric and

$$e \in A \Longleftrightarrow d_{f(e)} \text{ is a metric.}$$

If $e \in A$, then for each $n \in \mathbb{N}$, there is the least $s_n \in \mathbb{N}$ such that $R_A(e,n,s_n)$, and so we will do (1) for $(a_n,b_n)$ at stage $s = s_n$. Hence $\psi_e(a_n,b_n,k) = \psi_e(b_n,a_n,k) = 2^{-s_n}$ for all $k \geq s_n$. It follows that $\varphi_{f(e)}$ induces a pseudometric, namely $d_{f(e)}$, where $d_{f(e)}(a_n,b_n) = d_{f(e)}(b_n,a_n) = 2^{-s_n} > 0$. Therefore, $d_{f(e)}$ is a metric.

If $e \notin A$, then there is an $n \in \mathbb{N}$ such that $\forall s \neg R_A(e,n,s)$. So we will never do (1) for $(a_n, b_n)$, and we will do (2) for $(a_n, b_n)$ at every stage $s \geq n$. Hence $\varphi_{f(e)}(a_n, b_n, k) = \varphi_{f(e)}(b_n, a_n, k) = 2^{-k}$ for all $k \in \mathbb{N}$. It follows that $\varphi_{f(e)}$ induces a pseudometric, namely $d_{f(e)}$, where $d_{f(e)}(a_n, b_n) = \lim_{k\to\infty} \varphi_{f(e)}(a_n, b_n, k) = \lim_{k\to\infty} 2^{-k} = 0$ but $a_n \neq b_n$. Therefore, $d_{f(e)}$ is not a metric.

We conclude that $\{e : d_e \text{ is a metric}\}$ is $\Pi^0_2$-hard within *PolSp*. □

**Proposition 2.0.5.** *The set $\{e \in \mathbb{N} : M_e$ is infinite$\}$ is $\Pi^0_2$ within PolSp.*

*Proof.* Note that for all $e \in PolSp$, since $(\mathbb{N}, d_e)$ is dense in $M_e$, we have

$$\begin{aligned} M_e \text{ is infinite} &\iff \text{there are infinitely many } (i,j) \in \mathbb{N}^2 \text{ such that } d_e(i,j) \neq 0 \\ &\iff (\forall n \in \mathbb{N})\,(\exists i, j > n)\,(d_e(i,j) \neq 0) \\ &\iff (\forall n \in \mathbb{N})\,(\exists i, j > n)\,(\exists k \in \mathbb{N})\,(|\varphi_e(i,j,k)| > 2^{-k+1}). \end{aligned}$$

Therefore, $\{e : M_e \text{ is infinite}\}$ is $\Pi^0_2$ within *PolSp*. □

**Proposition 2.0.6.**

(1) *The set $\{e \in \mathbb{N} : M_e$ is infinite$\}$ is $\Pi^0_2$-hard within PolSp.*

(2) *The set $\{e \in \mathbb{N} : M_e$ is unbounded$\}$ is $\Pi^0_2$-hard within PolSp.*

*Proof.* For each $e \in \mathbb{N}$, we effectively construct a Polish metric space $(\{x_i : i \in \mathbb{N}\}, d)$ uniformly in $e$ (i.e. we construct a partial computable function $\psi_e$ that induces a metric $d$ ($d$ depends on $e$), and the construction is effective uniformly in $e$) as follows:

Stage 0: Let $s_0 := 0$, $d(x_{s_0}, x_{s_0}) := 0$, and go to stage 1.

(Formally, when we let $d(x_i, x_j) := q$, where $i, j \in \mathbb{N}$ and $q \in \mathbb{Q}$, it means that we define $\psi_e(i, j, t) := q$ and $\psi_e(j, i, t) := q$ for all $t \in \mathbb{N}$.)

Stage $n + 1$ where $n \in \mathbb{N}$: For each $s > s_n$, starting from $s = s_n + 1$, we check if $\varphi_{e,s}(n)\downarrow$ until we find (if ever) the least $s > s_n$ such that $\varphi_{e,s}(n)\downarrow$.

If $\varphi_{e,s}(n)\uparrow$, then we let

(1) $d(x_s, x_j) := 0$ for all $j \in \{s_n, \ldots, s\}$,

(2) $d(x_s, x_j) := n - i$ for all $j \in \{s_i, \ldots, s_{i+1} - 1\}$ and $i \in \{0, \ldots, n - 1\}$,

and then we check for the next value of $s$.

Whenever we find (if ever) the least $s > s_n$ such that $\varphi_{e,s}(n)\downarrow$, then we let $s_{n+1} := \min\{s' > s_n : \varphi_{e,s'}(n)\downarrow\} = s$, and we let

(1) $d(x_{s_{n+1}}, x_{s_{n+1}}) := 0$,

(2) $d(x_{s_{n+1}}, x_j) := n + 1 - i$ for all $j \in \{s_i, \ldots, s_{i+1} - 1\}$ and $i \in \{0, \ldots, n\}$,

and then go to stage $n + 2$.

This ends the construction.

Note that the construction is effective uniformly in $e$. So $\psi_e$ is a partial computable function, and by the *s-m-n* theorem, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{f(e)} = \psi_e$ for all $e \in \mathbb{N}$.

Note that if $e \in Tot$, then we obtain an infinite Polish metric space $(\{x_i : i \in \mathbb{N}\}, d)$ and an infinite set $\{s_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ where

(1) $x_{s_n} = x_{s_n+1} = \cdots = x_{s_{n+1}-1}$ for all $n \in \mathbb{N}$,

(2) $d(x_{s_{n+1}}, x_{s_i}) = n + 1 - i$ for all $i < n + 1$ and $n \in \mathbb{N}$.

So after identifying points with zero distance, we obtain the space $(\{x_{s_n} : n \in \mathbb{N}\}, d)$. Using the map $x_{s_n} \mapsto n$, this space is isometric to the Polish metric space $(\mathbb{N}, d_{\mathbb{N}})$ with $d_{\mathbb{N}}(i,j) := |i-j|$. Therefore, $M_{f(e)} \cong (\mathbb{N}, d_{\mathbb{N}})$ is an infinite Polish metric space, and it is also unbounded.

On the other hand, if $e \notin Tot$, say $N := \min\{n \in \mathbb{N} : \varphi_e(n)\uparrow\}$, then we obtain a finite Polish metric space $(\{x_i : i \in \mathbb{N}\}, d)$ and a finite set $\{s_0, s_1, \ldots, s_N\} \subseteq \mathbb{N}$ where

(1) $x_{s_n} = x_{s_n+1} = \cdots = x_{s_{n+1}-1}$ for all $n < N$,

(2) $x_s = x_{s_N}$ for all $s > s_N$,

(3) $d(x_{s_{n+1}}, x_{s_i}) = n + 1 - i$ for all $i < n + 1$ and $n < N$.

So after identifying points with zero distance, we obtain the space $(\{x_{s_0}, x_{s_1}, \ldots, x_{s_N}\}, d)$. Using the map $x_{s_n} \mapsto n$, this space is isometric to the finite Polish metric space $(\{0, 1, \ldots, N\}, d_{\mathbb{N}})$ with $d_{\mathbb{N}}(i,j) := |i-j|$. Therefore, $M_{f(e)} \cong (\{0, 1, \ldots, N\}, d_{\mathbb{N}})$ is a finite Polish metric space, and it is also bounded.

From the above argument, we have that there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $e \in \mathbb{N}$, we have $f(e) \in PolSp$,

$$e \in Tot \iff M_{f(e)} \text{ is infinite},$$

$$e \in Tot \iff M_{f(e)} \text{ is unbounded}.$$

Therefore, since $Tot$ is $\Pi^0_2$-hard, $\{e : M_e \text{ is infinite}\}$ and $\{e : M_e \text{ is unbounded}\}$ are $\Pi^0_2$-hard within $PolSp$. □

By Proposition 2.0.5, Proposition 2.0.6 and Remark 2.0.3, we have the following.

**Theorem 2.0.7.** *The set* $\{e \in \mathbb{N} : M_e \text{ is infinite}\}$ *is* $\Pi^0_2$*-complete within* $PolSp$*, and so it is* $\Pi^0_2$*-complete.*

The following theorem says that there is a uniform way to pass from an index $e$ such that $M_e$ is infinite, to an index $i$ such that $M_i \cong M_e$ and $d_i$ is a metric. Therefore, if $(p_i)_{i\in\mathbb{N}}$ is a computable presentation of an infinite metric space, then we can assume without loss of generality that $(p_i)_{i\in\mathbb{N}}$ has no repetitions, that is, $p_i \neq p_j$ for all $i \neq j$.

**Theorem 2.0.8.** *There is a computable function* $f$ *such that for all* $e \in PolSp$*, if* $M_e$ *is infinite, then* $M_{f(e)} \cong M_e$ *and* $d_{f(e)}$ *is a metric. Furthermore, the isometries between* $M_{f(e)} \cong M_e$ *are computable uniformly in* $e$ *in the sense that there is a sequence* $(g_e)_{e\in\mathbb{N}}$ *of partial computable functions, uniformly in* $e$*, such that for all* $e \in PolSp$*, if* $M_e$ *is infinite, then* $g_e : (\mathbb{N}, d_{f(e)}) \to (\mathbb{N}, d_e)$ *is a computable isometry, and so it extends (uniquely) to a computable isometry* $\widetilde{g}_e : M_{f(e)} \to M_e$*.*

*Proof.* For each $e \in \mathbb{N}$, we define a partial computable function $g : \mathbb{N} \to \mathbb{N}$ uniformly in $e$ by induction as follows:

Let $g_e(0) = 0$. For each $n \in \mathbb{N}$, assuming by induction that $g_e(m)\downarrow$ for all $m \leq n$, we search by dovetailing until we find the least pair (if exists) $\langle i, k\rangle$ such that and $\varphi_e(i, g_e(m), k) \downarrow > 2^{-k}$ for all $m \leq n$. Then we let $g_e(n + 1) = i$. Otherwise, we let $g_e(n + 1)\uparrow$.

**Claim.** For all $e \in PolSp$, if $M_e$ is infinite, then $g_e$ is total, $range(g_e)$ is dense in $M_e$ and $d_e(g_e(n), g_e(m)) > 0$ for all distinct $n, m \in \mathbb{N}$.

Let $e \in PolSp$. Assume that $M_e$ is infinite. Then $\varphi_e$ is total, $d_e$ is the pseudometric induced by $\varphi_e$ and $|\varphi_e(i,j,k) - d_e(i,j)| \leq 2^{-k}$ for all $i,j,k \in \mathbb{N}$.

First, we show that $g_e$ is total. From the definition of $g_e$, we have $g_e(0) \downarrow = 0$. Let $n \in \mathbb{N}$ and assume by induction that $g_e(m) \downarrow$ for all $m \leq n$. Since $M_e$ is infinite, there must be an $i \in \mathbb{N}$ such that $d_e(i, g_e(m)) > 0$ for all $m \leq n$. (Otherwise, we would have $|M_e| \leq n+1$, a contradiction.)

Suppose for a contradiction that $(\forall k \in \mathbb{N})(\exists m \leq n)(\varphi_e(i, g_e(m), k) \leq 2^{-k})$. Then there must be an $m \leq n$ such that $\varphi_e(i, g_e(m), k) \leq 2^{-k}$ for infinitely many $k$.

So for each $K \in \mathbb{N}$, there is a $k \geq K$ such that $\varphi_e(i, g_e(m), k) \leq 2^{-k}$, and so

$$0 \leq d_e(i, g_e(m)) \leq \varphi_e(i, g_e(m), k) + 2^{-k} \leq 2^{-k} + 2^{-k} = 2^{-k+1} \leq 2^{-K+1}.$$

Taking limit $K \to \infty$, we have $d_e(i, g_e(m)) = 0$, but $d_e(i, g_e(m)) > 0$, a contradiction.

Therefore, there is a $k \in \mathbb{N}$ such that $\varphi_e(i, g_e(m), k) > 2^{-k}$ for all $m \leq n$. So $g_e(n+1) \downarrow$. We conclude that $g_e$ is total, and so it is computable.

Since $g_e$ is total, the definition of $g_e$ ensures that for all $n \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $\varphi_e(g_e(n+1), g_e(m), k) > 2^{-k}$ for all $m \leq n$. Hence for all $m \leq n$,

$$d_e(g_e(n+1), g_e(m)) \geq \varphi_e(g_e(n+1), g_e(m), k) - 2^{-k} > 2^{-k} - 2^{-k} = 0.$$

It follows that $d_e(g_e(n), g_e(m)) > 0$ for all distinct $n, m \in \mathbb{N}$.

It remains to show that $range(g_e)$ is dense in $M_e$. Since $(\mathbb{N}, d_e)$ is dense in $M_e$, it suffices to show that for every $i \in \mathbb{N}$, $i \in cl(range(g_e))$. Let $i \in \mathbb{N}$. If $i \in range(g_e)$, then $i \in cl(range(g_e))$. Assume $i \notin range(g_e)$.

We will show that $(\forall k \in \mathbb{N})(\exists j_k \in range(g_e))(\varphi_e(i, j_k, k) \leq 2^{-k})$. Suppose for a contradiction that there is a $k \in \mathbb{N}$ such that $(\forall j \in range(g_e))(\varphi_e(i, j, k) > 2^{-k})$.

Let $A := \{i' \in \mathbb{N} : \langle i', k' \rangle < \langle i, k \rangle \text{ for some } k' \in \mathbb{N}\}$. Then $A$ is finite. Let $n \in \mathbb{N}$. Since $\varphi_e$ is total, for each pair $\langle i', k' \rangle < \langle i, k \rangle$ we can check in finitely many steps whether it works in the sense that $\varphi_e(i', g_e(m), k') > 2^{-k'}$ for all $m \leq n$. If there is the least such pair, say $\langle i_n, k_n \rangle < \langle i, k \rangle$, we will let $g_e(n+1) := i_n$. Then $i_n \in A$ and $i_n \neq g_e(m)$ for all $m \leq n$. Since $i \notin range(g_e)$, $g_e(n+1) \neq i$, and so there must be the least such pair. (Otherwise, since the pair $\langle i, k \rangle$ works (by assumption), we will let $g_e(n+1) = i$, a contradiction.) It follows that $i_n$'s are all distinct and $i_n \in A$ for all $n \in \mathbb{N}$. But $A$ is finite, a contradiction.

Therefore, for every $k \in \mathbb{N}$, there is a $j_k \in range(g_e)$ such that $\varphi_e(i, j_k, k) \leq 2^{-k}$. Taking limit $k \to \infty$, we have that $\lim_{k \to \infty} d_e(i, j_k) = 0$. So $(j_k)_{k \in \mathbb{N}}$ is a sequence in $range(g_e)$ that converges to $i$. Hence $i \in cl(range(g_e))$. We conclude that $range(g_e)$ is dense in $M_e$.

This ends the proof of the claim.

Since $g_e$ is partial computable uniformly in $e$, by the *s-m-n* Theorem, there is a computable function $f$ such that for all $e, i, j, k \in \mathbb{N}$,

$$\varphi_{f(e)}(i, j, k) = \varphi_e(g_e(i), g_e(j), k).$$

We claim that for all $e \in PolSp$, if $M_e$ is infinite, then $M_{f(e)} \cong M_e$ and $d_{f(e)}$

is a metric. Let $e \in PolSp$ and assume that $M_e$ is infinite. By Claim, $g_e$ is total, $range(g_e)$ is dense in $M_e$ and $d_e(g_e(n), g_e(m)) > 0$ for all distinct $n, m \in \mathbb{N}$. So for all distinct $n, m \in \mathbb{N}$,

$$d_{f(e)}(n, m) = \lim_{k\to\infty} \varphi_{f(e)}(n, m, k) = \lim_{k\to\infty} \varphi_e(g_e(n), g_e(m), k) = d_e(g_e(n), g_e(m)) > 0.$$

Hence $d_{f(e)}$ is a metric and $g_e : (\mathbb{N}, d_{f(e)}) \to (\mathbb{N}, d_e)$ is distance-preserving. Therefore, since $range(g_e)$ is dense in $M_e$ and $g_e$ is computable, $g_e$ extends uniquely to a computable isometry $\widetilde{g}_e : M_{f(e)} \to M_e$, and so $M_{f(e)} \cong M_e$. □

**Theorem 2.0.9.** *The set $\{e \in \mathbb{N} : M_e$ is bounded$\}$ is $\Sigma^0_2$-complete within PolSp.*

*Proof.* By Proposition 2.0.6, $\{e : M_e$ is bounded$\}$ is $\Sigma^0_2$-hard within $PolSp$.

Note that for all $e \in PolSp$, since $(\mathbb{N}, d_e)$ is dense in $M_e$, we have

$$\begin{aligned} M_e \text{ is bounded} &\iff (\exists N \in \mathbb{N})(\forall x, y \in M_e)(d_e(x, y) \le N) \\ &\iff (\exists N \in \mathbb{N})(\forall i, j \in \mathbb{N})(d_e(i, j) \le N) \end{aligned}$$

Since "$d_e(i, j) \le N$" is a $\Pi^0_1$ statement, $\{e : M_e$ is bounded$\}$ is $\Sigma^0_2$ within $PolSp$. □

Next, we will show that the set $\{e \in \mathbb{N} : M_e$ is bounded$\}$ is $d$-$\Sigma^0_2$-complete by using the following fact.

**Fact 2.0.10.** *For every $\Sigma^0_2$ set $A$, there is a computable sequence $(A_s)_{s\in\mathbb{N}}$ of sets such that for all $e \in \mathbb{N}$,*

$$e \in A \iff \exists t(\forall s \ge t)(e \in A_s).$$

*Proof.* Let $A$ be a $\Sigma^0_2$ set. Since $Fin := \{e : dom(\varphi_e) \text{ is finite}\}$ is $\Sigma^0_2$-complete, there is a computable function $f$ such that for all $e \in \mathbb{N}$,

$$e \in A \Longleftrightarrow f(e) \in Fin.$$

Note that the projections $\pi_0, \pi_1 : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi_0 : \langle n, s\rangle \mapsto n \text{ and } \pi_1 : \langle n, s\rangle \mapsto s$$

are computable functions. Also note that for all $m \in \mathbb{N}$, there exists a $u \in \mathbb{N}$ such that for all $n, s \in \mathbb{N}$, if $\langle n, s\rangle \geq u$, then $n \geq m$. So for all $e \in \mathbb{N}$,

$$\begin{aligned}
e \in A &\Longleftrightarrow dom(\varphi_{f(e)}) \text{ is finite} \\
&\Longleftrightarrow \exists m(\forall n \geq m)(\varphi_{f(e)}(n)\uparrow) \\
&\Longleftrightarrow \exists m(\forall n \geq m)(\forall s)(\varphi_{f(e),s}(n)\uparrow) \\
&\Longleftrightarrow \exists u(\forall v \geq u)(v = \langle n, s\rangle \Longrightarrow \varphi_{f(e),s}(n)\uparrow) \\
&\Longleftrightarrow \exists u(\forall v \geq u)(\varphi_{f(e),\pi_1(v)}(\pi_0(v))\uparrow).
\end{aligned}$$

For each $v \in \mathbb{N}$, let $A_v := \{e : \varphi_{f(e),\pi_1(v)}(\pi_0(v))\uparrow\}$. Then $(A_v)_{v\in\mathbb{N}}$ is a computable sequence of sets and for all $e \in \mathbb{N}$,

$$e \in A \Longleftrightarrow \exists u(\forall v \geq u)(e \in A_v).$$

□

**Theorem 2.0.11.** *The set* $\{e \in \mathbb{N} : M_e$ *is bounded*$\}$ *is d-*$\Sigma^0_2$*-complete*

*Proof.* Since $\{e : M_e \text{ is bounded}\}$ is $\Sigma^0_2$ within *PolSp* and *PolSp* is $\Pi^0_2$, we have that $\{e : M_e \text{ is bounded}\}$ is $d$-$\Sigma^0_2$. It remains to show that it is $d$-$\Sigma^0_2$-hard.

For each $n \in \mathbb{N}$, let $a_n := 2n$ and $b_n := 2n + 1$. Then

$$\mathbb{N} = \{a_n : n \in \mathbb{N}\} \sqcup \{b_n : n \in \mathbb{N}\}.$$

Let $C$ be a $d$-$\Sigma^0_2$ set, say $C = A \setminus B$ where $A, B$ are $\Sigma^0_2$. By Fact 2.0.10, there exist a computable sequence $(A_s)_{s\in\mathbb{N}}$ of sets and a computable relation $R_B$ such that for all $e \in \mathbb{N}$,

$$e \in A \iff \exists t(\forall s \geq t)(e \in A_s),$$

$$e \in B \iff \exists n \forall s R_B(e, n, s).$$

For each $e \in \mathbb{N}$, we define a partial computable function $\psi_e$ uniformly in $e$ as follows.

Let

- $\psi_e(a_0, b_n, k) = \psi_e(b_n, a_0, k) = 1$ for all $n, k \in \mathbb{N}$,
- $\psi_e(a_n, a_n, k) = 0$ for all $n, k \in \mathbb{N}$,
- $\psi_e(b_n, b_m, k) = 1$ for all $n, m, k \in \mathbb{N}$ with $n \neq m$.

At stage $s$ where $s \in \mathbb{N}$: We do the following:

(1) For each $n \leq s$, if $(\exists t \leq s)\neg R_B(e, n, t)$, then let $\psi_e(b_n, b_n, k) = 0$ for all $k \in \mathbb{N}$.

(2) If $s > 0$ and $e \in A_s$, then for all $k \in \mathbb{N}$ and $u \in \mathbb{N} \setminus \{a_i : i \geq s\}$, let

$$\psi_e(a_s, u, k) = \psi_e(u, a_s, k) = \psi_e(a_{s-1}, b_0, k).$$

(3) If $s > 0$ and $e \notin A_s$, then for all $k \in \mathbb{N}$ and $u \in \mathbb{N} \setminus \{a_i : i \geq s\}$, let

$$\psi_e(a_s, u, k) = \psi_e(u, a_s, k) = \psi_e(a_{s-1}, b_0, k) + 1.$$

This ends the construction.

By the *s-m-n* Theorem, there is a computable function $f$ such that $\varphi_{f(e)} = \psi_e$ for all $e \in \mathbb{N}$.

We claim that for all $e \in \mathbb{N}$, $e \in A \backslash B \iff M_{f(e)}$ is a bounded Polish metric space.

If $e \in B$, then there is an $n \in \mathbb{N}$ such that $\forall s R_B(e, n, s)$. So we will never do (1), that is, for all $k \in \mathbb{N}$, $\psi_e(b_n, b_n, k)$ is never defined in the construction. Hence $\varphi_{f(e)} = \psi_e$ is not total, and so $\varphi_{f(e)}$ does not induce a pseudometric.

If $e \notin B$, then for each $n \in \mathbb{N}$, there is an $s_n \in \mathbb{N}$ such that $\neg R_B(e, n, s_n)$, and so we will define $\psi_e(b_n, b_n, k) = 0$ for all $k \in \mathbb{N}$ at or before stage $s = \max\{n, s_n\}$. It follows that $\varphi_{f(e)} = \psi_e$ is total and it is easy to see that $\varphi_{f(e)}$ induces a pseudometric, namely $d_{f(e)}$. In fact, $d_{f(e)}$ is a metric and $d_{f(e)}(u, v) \in \mathbb{N}$ for all $u, v \in \mathbb{N}$.

If $e \notin B$ and $e \in A$, then since $e \notin B$, $\varphi_{f(e)}$ induces a pseudometric. Since $e \in A$, there is a $t \in \mathbb{N}$ such that $(\forall s \geq t)(e \in A_s)$, and so we will do (2) and never do (3) at every stage $s > t$. It follows that $\{d_{f(e)}(a_s, u) : s, u \in \mathbb{N}\}$ is bounded. Now it is clear from the construction that $\{d_{f(e)}(u, v) : u, v \in \mathbb{N}\}$ is bounded, and so $M_{f(e)}$ is bounded.

If $e \notin B$ and $e \notin A$, then since $e \notin B$, $\varphi_{f(e)}$ induces a pseudometric. Since $e \notin A$, we have $\forall t(\exists s \geq t)(e \notin A_s)$, and so we will do (3) infinitely many times. It follows that $\lim_{s \to \infty} d_{f(e)}(a_s, b_0) = \infty$. Therefore, $M_{f(e)}$ is unbounded.

From all cases, we conclude that for all $e \in \mathbb{N}$,

$$e \in A \setminus B \iff M_{f(e)} \text{ is a bounded Polish metric space.}$$

Therefore, $\{e : M_e \text{ is bounded}\}$ is $d$-$\Sigma^0_2$-hard. $\square$

**Theorem 2.0.12.**

(1) *The set* $\{e \in \mathbb{N} : |M_e| \geq 1\} = PolSp$ *is* $\Pi^0_2$*-complete.*

(2) *The set* $\{e \in \mathbb{N} : |M_e| = 1\} = \{e \in \mathbb{N} : M_e \cong \{0\}\}$ *is* $\Pi^0_1$*-complete within* $PolSp$*, and so it is* $\Pi^0_2$*-complete.*

*Proof.* It is clear that $\{e : |M_e| \geq 1\} = PolSp$ and $\{e : |M_e| = 1\} = \{e : M_e \cong \{0\}\}$. So (1) follows from Theorem 2.0.1.

Note that for all $e \in PolSp$,

$$|M_e| = 1 \iff (\forall i, j \in \mathbb{N})(d_e(i, j) = 0).$$

Thus, since "$d_e(i, j) = 0$" is a $\Pi^0_1$ statement, $\{e : |M_e| = 1\}$ is $\Pi^0_1$ within $PolSp$.

It remains to show that $\{e : |M_e| = 1\}$ is $\Pi^0_1$-hard within $PolSp$. Let $A$ be a $\Pi^0_1$ set. Then there is a computable relation $R(e, y)$ such that for all $e \in \mathbb{N}$,

$$e \in A \iff \forall y R(e, y).$$

For each $e \in \mathbb{N}$, we construct a computable metric space $(X_e, d)$ uniformly in $e$ as follows.

For each $y \in \mathbb{N}$, starting from $y = 0$, we check if $R(e, y)$ until we find (if ever) the least $y$ such that $\neg R(e, y)$.

For each $y \in \mathbb{N}$ such that $R(e, y)$, we let $d(i, j) := 0$ for all $i, j \in \{0, \ldots, y\}$.

Whenever we find (if ever) the least $y$ such that $\neg R(e, y)$, we let

- $d(i, y) = 1$ for all $i \in \mathbb{N} \setminus \{y\}$.
- $d(y, y) = 0$.
- $d(i, j) = 0$ for all $i, j \in \mathbb{N} \setminus \{y\}$.

Then we stop the construction.

This ends the construction.

Let $(X_e, d)$ be the completion of the resulting pseudometric space $(\mathbb{N}, d)$. The key point is that, whenever we find (if ever) the least $y$ such that $\neg R(e, y)$, we make sure that there are $i, y \in X_e$ such that $d(i, y) > 0$, and so $|X_e| > 1$.

Note that for all $e \in \mathbb{N}$,

$$
\begin{aligned}
&e \in A \Longrightarrow \forall y R(e, y) \Longrightarrow (\forall i, j \in \mathbb{N})(d(i, j) = 0) \Longrightarrow |X_e| = 1 \\
&e \notin A \Longrightarrow \exists y \neg R(e, y) \Longrightarrow (\exists i, y \in \mathbb{N})(d(i, y) = 1) \Longrightarrow (\exists i, y \in X_e)(d(i, y) = 1) \\
&\qquad \Longrightarrow |X_e| > 1.
\end{aligned}
$$

It follows that $A$ is many-one reducible to $\{e : |M_e| = 1\}$. Therefore, $\{e : |M_e| = 1\}$ is $\Pi^0_1$-hard within $PolSp$. □

**Theorem 2.0.13.** *Let $n \geq 2$.*

(1) *The set* $\{e \in \mathbb{N} : |M_e| \geq n\}$ *is* $\Sigma^0_1$*-complete within PolSp, and so it is* $\Pi^0_2$*-complete.*

(2) *The set* $\{e \in \mathbb{N} : |M_e| = n\}$ *is d-*$\Sigma^0_1$*-complete within PolSp, and so it is* $\Pi^0_2$*-complete.*

*Proof.* Note that for all $e \in PolSp$,

$$|M_e| \geq n \iff (\exists x_1, \ldots, x_n \in \mathbb{N})(\forall i, j \in \{1, \ldots, n\})(i \neq j \implies d_e(x_i, x_j) > 0),$$
$$|M_e| = n \iff |M_e| \geq n \wedge \neg(|M_e| \geq n + 1).$$

Thus, since "$d_e(x_i, x_j) > 0$" is a $\Sigma^0_1$ statement, $\{e : |M_e| \geq n\}$ is $\Sigma^0_1$ within $PolSp$, and so $\{e : |M_e| = n\}$ is $d$-$\Sigma^0_1$ within $PolSp$.

It remains to show that $\{e : |M_e| \geq n\}$ is $\Sigma^0_1$-hard within $PolSp$, and $\{e : |M_e| = n\}$ is $d$-$\Sigma^0_1$-hard within $PolSp$. Let $(X, d_X)$ be any finite computable metric space with $|X| = n - 1$, say $X = \{x_0, \ldots, x_{n-2}\}$.

Let $C$ be a $d$-$\Sigma^0_1$ set. Then $C = A \setminus B$ where $A, B$ are $\Sigma^0_1$ sets. So there are computable relations $R^0_A(e, y)$ and $R^0_B(e, y)$ such that for all $e \in \mathbb{N}$,

$$e \in A \iff \exists y R^0_A(e, y),$$
$$e \in B \iff \exists y R^0_B(e, y).$$

Define computable relations $R_A(e, y)$ and $R_B(e, y)$ by

$$R_A(e, y) \iff (\exists z \leq y) R^0_A(e, z),$$
$$R_B(e, y) \iff (\exists z \leq y) R^0_B(e, z).$$

Then for all $e, m \in \mathbb{N}$,

$$e \in A \iff (\exists y \geq m) R_A(e, y),$$

$$e \in B \iff (\exists y \geq m) R_B(e, y).$$

Since $X$ is finite, there is an $r \in \mathbb{N}$ such that $d_X(x, y) < r$ for all $x, y \in X$. For example, we can choose any $r \in \mathbb{N}$ such that $r > diam(X)$.

For each $e \in \mathbb{N}$, we construct a computable metric space $(X_e, d)$ uniformly in $e$ as follows.

Step 1: Let $d(i, j) = d_X(x_i, x_j)$ for all $i, j \in \{0, \ldots, n-2\}$.

Step 2: For each $y \geq n-1$, starting from $y = n-1$, we check if $R_A(e, y)$ until we find (if ever) the least $y \geq n-1$ such that $R_A(e, y)$.

For each $y \geq n-1$ such that $\neg R_A(e, y)$, we let

- $d(i, j) = 0$ for all $i, j \in \{n-1, \ldots, y\}$,
- $d(i, j) = d(i, n-2) = d_X(x_i, x_{n-2})$ for all $i \leq n-2$ and $j \in \{n-1, \ldots, y\}$.

Whenever we find (if ever) the least $y \geq n-1$ such that $R_A(e, y)$, we call it $y_A$ and let

- $d(i, y_A) = r$ for all $i \in \{0, \ldots, y_A - 1\}$,
- $d(y_A, y_A) = 0$.

Then we go to Step 3.

Step 3: We have found $y_A$ from Step 2. For each $y \geq y_A + 1$, starting from

$y = y_A + 1$, we check if $R_B(e, y)$ until we find (if ever) the least $y \geq y_A + 1$ such that $R_B(e, y)$.

For each $y \geq y_A + 1$ such that $\neg R_B(e, y)$, we let

- $d(i, j) = 0$ for all $i, j \in \{y_A + 1, \ldots, y\}$,
- $d(i, j) = d(i, y_A)$ for all $i \in \{0, \ldots, y_A\}$ and $j \in \{y_A + 1, \ldots, y\}$.

Whenever we find (if ever) the least $y \geq y_A + 1$ such that $R_B(e, y)$, we call it $y_B$ and let

- $d(i, j) = d(i, y_B) = r$ for all $i < y_B$ and $j \geq y_B$,
- $d(i, j) = 0$ for all $i, j \geq y_B$.

Then we stop the construction.

This ends the construction.

Let $(X_e, d)$ be the completion of the resulting pseudometric space $(\mathbb{N}, d)$. The idea is that, we put a copy of $X$ into $X_e$ at Step 1. Then, at Step 2, we search for the least $y \geq n - 1$ such that $R_A(e, y)$. Whenever find such a $y$, we call it $y_A$, put $y_A$ into $X_e$ as a new element, and go to Step 3. At Step 3, we search for the least $y \geq y_A + 1$ such that $R_B(e, y)$. Whenever find such a $y$, we call it $y_B$, put $y_B$ into $X_e$ as a new element ($y_B \neq y_A$), and stop the construction.

If $e \notin A$, then there is no $y \geq n - 1$ such that $R_A(e, y)$. So we will never put a new element $y_A$ into $X_e$ in Step 2, and never go to Step 3. From the construction, we identify all points $i \geq n - 1$ with the point $n - 2$. So $(X_e, d) \cong (X, d_X)$, and so $|X_e| = |X| = n - 1 < n$.

If $e \in A$, then there is a $y \geq n-1$ such that $R_A(e, y)$, and we will call it $y_A$. From the construction, we will put $y_A$ into $X_e$ as a new element by letting $d(i, y_A) = r$ for all $i < y_A$. The fact that $d_X(x, y) < r$ for all $x, y \in X$ ensures that $d$ satisfies the triangle inequality. Then we identify all points $i \in \{n-1, \ldots, y_A - 1\}$ with the point $n-2$. So $|X_e| \geq |X| + 1 = n$. After we have found $y_A$, we go to Step 3 to search for the least $y \geq y_A + 1$ such that $R_B(e, y)$.

If $e \in A \setminus B$, then, since $e \in A$, we will put $y_A$ into $X_e$ and go to Step 3. Since $e \notin B$, there is no $y \geq n-1$ such that $R_B(e, y)$. So we will never put a new element $y_B$ into $X_e$ in Step 3. From the construction, we identify all points $i \geq y_A + 1$ with $y_A$. So $X_e$ is isometric to the space $X$ with one extra element $y_A$. Hence $|X_e| = |X| + 1 = n$.

If $e \in A \cap B$, then, since $e \in A$, we will put $y_A$ into $X_e$ and go to Step 3. Since $e \in B$, there is a $y \geq y_A + 1$ such that $R_B(e, y)$, and we will call it $y_B$. From the construction, we will put $y_B$ into $X_e$ as a new element by letting $d(i, j) = d(i, y_B) = r$ for all $i < y_B$ and $j \geq y_B$. Then we identify all points $i \geq y_B + 1$ with $y_B$. So $X_e$ is isometric to the space $X$ with two extra elements $y_A$ and $y_B$. Hence $|X_e| = |X| + 2 = n + 1 > n$.

We can conclude as follows:

- $e \notin A \Longrightarrow (X_e, d) \cong (X, d_X) \Longrightarrow |X_e| = |X| = n - 1$.
- $e \in A \Longrightarrow |X_e| \geq |X| + 1 = n$.
- $e \in A \setminus B \Longrightarrow X_e \cong X \sqcup \{y_A\} \Longrightarrow |X_e| = |X| + 1 = n$.
- $e \in A \cap B \Longrightarrow X_e \cong X \sqcup \{y_A, y_B\} \Longrightarrow |X_e| = |X| + 2 = n + 1$.

It follows that $C = A \setminus B$ is many-one reducible to $\{e : |M_e| = n\}$. Therefore, $\{e : |M_e| = n\}$ is $d$-$\Sigma^0_1$-hard within $PolSp$.

If we let $A$ be any $\Sigma^0_1$-complete set and $B$ be any $\Sigma^0_1$ set, and use the same construction, then we will have that $A$ is many-one reducible to $\{e : |M_e| \geq n\}$, and so $\{e : |M_e| \geq n\}$ is $\Sigma^0_1$-hard within $PolSp$. $\square$

# Chapter 3

# Isomorphism Problems and Embedding Problems

In this chapter, we consider isomorphism problems and the following *embedding problems* for computable metric spaces:

- $\{(i,j) \in \mathbb{N}^2 : M_i \hookrightarrow M_j\}$: Given a pair of computable metric spaces $M_i$ and $M_j$, determine if $M_i$ embeds into $M_j$.
- $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$: For a fixed computable metric space $X$, given a computable metric space $M_e$, determine if $X$ embeds into $M_e$.
- $\{e \in \mathbb{N} : M_e \hookrightarrow X\}$: For a fixed computable metric space $X$, given a computable metric space $M_e$, determine if $M_e$ embeds into $X$.

Throughout this chapter, we fix an effective list $(T_e)_{e\in\mathbb{N}}$ of all primitive recursive trees $T_e \subseteq \mathbb{N}^{<\mathbb{N}}$. By Fact 1.1.19, the set $\mathcal{T} := \{e \in \mathbb{N} : T_e \text{ has an infinite path}\}$ is $\Sigma^1_1$-complete. Thus, by the *s-m-n* Theorem, to show that an index set of the form

$I = \{e \in \mathbb{N} : M_e$ has property $P\}$ is $\Sigma^1_1$-hard, it is enough to build a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$,

$$T_e \text{ has an infinite path} \iff X_e \text{ has property } P.$$

## 3.1 Basic Results

First note that Theorem 2.0.2 gives the following remark.

**Remark 3.1.1.** *For every computably presentable metric space $X$, the sets $\{e \in \mathbb{N} : M_e \cong X\}$, $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ and $\{e \in \mathbb{N} : M_e \hookrightarrow X\}$ are $\Pi^0_2$-hard.*

**Proposition 3.1.2.** *The set $\{(i,j) \in \mathbb{N}^2 : M_i \cong M_j\}$ is $\Sigma^1_1$.*

*Proof.* Let $I := \{(i,j) \in \mathbb{N}^2 : M_i \cong M_j\}$. Then $(i,j) \in I$ if and only if $M_i$ and $M_j$ are Polish metric spaces and $M_i \cong M_j$. Since *PolSp* is $\Pi^0_2$, it remains to show that "$M_i \cong M_j$" is a $\Sigma^1_1$ statement. The argument below is essentially the same as in the proof of Lemma 3.2 in [4].

For a function $f : \mathbb{N}^2 \to \mathbb{N}$, we define $f_m(n) := f(m,n)$ for all $m, n \in \mathbb{N}$. Let $p_m$ denote the $m$-th rational point in $M_i$, and $q_m$ denote the $m$-th rational point in $M_j$. We will show that $M_i \cong M_j$ if and only if there are functions $f, g : \mathbb{N}^2 \to \mathbb{N}$ that satisfy the following conditions.

(1) For all $m, n, l \in \mathbb{N}$,

$$d_j(q_{f(m,n)}, q_{f(m,n+l)}) \leq 2^{-n} \text{ and } d_i(p_{g(m,n)}, p_{g(m,n+l)}) \leq 2^{-n}.$$

(2) For all $m, m', n, n' \in \mathbb{N}$,

$$|d_i(p_m, p_{m'}) - d_j(q_{f(m,n)}, q_{f(m',n')})| \leq 2^{-n} + 2^{-n'} \text{ and}$$
$$|d_j(q_m, q_{m'}) - d_i(p_{g(m,n)}, p_{g(m',n')})| \leq 2^{-n} + 2^{-n'}.$$

(3) For all $m, n, n' \in \mathbb{N}$,

$$d_i(p_m, p_{g(f(m,n),n')}) \leq 2^{-n} + 2^{-n'} \text{ and } d_j(q_m, q_{f(g(m,n),n')}) \leq 2^{-n} + 2^{-n'}.$$

Note that conditions (1), (2) and (3) are $\Pi^0_1$ statements.

($\Rightarrow$) Assume that $M_i \cong M_j$ via an isometry $\alpha : M_i \to M_j$. Let $\beta := \alpha^{-1}$. Then there exist $f, g : \mathbb{N}^2 \to \mathbb{N}$ such that for every $m \in \mathbb{N}$, $f_m$ is a Cauchy name of $\alpha(p_m)$ in $M_j$ and $g_m$ is a Cauchy name of $\beta(q_m)$ in $M_i$. So $f$ and $g$ satisfy condition (1), $\lim_{n\to\infty} q_{f(m,n)} = \alpha(p_m)$ and $\lim_{n\to\infty} p_{g(m,n)} = \beta(q_m)$. Note that for all $m, m', n, n' \in \mathbb{N}$, since $\alpha$ is distance-preserving, we have

$$\begin{aligned} d_i(p_m, p_{m'}) &= d_j(\alpha(p_m), \alpha(p_{m'})) \\ &\leq d_j(\alpha(p_m), q_{f(m,n)}) + d_j(q_{f(m.n)}, q_{f(m',n')}) + d_j(q_{f(m',n')}, \alpha(p_{m'})) \\ &\leq 2^{-n} + d_j(q_{f(m.n)}, q_{f(m',n')}) + 2^{-n'}. \end{aligned}$$

Similarly, we have that $d_j(q_{f(m.n)}, q_{f(m',n')}) \leq d_i(p_m, p_{m'}) + 2^{-n} + 2^{-n'}$. Hence $|d_i(p_m, p_{m'}) - d_j(q_{f(m,n)}, q_{f(m',n')})| \leq 2^{-n} + 2^{-n'}$. Since $\beta$ is distance-preserving, by the same argument, we have that $|d_j(q_m, q_{m'}) - d_i(p_{g(m,n)}, p_{g(m',n')})| \leq 2^{-n} + 2^{-n'}$. So condition (2) is satisfied.

Now note that for all $m, n, n' \in \mathbb{N}$, since $\beta(\alpha(p_m)) = p_m$ and $\beta$ is distance-preserving, we have

$$\begin{aligned} d_i(p_m, p_{g(f(m,n),n')}) &= d_i(\beta(\alpha(p_m)), p_{g(f(m,n),n')}) \\ &\leq d_i(\beta(\alpha(p_m)), \beta(q_{f(m,n)})) + d_i(\beta(q_{f(m,n)}), p_{g(f(m,n),n')}) \\ &= d_j(\alpha(p_m), q_{f(m,n)}) + d_i(\beta(q_{f(m,n)}), p_{g(f(m,n),n')}) \\ &\leq 2^{-n} + 2^{-n'}. \end{aligned}$$

Similarly, we have $d_j(q_m, q_{f(g(m,n),n')}) \leq 2^{-n} + 2^{-n'}$. So condition (3) is satisfied.

($\Leftarrow$) Assume that there are $f, g : \mathbb{N}^2 \to \mathbb{N}$ that satisfy condition (1), (2) and (3). By condition (1), we have that for all $m \in \mathbb{N}$, $f_m$ is a Cauchy name in $M_j$ and $g_m$ is a Cauchy name in $M_i$, and so, in particular, $\lim_{n\to\infty} q_{f(m,n)}$ exists in $M_j$ and $\lim_{n\to\infty} p_{g(m,n)}$ exists in $M_i$. Define $\alpha(p_m) = \lim_{n\to\infty} q_{f(m,n)}$ and $\beta(q_m) = \lim_{n\to\infty} p_{g(m,n)}$. From condition (2), by taking $n, n' \to \infty$, we have that $\alpha$ is distance-preserving on the rational points $p_m$, and so it has an isometric extension to $M_i$, also denoted by $\alpha$. Similarly for $\beta$.

We claim that $\beta(\alpha(p_m)) = p_m$ for all $m \in \mathbb{N}$. Suppose for a contradiction that $\beta(\alpha(p_m)) \neq p_m$ for some $m \in \mathbb{N}$. Then, by continuity of $\beta$, $\lim_{n\to\infty} \beta(q_{f(m,n)}) = \beta(\alpha(p_m)) \neq p_m$. So there must be an $n \in \mathbb{N}$ such that $d_i(p_m, \beta(q_{f(m,n)})) > 2^{-n}$. Hence there is an $n' \in \mathbb{N}$ such that $d(p_m, p_{g(f(m,n),n')}) > 2^{-n} + 2^{-n'}$, contradicting condition (3). Therefore, $\beta(\alpha(p_m)) = p_m$ for all $m \in \mathbb{N}$. Similarly, we have $\alpha(\beta(q_m)) = q_m$ for all $m \in \mathbb{N}$. Thus, by continuity, $\alpha^{-1} = \beta$. We now conclude that $\alpha : M_i \to M_j$ is an isometry, and so $M_i \cong M_j$.

Now we have proved that $M_i \cong M_j$ if and only if there are functions $f, g : \mathbb{N}^2 \to \mathbb{N}$ that satisfy condition (1), (2) and (3). Thus, since condition (1), (2) and (3) are

$\Pi^0_1$ statements, it follows that "$M_i \cong M_j$" is a $\Sigma^1_1$ statement. Therefore, the set $\{(i,j) \in \mathbb{N}^2 : M_i \cong M_j\}$ is $\Sigma^1_1$. □

**Proposition 3.1.3.** *The set* $\{(i,j) \in \mathbb{N}^2 : M_i \cong M_j\}$ *is* $\Sigma^1_1$*-hard.*

*Proof.* We use the fact that there is a computable sequence $(H_e)_{e\in\mathbb{N}}$ of directed graphs such that the set $\{(i,j) \in \mathbb{N}^2 : H_i \cong H_j\}$ is $\Sigma^1_1$-complete (see [5]). By applying the coding method in Section 3.1 of [6] to the sequence $(H_e)_{e\in\mathbb{N}}$, we obtain a computable sequence $(G_e)_{e\in\mathbb{N}}$ of connected undirected graphs such that the set $J := \{(i,j) \in \mathbb{N}^2 : G_i \cong G_j\}$ is $\Sigma^1_1$-complete.

Let $I := \{(i,j) \in \mathbb{N}^2 : M_i \cong M_j\}$. To show that $I$ is $\Sigma^1_1$-hard, it is enough to show that $J$ is many-one reducible to $I$. Recall that we can think of a connected graph $G$ as the Polish metric space $(G, d_G)$ where $d_G$ is the shortest path metric. Then, by the *s-m-n* Theorem, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $e \in \mathbb{N}$, $M_{f(e)} = (G_e, d_{G_e})$.

For a graph $G$, we let $V_G$ denote the set of vertices in $G$, and $E_G$ denote the set of edges in $G$. We claim that for every $i, j \in \mathbb{N}$, $G_i \cong G_j$ (as graphs) $\iff M_{f(i)} \cong M_{f(j)}$.

($\Leftarrow$) Assume $M_{f(i)} \cong M_{f(j)}$ via an isometry $\alpha : (G_i, d_{G_i}) \to (G_j, d_{G_j})$. Then for every $u, v \in \mathbb{N}$,

$$(u,v) \in E_{G_i} \Leftrightarrow d_{G_i}(u,v) = 1 \Leftrightarrow d_{G_j}(\alpha(u),\alpha(v)) = 1 \Leftrightarrow (\alpha(u),\alpha(v)) \in E_{G_j}.$$

Therefore, $\alpha : G_i \to G_j$ is a graph isomorphism, and so $G_i \cong G_j$.

($\Rightarrow$) Assume $G_i \cong G_j$ via a graph isomorphism $\alpha : G_i \to G_j$. Then for any path $(v_0, v_1, \ldots, v_k)$ in $G_i$, the sequence $(\alpha(v_0), \alpha(v_1), \ldots, \alpha(v_k))$ is a path in $G_j$.

This implies that for every $u, v \in V_{G_i}$, $d_{G_j}(\alpha(u), \alpha(v)) \leq d_{G_i}(u, v)$. Similarly, since $\alpha^{-1} : G_j \to G_i$ is a graph isomorphism, we have that $d_{G_i}(\alpha^{-1}(x), \alpha^{-1}(y)) \leq d_{G_j}(x, y)$ for every $x, y \in V_{G_j}$, and so $d_{G_i}(u, v) \leq d_{G_j}(\alpha(u), \alpha(v))$ for every $u, v \in V_{G_i}$. Hence $d_{G_i}(u, v) = d_{G_j}(\alpha(u), \alpha(v))$ for every $u, v \in V_{G_i}$. Therefore, $\alpha : (G_i, d_{G_i}) \to (G_j, d_{G_j})$ is an isometry, and so $M_{f(i)} \cong M_{f(j)}$.

Define $g(i, j) := (f(i), f(j))$. Then $g$ is a computable function and by the claim, we have that $(i, j) \in J \iff g(i, j) \in I$ for every $i, j \in \mathbb{N}$. Therefore, $J$ is many-one reducible to $I$, and so $I$ is $\Sigma^1_1$-hard. □

By Proposition 3.1.2 and Proposition 3.1.3, we have the following.

**Theorem 3.1.4.** *The set* $\{(i, j) \in \mathbb{N}^2 : M_i \cong M_j\}$ *is* $\Sigma^1_1$*-complete.*

**Proposition 3.1.5.** *The set* $\{(i, j) \in \mathbb{N}^2 : M_i \hookrightarrow M_j\}$ *is* $\Sigma^1_1$.

*Proof.* From the argument in the proof of Proposition 3.1.2, we have that $M_i \hookrightarrow M_j$ if and only if there is a function $f : \mathbb{N}^2 \to \mathbb{N}$ that satisfies the following conditions.

(1) For all $m, n, l \in \mathbb{N}$,

$$d_j(q_{f(m,n)}, q_{f(m,n+l)}) \leq 2^{-n}.$$

(2) For all $m, m', n, n' \in \mathbb{N}$,

$$|d_i(p_m, p_{m'}) - d_j(q_{f(m,n)}, q_{f(m',n')})| \leq 2^{-n} + 2^{-n'}.$$

Thus, since conditions (1) and (2) are $\Pi^0_1$ statements, the set $\{(i, j) \in \mathbb{N}^2 : M_i \hookrightarrow M_j\}$ is $\Sigma^1_1$. □

**Proposition 3.1.6.** *For every computably presentable Polish metric space $X$, the sets $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ and $\{e \in \mathbb{N} : M_e \hookrightarrow X\}$ are $\Sigma^1_1$.*

*Proof.* Follows immediately from Proposition 3.1.5. □

**Proposition 3.1.7.** *The set $\{e \in \mathbb{N} : (\mathbb{N}, d_{\mathbb{N}}) \hookrightarrow M_e\}$ is $\Sigma^1_1$-complete, where $d_{\mathbb{N}}$ is the usual metric on $\mathbb{N}$.*

*Proof.* Let $I := \{e \in \mathbb{N} : (\mathbb{N}, d_{\mathbb{N}}) \hookrightarrow M_e\}$. Since $d_{\mathbb{N}}$ is computable, there is an $e_0 \in \mathbb{N}$ such that $d_{\mathbb{N}} = d_{e_0}$ (i.e. $d_{\mathbb{N}}$ is the metric induced by $\varphi_{e_0}$). So $M_{e_0} \cong (\mathbb{N}, d_{\mathbb{N}})$. Thus, by Proposition 3.1.6, $I$ is $\Sigma^1_1$.

To show $I$ is $\Sigma^1_1$-hard, it is enough to build a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that

$$T_e \text{ has an infinite path} \iff (\mathbb{N}, d_{\mathbb{N}}) \hookrightarrow X_e.$$

Fix a computable bijection $i \mapsto \sigma_i$ from $\mathbb{N}$ onto $\mathbb{N}^{<\mathbb{N}}$ such that $\sigma_0 = \lambda$. For each $e \in \mathbb{N}$, we define a pseudometric $d$ (depending on $e$) on $\mathbb{N}$ as follows.

- If $\sigma_i \notin T_e$, let $d(i, 0) := 0$ and $d(i, j) := d(0, j)$ for all $j \in \mathbb{N}$.
- If $\sigma_i, \sigma_j \in T_e$, let $d(i, j)$ be the length of the shortest path in $T_e$ from $\sigma_i$ to $\sigma_j$.
- Let $d(i, j) = d(j, i)$ for all $i, j \in \mathbb{N}$.

Let $X_e$ be the completion of $(\mathbb{N}, d)$. Note that the construction of $X_e$ is effective uniformly in $e$. Thus, $(X_e)_{e\in\mathbb{N}}$ is a computable sequence of Polish metric spaces.

It is easy to see that for every $e \in \mathbb{N}$,

$$T_e \text{ has an infinite path} \iff (\mathbb{N}, d_{\mathbb{N}}) \hookrightarrow X_e.$$

Therefore, $I$ is $\Sigma^1_1$-hard, and so it is $\Sigma^1_1$-complete. $\square$

**Theorem 3.1.8.** *The set* $\{(i,j) \in \mathbb{N}^2 : M_i \hookrightarrow M_j\}$ *is* $\Sigma^1_1$*-complete.*

*Proof.* Let $I := \{(i,j) \in \mathbb{N}^2 : M_i \hookrightarrow M_j\}$ and $J := \{e \in \mathbb{N} : (\mathbb{N}, d_{\mathbb{N}}) \hookrightarrow M_e\}$. Fix an index $e_0 \in \mathbb{N}$ such that $(\mathbb{N}, d_{\mathbb{N}}) \cong M_{e_0}$. So for every $e \in \mathbb{N}$, $e \in I \iff (e_0, e) \in J$. Thus, since $J$ is $\Sigma^1_1$-hard by Proposition 3.1.7, $I$ is also $\Sigma^1_1$-hard. Therefore, by Proposition 3.1.5, $I$ is $\Sigma^1_1$-complete. $\square$

## 3.2 Results on Finite Metric Spaces

For a finite computable metric space $X$, the embedding problem $M_e \hookrightarrow X$ is quite simple. This is because $\mathcal{P}(X)$ is finite, and so there are only finitely many metric spaces (up to isometry) that embeds into $X$.

**Theorem 3.2.1.** *For every computably presentable finite metric space* $X$*, the set* $\{e \in \mathbb{N} : M_e \hookrightarrow X\}$ *is* $\Pi^0_1$*-complete within PolSp, and so it is* $\Pi^0_2$*-complete.*

*Proof.* Let $(X, d_X)$ be a computably presentable finite metric space. Note that for all $e \in PolSp$,

$$\begin{aligned} M_e \hookrightarrow X &\iff M_e \text{ is finite and } (\exists f : M_e \to X)(f \text{ is distance-preserving}) \\ &\iff (\forall \text{ finite } Y \subseteq M_e)(\exists f : Y \to X)(f \text{ is distance-preserving}) \\ &\iff (\forall \text{ finite } Y \subseteq \mathbb{N})(\exists f : Y \to X)(\forall i,j \in Y)(d_e(i,j) = d_X(f(i), f(j))). \end{aligned}$$

Since $X$ and $Y$ are finite, the quantifier $(\exists f : Y \to X)$ is bounded. Therefore, $\{e : M_e \hookrightarrow X\}$ is $\Pi^0_1$ within $PolSp$.

It remains to show that $\{e : M_e \hookrightarrow X\}$ is $\Pi^0_1$-hard within $PolSp$. We use a similar argument as in the proof of Theorem 2.0.12. In fact, Theorem 2.0.12(2) is the special case of Theorem 3.2.1 when $|X| = 1$.

Let $A$ be a $\Pi^0_1$-complete set. Then there is a computable relation $R(e, y)$ such that for all $e \in \mathbb{N}$,

$$e \in A \iff \forall y R(e, y).$$

Since $X$ is finite, there is an $r \in \mathbb{N}$ such that $d_X(x, y) \neq r$ for all $x, y \in X$. (For example, we can choose any $r \in \mathbb{N}$ such that $r > diam(X)$.) So for any metric space $(Y, d_Y)$ such that $d_Y(a, b) = r$ for some $a, b \in Y$, we have $Y \not\hookrightarrow X$.

For each $e \in \mathbb{N}$, we effectively construct a computable metric space $(X_e, d)$ uniformly in $e$ as follows.

For each $y \in \mathbb{N}$, starting from $y = 0$, we check if $R(e, y)$ until we find (if ever) the least $y$ such that $\neg R(e, y)$.

For each $y \in \mathbb{N}$ such that $R(e, y)$, we let $d(i, j) := 0$ for all $i, j \in \{0, \ldots, y\}$.

Whenever we find (if ever) the least $y$ such that $\neg R(e, y)$, we let

- $d(i, y) = r$ for all $i \in \mathbb{N} \setminus \{y\}$,
- $d(y, y) = 0$,
- $d(i, j) = 0$ for all $i, j \in \mathbb{N} \setminus \{y\}$.

Then we stop the construction.

This ends the construction.

Let $(X_e, d)$ be the completion of the resulting pseudometric space $(\mathbb{N}, d)$. The key point is that, whenever we find (if ever) the least $y$ such that $\neg R(e, y)$, we make sure that there are $i, y \in X_e$ such that $d(i, y) = r$, which is not equal to any distance in $X$, and so $X_e \not\hookrightarrow X$.

Note that for all $e \in \mathbb{N}$,

$$e \in A \Longrightarrow \forall y R(e,y) \Longrightarrow (\forall i, j \in \mathbb{N})(d(i,j) = 0) \Longrightarrow (X_e, d) \cong \{0\} \Longrightarrow X_e \hookrightarrow X$$

$$e \notin A \Longrightarrow \exists y \neg R(e,y) \Longrightarrow (\exists i, y \in \mathbb{N})(d(i,y) = r) \Longrightarrow (\exists i, y \in X_e)(d(i,y) = r)$$
$$\Longrightarrow X_e \not\hookrightarrow X.$$

It follows that $A$ is many-one reducible to $\{e : M_e \hookrightarrow X\}$. Therefore, $\{e : M_e \hookrightarrow X\}$ is $\Pi^0_1$-hard within *PolSp*. □

**Corollary 3.2.2.** *Let $X = \{x\}$ be the one-point metric space.*

(1) *The set $\{e \in \mathbb{N} : M_e \cong X\} = \{e \in \mathbb{N} : M_e \hookrightarrow X\}$ is $\Pi^0_1$-complete within PolSp, and so it is $\Pi^0_2$-complete.*

(2) *The set $\{e \in \mathbb{N} : X \hookrightarrow M_e\} = PolSp$ is $\Pi^0_2$-complete.*

*Proof.* Since $X$ is the one-point metric space, it is clear that

$$\{e : M_e \cong X\} = \{e : M_e \hookrightarrow X\} \text{ and } \{e : X \hookrightarrow M_e\} = PolSp.$$

Then (1) follows from Theorem 3.2.1, and (2) follows from Theorem 2.0.1. □

Note that the embedding problem $M_e \hookrightarrow X$ is also simple for some infinite computable metric spaces $X$. For example, if $X$ contains an isometric copy of every Polish metric space (e.g. the Urysohn space $\mathbb{U}$, the space $C[0,1]$, etc), then $\{e \in \mathbb{N} : M_e \hookrightarrow X\} = PolSp$, which is $\Pi^0_2$-complete.

Another example is the metric space $(\mathbb{N}, \widetilde{d}_1)$ defined by $\widetilde{d}_1(i,j) = 1$ for all $i \neq j$. Then for all $e \in PolSp$,

$$M_e \hookrightarrow (\mathbb{N}, \widetilde{d}_1) \iff (\forall i,j \in \mathbb{N})(d_e(i,j) = 1 \vee d_e(i,j) = 0).$$

Therefore, $\{e : M_e \hookrightarrow (\mathbb{N}, \widetilde{d}_1)\}$ is $\Pi^0_1$ within $PolSp$. By the same argument as in the proof of Theorem 3.2.1 (we can choose any $r \in \mathbb{N} \setminus \{0,1\}$), $\{e : M_e \hookrightarrow (\mathbb{N}, \widetilde{d}_1)\}$ is $\Pi^0_1$-hard within $PolSp$, and so it is $\Pi^0_1$-complete within $PolSp$.

**Theorem 3.2.3.** *For every computably presentable finite metric space $X$ with $|X| > 1$, the set $\{e \in \mathbb{N} : M_e \cong X\}$ is $d$-$\Sigma^0_1$-complete within PolSp, and so it is $\Pi^0_2$-complete.*

*Proof.* Let $X$ be a computably presentable finite metric space with $|X| > 1$, say $X = \{x_0, \ldots, x_{n-1}\}$. Since $|X| = n$, we have that for all $e \in PolSp$,

$$M_e \cong X \iff |M_e| = n \wedge M_e \hookrightarrow X.$$

By Theorem 3.2.1, $\{e : M_e \hookrightarrow X\}$ is $\Pi^0_1$ within $PolSp$. By Theorem 2.0.13, $\{e : |M_e| = n\}$ is $d$-$\Sigma^0_1$ within $PolSp$. Therefore, $\{e : M_e \cong X\}$ is $d$-$\Sigma^0_1$ within $PolSp$.

It remains to show that $\{e : M_e \cong X\}$ is $d$-$\Sigma^0_1$-hard within $PolSp$. Let $\widetilde{X}$ be the Polish metric space obtained by removing the point $x_{n-1}$ from $X$. So we have

$\widetilde{X} = \{x_0, x_1, \ldots, x_{n-2}\}$ and $|\widetilde{X}| = n-1$. Since $X$ is computably presentable, so is $\widetilde{X}$. Let $C$ be a $d$-$\Sigma^0_1$ set. Then $C = A \setminus B$ where $A$ and $B$ are $\Sigma^0_1$ sets. Then there are computable relations $R_A(e, y)$ and $R_B(e, y)$ such that for all $e \in \mathbb{N}$,

$$e \in A \Longleftrightarrow (\exists y \geq n-1) R_A(x, y),$$
$$e \in B \Longleftrightarrow (\exists y \geq n-1) R_B(x, y).$$

Since $X$ is finite, there is an $r \in \mathbb{N}$ such that $d_X(x, y) < r$ for all $x, y \in X$.

We construct a computable sequence $(X_e)_{e \in \mathbb{N}}$ of Polish metric spaces by using the same construction as in the proof of Theorem 2.0.13, but we work with $\widetilde{X}$ instead of $X$, and we let $y_A$ be a copy of the point $x_{n-1}$ in $X$ instead of just some new point. That is, we modify Step 2 and leave Step 1 and Step 3 the same.

For each $e \in \mathbb{N}$, we effectively construct a computable metric space $(X_e, d)$ uniformly in $e$ as follows.

Step 1: Let $d(i, j) = d_{\widetilde{X}}(x_i, x_j)$ for all $i, j \in \{0, \ldots, n-2\}$.

Step 2: For each $y \geq n-1$, starting from $y = n-1$, we check if $R_A(e, y)$ until we find (if ever) the least $y \geq n-1$ such that $R_A(e, y)$.

For each $y \geq n-1$ such that $\neg R_A(e, y)$, we let

- $d(i, j) = 0$ for all $i, j \in \{n-1, \ldots, y\}$,
- $d(i, j) = d(i, n-2) = d_{\widetilde{X}}(x_i, x_{n-2})$ for all $i \leq n-2$ and $j \in \{n-1, \ldots, y\}$.

Whenever we find (if ever) the least $y \geq n-1$ such that $R_A(e, y)$, we call it $y_A$ and let

- $d(i, y_A) = d_{\widetilde{X}}(i, x_{n-1})$ for all $i \in \{0, \ldots, y_A - 1\}$,
- $d(y_A, y_A) = 0$.

Then we go to Step 3.

Step 3: We have found $y_A$ from Step 2. For each $y \geq y_A + 1$, starting from $y = y_A + 1$, we check if $R_B(e, y)$ until we find (if ever) the least $y \geq y_A + 1$ such that $R_B(e, y)$.

For each $y \geq y_A + 1$ such that $\neg R_B(e, y)$, we let

- $d(i, j) = 0$ for all $i, j \in \{y_A + 1, \ldots, y\}$,
- $d(i, j) = d(i, y_A)$ for all $i \in \{0, \ldots, y_A\}$ and $j \in \{y_A + 1, \ldots, y\}$.

Whenever we find (if ever) the least $y \geq y_A + 1$ such that $R_B(e, y)$, we call it $y_B$ and let

- $d(i, j) = d(i, y_B) = r$ for all $i < y_B$ and $j \geq y_B$,
- $d(i, j) = 0$ for all $i, j \geq y_B$.

Then we stop the construction.

This ends the construction.

From the construction, we have that for all $e \in \mathbb{N}$,

- $e \notin A \Longrightarrow (X_e, d) \cong (\widetilde{X}, d_{\widetilde{X}}) \Longrightarrow |X_e| = |\widetilde{X}| = n - 1 < |X| \Longrightarrow X_e \ncong X$,
- $e \in A \setminus B \Longrightarrow X_e \cong \widetilde{X} \sqcup \{y_A\} \Longrightarrow X_e \cong X$,
- $e \in A \cap B \Longrightarrow X_e \cong \widetilde{X} \sqcup \{y_A, y_B\} \Longrightarrow |X_e| = |\widetilde{X}| + 2 = n + 1 > |X| \Longrightarrow X_e \ncong X$.

We conclude that for all $e \in \mathbb{N}$,

$$e \in A \setminus B \iff (X_e, d) \cong (X, d_X).$$

It follows that $C = A \setminus B$ is many-one reducible to $\{e : M_e \cong X\}$. Therefore, $\{e : M_e \cong X\}$ is $d$-$\Sigma^0_1$-hard within *PolSp*. □

To find the complexity of the embedding problem $X \hookrightarrow M_e$ when $1 < |X| < \infty$, we use the following theorem as the main tool.

**Theorem 3.2.4.** *Fix an effective list $(T_e)_{e\in\mathbb{N}}$ of all primitive recursive trees $T_e \subseteq \mathbb{N}^{<\mathbb{N}}$. Let $r > 0$ be a computable real. Then there is a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for every $e \in \mathbb{N}$, the following conditions hold:*

(1) *$X_e$ is the completion of the tree $T_e$, where we add weights to $T_e$ and use the shortest path metric $d$.*

(2) *$d(\lambda, x) \leq r$ for all $x \in X_e$.*

(3) *If $T_e$ has an infinite path, then there is an $\widehat{x} \in X_e$ such that $d(\lambda, \widehat{x}) = r$.*

(4) *If $T_e$ has no infinite paths, then there are no $x, y \in X_e$ such that $d(x, y) = r$, and so $d(\lambda, x) < r$ for all $x \in X_e$.*

*Proof.* First, we show the following.

**Claim 1.** There is a computable strictly increasing sequence $(r_n)_{n\in\mathbb{N}}$ of rationals converging to $r$ such that $r_0 = 0$ and $r_{n+1} > r_n + \frac{1}{2}(r - r_n)$ for all $n \in \mathbb{N}$.

Since $r > 0$ is a computable real, there is a computable strictly increasing sequence $(q_n)_{n\in\mathbb{N}}$ of rationals converging to $r$ such that $q_0 = 0$ and $q_n < r < q_n + 2^{-n}$ for all $n > 0$.

We define a subsequence $(r_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ by induction as follows.

Let $r_0 := q_0 = 0$.

Now let $n \in \mathbb{N}$ and assume that we have defined $r_n$ such that $r_n = q_m$ for some $m \in \mathbb{N}$. Suppose that $(\forall N > m)(\forall k \geq 1)(q_N \leq q_m + \frac{1}{2}(q_k + 2^{-k} - q_m))$. Then, by taking limit $k \to \infty$, we have that $(\forall N > m)(q_N \leq q_m + \frac{1}{2}(r - q_m))$. Taking $N \to \infty$, we have $r \leq q_m + \frac{1}{2}(r - q_m)$, but $q_m + \frac{1}{2}(r - q_m) < q_m + (r - q_m) = r$, a contradiction. So there must be the least pair $(N, k)$ such that $N > m$, $k \geq 1$ and $q_N > q_m + \frac{1}{2}(q_k + 2^{-k} - q_m)$. Let $r_{n+1} := q_N$.

Since $k \geq 1$, we have $q_k + 2^{-k} > r$, and so

$$r_{n+1} = q_N > q_m + \frac{1}{2}(q_k + 2^{-k} - q_m) > q_m + \frac{1}{2}(r - q_m) = r_n + \frac{1}{2}(r - r_n).$$

This ends the construction of $(r_n)_{n\in\mathbb{N}}$.

From the construction, it is clear that $(r_n)_{n\in\mathbb{N}}$ is a subsequence of $(q_n)_{n\in\mathbb{N}}$. So $(r_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of rationals converging to $r$. It is also clear that $r_{n+1} > r_n + \frac{1}{2}(r - r_n)$ for all $n \in \mathbb{N}$.

This ends the proof of Claim 1.

For each $i \in \mathbb{N}$, let $w_i := r_{i+1} - r_i > 0$. Then $\sum_{i=0}^{n} w_i = r_{n+1}$ for all $n \in \mathbb{N}$ and $\sum_{i=0}^{\infty} w_i = \lim_{n\to\infty} r_n = r$.

Note that $(w_n)_{n\in\mathbb{N}}$ is strictly decreasing. This is because for all $n \in \mathbb{N}$, since

$r_{n+1} > r_n + \frac{1}{2}(r - r_n)$, we have $2r_{n+1} > 2r_n + r - r_n = r_n + r$, and so

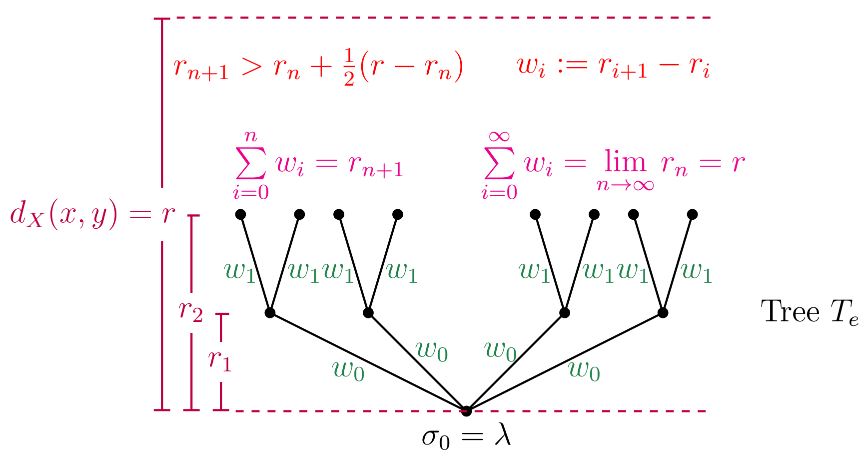
$$w_{n+1} = r_{n+2} - r_{n+1} < r - r_{n+1} < r_{n+1} - r_n = w_n.$$

Fix an effective list $(\sigma_i)_{i\in\mathbb{N}}$ of all finite strings in $\mathbb{N}^{<\mathbb{N}}$ with $\sigma_0 = \lambda$. We construct a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces as follows.

For each $e \in \mathbb{N}$, we put weight $w_n$ on each edge between level $n$ and level $n+1$ in the tree $T_e$. Then we define a pseudometric $d$ (depending on $e$) on $\mathbb{N}$ by

- if $\sigma_i \notin T_e$, let $d(i, 0) := 0$ and $d(i, j) := d(0, j)$ for all $j \in \mathbb{N}$, that is, we identify every $\sigma_i \in \mathbb{N}^{<\mathbb{N}} \setminus T_e$ with $\sigma_0 = \lambda$,
- if $\sigma_i, \sigma_j \in T_e$, let $d(i, j) :=$ the length of the shortest path in $T_e$ from $\sigma_i$ to $\sigma_j$.

Let $(X_e, d)$ be the completion of the resulting pseudometric space $(\mathbb{N}, d)$. Therefore, $X_e$ is the completion of the tree $T_e$ with the shortest path metric $d$, where we put weight $w_n$ on each edge between level $n$ and level $n+1$. Note that the height of each weighted tree $T_e$ is at most $\sum_{i=0}^{\infty} w_i = r$.



We can compute the distances $d(i,j)$ in term of $w_k$'s and $r_n$'s as follows. For any $\sigma_i, \sigma_j \in T_e$, if $\tau$ is the longest common initial segment of $\sigma_i$ and $\sigma_j$, i.e. the longest string in $T_e$ such that $\tau \subseteq \sigma_i$ and $\tau \subseteq \sigma_j$, then we have that

$$d(i,j) = \sum_{k=|\tau|}^{|\sigma_i|-1} w_k + \sum_{k=|\tau|}^{|\sigma_j|-1} w_k = (r_{|\sigma_i|} - r_{|\tau|}) + (r_{|\sigma_j|} - r_{|\tau|}) = r_{|\sigma_i|} + r_{|\sigma_j|} - 2r_{|\tau|}.$$

(If $|\tau| = |\sigma_i|$, then $\sum_{k=|\tau|}^{|\sigma_i|-1} w_k = 0$.)

In particular, for any $\sigma_i, \sigma_j \in T_e$, if $\sigma_i \subseteq \sigma_j$, then $\tau = \sigma_i$, and so

$$d(i,j) = \sum_{k=|\sigma_i|}^{|\sigma_j|-1} w_k = r_{|\sigma_j|} - r_{|\sigma_i|}.$$

**Claim 2.** $d(\lambda, x) \leq r$ for all $x \in X_e$.

Since $\sigma_0 = \lambda$, $\lambda$ is the rational point 0 in $(\mathbb{N}, d)$. Note that for all $i \in \mathbb{N}$,

$$d(\lambda, i) = \sum_{k=0}^{|\sigma_i|-1} w_k = r_{|\sigma_i|} < r.$$

Thus, since $(\mathbb{N}, d)$ is dense in $X_e$, we have that $d(\lambda, x) \leq r$ for all $x \in X_e$.

This ends the proof of Claim 2.

**Claim 3.** For all $e \in \mathbb{N}$, if $T_e$ has an infinite path, then there is an $\widehat{x} \in X_e$ such that $d(\lambda, \widehat{x}) = r$.

Let $e \in \mathbb{N}$ and assume that $T_e$ has an infinite path, say $g \in \mathbb{N}^{\mathbb{N}}$. Then for each

$n \in \mathbb{N}$, there is a unique $i_n \in \mathbb{N}$ such that $\sigma_{i_n} = g \upharpoonright n$. So $i_0 = 0$ and for all $m, n \in \mathbb{N}$,

$$d(i_m, i_n) = |r_{|\sigma_{i_m}|} - r_{|\sigma_{i_n}|}| = |r_{|g\upharpoonright m|} - r_{|g\upharpoonright n|}| = |r_m - r_n|.$$

Thus, since $(r_n)_{n\in\mathbb{N}}$ converges, $(i_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X_e, d)$. Hence, since $(X_e, d)$ is complete, $(i_n)_{n\in\mathbb{N}}$ converges to a point $\widehat{x} \in X_e$. Therefore,

$$d(\lambda, \widehat{x}) = \lim_{m\to\infty} d(i_0, i_m) = \lim_{m\to\infty} |r_0 - r_m| = |r_0 - r| = |0 - r| = r.$$

This ends the proof of Claim 3.

**Claim 4.** For all $e \in \mathbb{N}$, if $T_e$ has no infinite paths, then there are no $x, y \in X_e$ such that $d(x, y) = r$, and so $d(\lambda, x) < r$ for all $x \in X_e$.

Let $e \in \mathbb{N}$. Since $d(\lambda, x) \leq r$ for all $x \in X_e$, to prove Claim 4, we assume that there are $x, y \in X_e$ with $d(x, y) = r$, and then show that $T_e$ has an infinite path.

Since $x, y \in X_e$, there are Cauchy sequences $(i_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$ in $(\mathbb{N}, d)$ such that $(i_n)_{n\in\mathbb{N}}$ converges to $x$ in $X_e$ and $(j_n)_{n\in\mathbb{N}}$ converges to $y$ in $X_e$. Then $\lim\limits_{n\to\infty} d(i_n, j_n) = d(x, y) = r$.

From the definition of $d$, we identify each $\sigma_i \in \mathbb{N}^{<\mathbb{N}} \setminus T_e$ with $\sigma_0 = \lambda$. So we can assume without loss of generality that $\sigma_{i_n}, \sigma_{j_n} \in T_e$ for all $n \in \mathbb{N}$.

Note that $r_1 > r_0 + \frac{1}{2}(r - r_0) = \frac{1}{2}r$, and so $2r_1 > r$. Hence we can fix a $\delta \in \mathbb{Q}$ such that $0 < \delta < \min\{\frac{1}{2}r, 2r_1 - r\}$.

Then there is an $N \in \mathbb{N}$ such that

$$(\forall m, n \geq N)(|d(i_n, j_n) - r| < \delta \wedge d(i_m, i_n) < \delta \wedge d(j_m, j_n) < \delta). \tag{4.1}$$

**Claim 4.1.** For all $i, j \in \mathbb{N}$, if $\sigma_i, \sigma_j \in T_e$ are incomparable, then $|d(i,j) - r| > \delta$.

Let $i, j \in \mathbb{N}$ and assume $\sigma_i, \sigma_j \in T_e$ are incomparable. Then $|\sigma_i| \geq 1$ and $|\sigma_j| \geq 1$. Let $\tau$ be the longest common initial segment of $\sigma_i$ and $\sigma_j$.

Case $\tau = \lambda$: Then $|\tau| = 0$, and so

$$d(i,j) = r_{|\sigma_i|} + r_{|\sigma_j|} - 2r_{|\tau|} = r_{|\sigma_i|} + r_{|\sigma_j|} - 2r_0 = r_{|\sigma_i|} + r_{|\sigma_j|} \geq r_1 + r_1 = 2r_1.$$

Hence $d(i,j) - r \geq 2r_1 - r > \delta$, and so $|d(i,j) - r| > \delta$.

Case $\tau \neq \lambda$: Then $|\tau| \geq 1$, and so

$$d(i,j) = r_{|\sigma_i|} + r_{|\sigma_j|} - 2r_{|\tau|} < r + r - 2r_{|\tau|} \leq r + r - 2r_1,$$

Hence $r - d(i,j) > 2r_1 - r > \delta$, and so $|d(i,j) - r| > \delta$.

This ends the proof of Claim 4.1.

**Claim 4.2.** For all $n \geq N$,

$$\text{either } (\lambda = \sigma_{i_n} \subsetneq \sigma_{j_n} \wedge i_n = 0 \wedge j_n > 0) \text{ or } (\lambda = \sigma_{j_n} \subsetneq \sigma_{i_n} \wedge j_n = 0 \wedge i_n > 0).$$

Let $n \geq N$. By (4.1), $|d(i_n, j_n) - r| < \delta$, and so, by Claim 4.1, $\sigma_{i_n}$ and $\sigma_{j_n}$ are comparable.

<u>Case $\sigma_{i_n} \subseteq \sigma_{j_n}$</u>: If $\sigma_{i_n} \neq \lambda$, then

$$d(i_n, j_n) = r_{|\sigma_{j_n}|} - r_{|\sigma_{i_n}|} < r - r_{|\sigma_{i_n}|} \leq r - r_1 < r - \frac{1}{2}r < r - \delta,$$

but $|d(i_n, j_n) - r| < \delta$, a contradiction. Hence $\sigma_{i_n} = \lambda = \sigma_0$, and so $i_n = 0$.

If $\sigma_{j_n} = \lambda$, then $j_n = 0 = i_n$, and so

$$|d(i_n, j_n) - r| = |0 - r| = r > \frac{1}{2}r > \delta,$$

but $|d(i_n, j_n) - r| < \delta$, a contradiction. Hence $\sigma_{j_n} \neq \lambda$. Therefore, $j_n > 0$ and $\lambda = \sigma_{i_n} \subsetneq \sigma_{j_n}$.

<u>Case $\sigma_{j_n} \subseteq \sigma_{i_n}$</u>: Similarly, we have that $\lambda = \sigma_{j_n} \subsetneq \sigma_{i_n}$, $j_n = 0$, and $i_n > 0$.

This ends the proof of Claim 4.2.

By Claim 4.2, without loss of generality, we can assume that $\lambda = \sigma_{i_N} \subsetneq \sigma_{j_N}$, $i_N = 0$, and $j_N > 0$.

If $\sigma_{i_n} \neq \lambda$ for some $n \geq N$, then

$$d(i_n, i_N) = r_{|\sigma_{i_n}|} - r_{|\sigma_{i_N}|} = r_{|\sigma_{i_n}|} - r_0 = r_{|\sigma_{i_n}|} \geq r_1 > \frac{1}{2}r > \delta,$$

which contradicts (4.1). So $\sigma_{i_n} = \lambda$ for all $n \geq N$. Thus, by Claim 4.2, we have that for all $n \geq N$, $\lambda = \sigma_{i_n} \subsetneq \sigma_{j_n}$, $i_n = 0$, and $j_n > 0$. So for all $n \geq N$, $d(i_n, j_n) = d(0, j_n) = r_{|\sigma_{j_n}|} < r$. Thus, since $\lim_{n\to\infty} r_{|\sigma_{j_n}|} = \lim_{n\to\infty} d(i_n, j_n) = r$ and $(r_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence converging to $r$, we must have that $\lim_{n\to\infty} |\sigma_{j_n}| = \infty$.

For each $k \in \mathbb{N}$, since $w_k = r_{k+1} - r_k > 0$ and $(j_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in

$(\mathbb{N}, d)$, we have that there is an $N_k \in \mathbb{N}$ such that $d(j_m, j_n) < w_k$ for all $m, n \geq N_k$. We can choose $N_k$ so that $N < N_0 < N_1 < \dots$.

For each $m, n \in \mathbb{N}$, let $\tau_{m,n}$ be the longest common initial segment of $\sigma_{j_m}$ and $\sigma_{j_n}$. Note that for all $k \in \mathbb{N}$ and $m, n \geq N_k$, if $\sigma_{j_m} \neq \sigma_{j_n}$, then $|\sigma_{j_m}| > |\tau_{m,n}|$ or $|\sigma_{j_n}| > |\tau_{m,n}|$, so

$$w_k > d(j_m, j_n) = \sum_{i=|\tau_{m,n}|}^{|\sigma_{j_m}|-1} w_i + \sum_{i=|\tau_{m,n}|}^{|\sigma_{j_m}|-1} w_i \geq w_{|\tau_{m,n}|},$$

and so, since $(w_i)_{i\in\mathbb{N}}$ is strictly decreasing, we have $|\tau_{m,n}| > k$. Therefore,

$$(\forall k \in \mathbb{N})(\forall m, n \geq N_k)(\sigma_{j_m} \neq \sigma_{j_n} \Longrightarrow |\tau_{m,n}| > k). \tag{4.2}$$

**Claim 4.3.** There is a sequence $(\rho_k)_{k\in\mathbb{N}} \subseteq T_e$ such that

(i) $(\forall k \in \mathbb{N})(\forall n \geq N_k)(\rho_k \subseteq \sigma_{j_n})$,

(ii) $|\rho_k| = k + 1$,

(iii) $(\forall k > 0)(\rho_{k-1} \subsetneq \rho_k)$.

We prove Claim 4.3 by induction on $k$ as follows.

Let $k \in \mathbb{N}$ and assume by induction that we have defined $\rho_0, \dots, \rho_{k-1} \in T_e$ that satisfy (i),(ii) and (iii).

Recall that $\lim_{n\to\infty} |\sigma_{j_n}| = \infty$. So there is an $m > N_k$ such that $|\sigma_{j_m}| > |\sigma_{j_{N_k}}|$, and so $\sigma_{j_{N_k}} \neq \sigma_{j_m}$. Since $m \geq N_k$ and $\sigma_{j_{N_k}} \neq \sigma_{j_m}$, by (4.2), we have $|\tau_{N_k,m}| > k$. Thus, since $\tau_{N_k,m} \subseteq \sigma_{j_{N_k}}$, we have $|\sigma_{j_{N_k}}| \geq |\tau_{N_k,m}| > k$.

Hence we can define $\rho_k := \sigma_{j_{N_k}} \upharpoonright (k+1) \in T_e$. Then $\rho_k \subseteq \sigma_{j_{N_k}}$ and $|\rho_k| = k+1$.

We claim that $\rho_k \subseteq \sigma_{j_n}$ for all $n \geq N_k$. Suppose for a contradiction that $\rho_k \not\subseteq \sigma_{j_n}$ for some $n \geq N_k$. Then, since $\rho_k \subseteq \sigma_{j_{N_k}}$, we have $\sigma_{j_{N_k}} \neq \sigma_{j_n}$ and $\tau_{N_k,n} \subsetneq \rho_k$. So $|\tau_{N_k,n}| \leq |\rho_k| - 1 = (k+1) - 1 = k$. But, since $n \geq N_k$ and $\sigma_{j_{N_k}} \neq \sigma_{j_n}$, by (4.2), we have $|\tau_{N_k,n}| > k$, a contradiction. Therefore, $\rho_k \subseteq \sigma_{j_n}$ for all $n \geq N_k$.

Since $\rho_{k-1}$ satisfies (ii), $|\rho_{k-1}| = k$. Since $\rho_k \subseteq \sigma_{j_{N_k}}$, $\rho_{k-1} \subseteq \sigma_{j_{N_k}}$ and $|\rho_{k-1}| = k < k+1 = |\rho_k|$, we must have $\rho_{k-1} \subsetneq \rho_k$.

This ends the proof of Claim 4.3.

By Claim 4.3, $\bigcup_{k\in\mathbb{N}} \rho_k$ is an infinite path in $T_e$. So we have proved Claim 4.

We conclude that the sequence $(X_e)_{n\in\mathbb{N}}$ satisfies conditions (1)-(4), and this completes the proof. $\square$

**Theorem 3.2.5.** *For every computably presentable finite metric space $X$ with $|X| > 1$, the set $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-complete.*

*Proof.* Let $(X, d_X)$ be a computably presentable finite metric space $X$ with $|X| > 1$. By Proposition 3.1.6, $\{e : X \hookrightarrow M_e\}$ is $\Sigma^1_1$.

To show that $\{e : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-hard, we consider three cases:

Case 1: $|X| = 2$.

Case 2: $|X| > 2$ and $X$ satisfies the strict triangle inequality, that is, for any distinct $a, b, c \in X$, $d_X(a,b) < d_X(a,c) + d_X(c,b)$.

Case 3: $X$ does not satisfy the strict triangle inequality.

**Case 1:** $|X| = 2$

Let $X = \{x, y\}$. Since $X$ is computably presentable, $r := d_X(x, y) > 0$ is a computable real. Let $(X_e)_{e\in\mathbb{N}}$ be the computable sequence of Polish metric spaces obtained from Theorem 3.2.4. Then for all $e \in \mathbb{N}$,

$$\begin{aligned} T_e \text{ has an infinite path} &\iff (\exists a, b \in X_e)(d(a, b) = r) \\ &\iff X \hookrightarrow X_e. \end{aligned}$$

Therefore, $\{e : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-hard.

This ends the proof for the case $|X| = 2$.

**Case 2:** $|X| > 2$ **and** $X$ **satisfies the strict triangle inequality**

Let

$$\begin{aligned} &\delta_X := \min\{d_X(a, b) : a \neq b \in X\}, \\ &\varepsilon_X := \min\{d_X(a, c) + d_X(c, b) - d_X(a, b) : a, b, c \in X \text{ are distinct}\}. \end{aligned}$$

Since $1 < |X| < \infty$, $\delta_X > 0$. Since $X$ satisfies the strict triangle inequality and $X$ is finite, $\varepsilon_X > 0$. So we can fix an $l \in \mathbb{Q}$ such that $0 < l < \min\{\frac{1}{2}\delta_X, \frac{1}{2}\varepsilon_X\}$.

For any distinct $a, b \in X$, let $I_{a,b} := d_X(a, b) - 2l$. We construct a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces as follows.

First, we shrink the metric space $X$ by $l$ to obtain a finite computable metric space $(\widetilde{X}, d)$ where $\widetilde{X} = \{\widetilde{x} : x \in X\}$ and $d$ is a metric defined by

$$d(\widetilde{a}, \widetilde{b}) := I_{a,b} = d_X(a, b) - 2l \quad \text{for all distinct } a, b \in X.$$

($d(\widetilde{a}, \widetilde{b}) := 0$ if $a = b$.)

We need to check that $d$ is a metric. Since $l < \frac{1}{2}\delta_X$, we have that for all distinct $a, b \in X$,

$$d(\widetilde{a}, \widetilde{b}) = d_X(a, b) - 2l \geq \delta_X - 2l > 0.$$

Since $l < \frac{1}{2}\varepsilon_X$, we have that for all distinct $a, b, c \in X$,

$$\begin{aligned} 2l < \varepsilon_X &\leq d_X(a, c) + d_X(c, b) - d_X(a, b), \text{and so} \\ d(\widetilde{a}, \widetilde{b}) &= d_X(a, b) - 2l \\ &< (d_X(a, c) + d_X(c, b) - 2l) - 2l \\ &= (d_X(a, c) - 2l) + (d_X(c, b) - 2l) \\ &= d(\widetilde{a}, \widetilde{c}) + d(\widetilde{c}, \widetilde{b}). \end{aligned}$$

It follows that $d$ is a metric on $\widetilde{X}$.

Note that $d(\widetilde{a}, \widetilde{b})$ is a computable real uniformly in $a, b$ because $d_X(a, b)$ is a computable real uniformly in $a, b$.

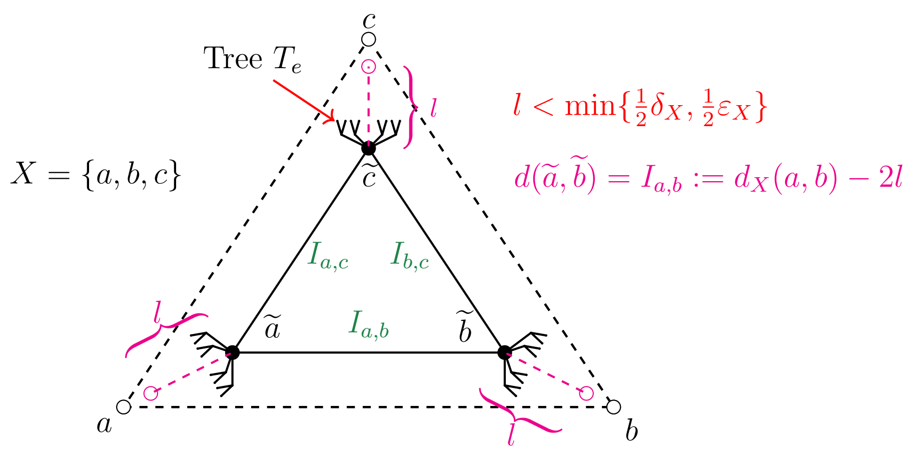
For each $a \in X$, we attach the weighted tree $T_{e,a}$ constructed in the proof of Theorem 3.2.4 to the point $\widetilde{a}$ by using $l$ as the maximum height of $T_e$. Let $(X_e, d)$ be the completion of the resulting space with the shortest path metric $d$.

More formally, we construct $(X_e, d)$ as follows.

For each $a \in X$, let $T_{e,a}$ be the weighted tree constructed in the proof of Theorem 3.2.4 by adding weights to $T_e$, using the point $\widetilde{a}$ as the root and using $l$ as the maximum height of $T_{e,a}$.

We can think of the metric space $(\widetilde{X}, d)$ as the weighted graph where the set of vertices is $\{\widetilde{a} : a \in X\}$ and every distinct $a, b \in X$ are connected by an edge of weight $d(\widetilde{a}, \widetilde{b})$. Note that $d$ is the shortest path metric in $X$.

Let $(X_e, d)$ be the completion of the weighted graph $\bigsqcup_{a\in X} T_{e,a}$, equipped with the shortest path metric $d$.




By the proof of Theorem 3.2.4, we have that for each $a \in X$, there is a computable metric space $(Y_{e,a}, d_{e,a})$ that satisfies the following conditions:

(1) $(Y_{e,a}, d_{e,a})$ is the completion of the tree $T_{e,a}$, equipped with the shortest path metric $d_{e,a}$.

(2) $d_{e,a}(\widetilde{a}, x) \leq l$ for all $x \in Y_{e,a}$.

(3) If $T_e$ has an infinite path, then there is an $\widehat{a} \in Y_{e,a}$ such that $d_{e,a}(\widetilde{a}, \widehat{a}) = l$.

(4) If $T_e$ has no infinite paths, then there are no $x, y \in Y_{e,a}$ such that $d_{e,a}(x, y) = l$, and for all $x \in Y_{e,a}$, $d_{e,a}(\widetilde{a}, x) < l$.

Then for all $x, y \in X_e$, there are $a, b \in X$ such that $x \in Y_{e,a}$ and $y \in Y_{e,b}$, and the distance between $x$ and $y$ is

$$d(x,y) = \begin{cases} d_{e,a}(x,y) & \text{if } a = b \\ d_{e,a}(x,\widetilde{a}) + d(\widetilde{a},\widetilde{b}) + d_{e,b}(\widetilde{b}, y) & \text{if } a \neq b \end{cases}$$

Since the construction of $X_e$ is effective uniformly in $e$, $(X_e)_{e\in\mathbb{N}}$ is a computable sequence of Polish metric spaces.

We claim that for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\Longleftrightarrow X \hookrightarrow X_e$.

($\Longrightarrow$) Assume that $T_e$ has an infinite path. By (3), for each $a \in X$, there is an $\widehat{a} \in Y_{e,a}$ such that $d_{e,a}(\widetilde{a}, \widehat{a}) = l$. Then for all distinct $a, b \in X$,

$$\begin{aligned} d(\widehat{a},\widehat{b}) &= d_{e,a}(\widehat{a},\widetilde{a}) + d(\widetilde{a},\widetilde{b}) + d_{e,b}(\widetilde{b},\widehat{b}) \\ &= l + (d_X(a,b) - 2l) + l \\ &= d_X(a,b). \end{aligned}$$

Therefore, $X \hookrightarrow X_e$ via the isometric embedding $f : a \mapsto \widehat{a}$.

($\Longleftarrow$) Assume that $T_e$ has no infinite paths. Since $X$ is finite, $diam(X) = \max\{d_X(x, y) : x, y \in X\} < \infty$.

We claim that for all $x, y \in X_e$, $d(x, y) < diam(X)$.

Let $x, y \in X_e$. Then there are $a, b \in X$ such that $x \in Y_{e,a}$ and $y \in Y_{e,b}$. Since $T_e$ has no infinite paths, by (4), $d_{e,a}(\widetilde{a}, x) < l$ and $d_{e,b}(\widetilde{b}, y) < l$.

If $a = b$, then

$$d(x, y) = d_{e,a}(x, y) \leq d_{e,a}(x, \widetilde{a}) + d_{e,a}(\widetilde{a}, y) < l + l < \delta_X \leq diam(X).$$

If $a \neq b$, then

$$\begin{aligned} d(x, y) &= d_{e,a}(x, \widetilde{a}) + d(\widetilde{a}, \widetilde{b}) + d_{e,b}(\widetilde{b}, y) < l + d(\widetilde{a}, \widetilde{b}) + l \\ &= l + (d_X(a, b) - 2l) + l = d_X(a, b) \leq diam(X). \end{aligned}$$

Therefore, $d(x, y) < diam(X)$ for all $x, y \in X_e$.

Since $X$ is finite, there are $a, b \in X$ such that $d_X(a, b) = diam(X)$. Thus, if $X \hookrightarrow X_e$ via an isometric embedding $f$, then $f(a), f(b) \in X_e$ and $d(f(a), f(b)) = d_X(a, b) = diam(X)$, but $d(x, y) < diam(X)$ for all $x, y \in X_e$, a contradiction. Therefore, $X \not\hookrightarrow X_e$.

We conclude that for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\iff X \hookrightarrow X_e$. Therefore, $\{e : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-hard.

This ends the proof for the case $|X| > 2$ and $X$ satisfies the strict triangle inequality.

**Case 3: $X$ does not satisfy the strict triangle inequality**

Since $X$ does not satisfy the strict triangle inequality, there are distinct $a, b, c \in X$

such that

$$d_X(a,b) = d_X(a,c) + d_X(c,b).$$

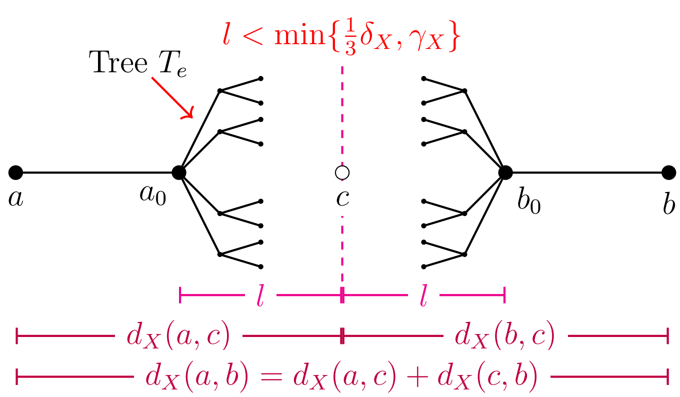
Let

$$\delta_X := \min\{d_X(x,y) : x \neq y \in X\},$$
$$\gamma_X := \min\{d_X(a,c) - d_X(x,c) : d_X(a,c) > d_X(x,c) \text{ and } x \in X \setminus \{c\}\}.$$

Since $1 < |X| < \infty$, $\delta_X > 0$ and $\gamma_X > 0$. So we can fix an $l \in \mathbb{Q}$ such that $0 < l < \min\{\frac{1}{3}\delta_X, \gamma_X\}$.

We will construct a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\iff X \hookrightarrow X_e$.

The idea of how to build $(X_e, d)$ is as follows. First, we remove the point $c$ from $X$, and add two new points $a_0$ and $b_0$. Define $d(a, a_0) := d_X(a,c) - l$ and $d(b, b_0) := d_X(b,c) - l$. Then, at the points $a_0$ and $b_0$, we attach the weighted tree $T_e$ constructed in the proof of Theorem 3.2.4 by using $l$ as the maximum height of $T_e$. We can define a metric $d$ on the resulting space such that $d$ "looks like" the shortest path metric. Let $(X_e, d)$ be the completion of the resulting metric space.



More formally, we construct $(X_e, d)$ as follows.

We use the same setting as in the proof of Theorem 3.2.4, where we use $l$ as the maximum height instead of $r$. So we have the following:

- $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ is a computable strictly increasing sequence converging to $l$ such that $r_0 = 0$ and $r_{n+1} > r_n + \frac{1}{2}(r - r_n)$ for all $n \in \mathbb{N}$.
- For each $i \in \mathbb{N}$, $w_i := r_{i+1} - r_i > 0$.
- $\sum_{i=0}^{n} w_i = r_{n+1}$ for all $n \in \mathbb{N}$ and $\sum_{i=0}^{\infty} w_i = \lim_{n\to\infty} r_n = l$.
- We think of $T_e$ as the weighted tree where we put the weight $w_n$ on each edge between level $n$ and level $n+1$. Then we equip $T_e$ with the shortest path metric.
- The maximum height of $T_e$ is $\sum_{i=0}^{\infty} w_i = l$.

Let $T_{e,a}$ be the weighted tree $T_e$ with the point $a_0$ as the root. We define $T_{e,b}$ in the same way. ($T_{e,a}$ and $T_{e,b}$ are disjoint.)

By the proof of Theorem 3.2.4, we have that there is a computable metric space $(Y_{e,a}, d_{e,a})$ that satisfies the following conditions:

(1) $(Y_{e,a}, d_{e,a})$ is the completion of the tree $T_{e,a}$, equipped with the shortest path metric $d_{e,a}$.

(2) $d_{e,a}(a_0, x) \le l$ for all $x \in Y_{e,a}$.

(3) If $T_e$ has an infinite path, then there is an $\widehat{a} \in Y_{e,a}$ such that $d_{e,a}(a_0, \widehat{a}) = l$.

(4) If $T_e$ has no infinite paths, then there are no $x, y \in Y_{e,a}$ such that $d_{e,a}(x, y) = l$, and for all $x \in Y_{e,a}$, $d_{e,a}(a_0, x) < l$.

Similarly, there is a computable metric space $(Y_{e,b}, d_{e,b})$ that satisfies (1)-(4).

To build $(X_e, d)$, we construct a weighted graph $G_e$ as follows.

Step 1: We start with the weighted graph $X \setminus \{c\}$, where for any distinct $x, y \in X \setminus \{c\}$, we put an edge of weight $d_X(x, y)$ between $x$ and $y$.

Step 2: We put two new points $a_0$ and $b_0$ into $X \setminus \{c\}$. Then put an edge of weight $d_X(a, c) - l$ between $a$ and $a_0$, and put an edge of weight $d_X(b, c) - l$ between $b$ and $b_0$.

Step 3: We attach the weighted trees $T_{e,a}$ and $T_{e,b}$ at $a_0$ and $b_0$, respectively.

Step 4: Let $G_e$ be the resulting weighted graph. Then the set of vertices of $G_e$ is

$$V(G_e) = (X \setminus \{c\}) \sqcup T_{e,a} \sqcup T_{e,b}.$$

Fix an index $e_0 \in \mathbb{N}$ such that $T_{e_0} = \mathbb{N}^{<\mathbb{N}}$.

We define a weighted graph $\widetilde{G}$ (does not depend on $e$) as follows.

Step 1: We start with the weighted graph $G_{e_0}$.

Step 2: For each $f \in \mathbb{N}^{\mathbb{N}}$, we add a new point $\widetilde{f}$ to $G_{e_0}$, and for each $x \in X \setminus \{a, b, c\}$, we put an edge of weight $d_X(x, c)$ between $x$ and $\widetilde{f}$.

Step 3: For each $\sigma \in T_{e_0,a} \setminus \{a_0\}$ and $f \in \mathbb{N}^{\mathbb{N}}$, if $\sigma$ is an initial segment of $f$, then we put an edge of weight $l - d_{e_0,a}(a_0, \sigma) = l - r_{|\sigma|}$ between $\sigma$ and $\widetilde{f}$. (Recall that $d_{e_0,a}(a_0, \sigma)$ is the length of the shortest path in $T_{e_0,a}$ from $a_0$ to $\sigma$.) We do the same for $T_{e_0,b}$.

Step 4: Let $\widetilde{G}$ be the resulting weighted graph. Then the set of vertices of $\widetilde{G}$ is

$$V(\widetilde{G}) = (X \setminus \{c\}) \sqcup T_{e_0,a} \sqcup T_{e_0,b} \sqcup \{\widetilde{f} : f \in \mathbb{N}^{\mathbb{N}}\} = V(G_{e_0}) \sqcup \{\widetilde{f} : f \in \mathbb{N}^{\mathbb{N}}\}.$$

Let $\widetilde{d}$ be the shortest path metric on $\widetilde{G}$. Let $(\widetilde{X}, \widetilde{d})$ be the completion of the metric space $(\widetilde{G}, \widetilde{d})$. The idea is that for each $f \in \mathbb{N}^{\mathbb{N}}$, we think of the point $\widetilde{f}$ as the limit of the sequence $(f \restriction n)_{n \in \mathbb{N}}$ in $\widetilde{X}$. If $T_e$ has an infinite path $f$, then there would be a copy of $f$ in the tree $T_{e,a}$ and another copy of $f$ in the tree $T_{e,b}$. We will show later that these two copies will give the same limit point $\widetilde{f}$ in $X_e$, and then $\widetilde{f}$ will be a copy of the point $c \in X$, and so $X \hookrightarrow X_e$.

Since $T_{e_0} = \mathbb{N}^{<\mathbb{N}}$, it is clear that $G_e$ is a subgraph of $\widetilde{G}$ for every $e \in \mathbb{N}$. Let $d$ be the restriction of $\widetilde{d}$ to $G_e$, i.e. $d := \widetilde{d}|_{G_e \times G_e}$. Thus, since $\widetilde{d}$ is a metric on $\widetilde{G}$ and $G_e \subseteq \widetilde{G}$, we have that $d$ is a metric on $G_e$. Let $(X_e, d)$ be the completion of the metric space $(G_e, d)$.

Note that the construction of $(X_e, d)$ is effective uniformly in $e$. So $(X_e)_{e \in \mathbb{N}}$ is a computable sequence of Polish metric spaces.

Also note that $(\widetilde{X}, \widetilde{d})$ is universal for $(X_e)_{e \in \mathbb{N}}$ in the sense that $(X_e, d) \hookrightarrow (\widetilde{X}, \widetilde{d})$ for all $e \in \mathbb{N}$. So we can think of each $X_e$ as a subset of $\widetilde{X}$.

The metric $d$ on $G_e$ might not be the shortest path metric on $G_e$. However, $d$ "looks like" the shortest path metric in the sense that $d$ is the restriction of the shortest path metric $\widetilde{d}$ on the extension $\widetilde{G}$ of $G_e$.

For each $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we let $\sigma_a$ denote the copy of the string $\sigma$ in the tree $T_{e_0,a}$. For each $f \in \mathbb{N}^{\mathbb{N}}$, we have that $f$ is an infinite path in $T_{e_0}$, and we let $f_a$ denote the copy of the infinite path $f$ in the tree $T_{e_0,a}$. That is, $f_a \upharpoonright n = (f \upharpoonright n)_a$ for all $n \in \mathbb{N}$. We define $\sigma_b$ and $f_b$ similarly.

For each $\sigma, \rho \in \mathbb{N}^{<\mathbb{N}}$, let $\tau_{\sigma,\rho}$ denote the longest initial segment of $\sigma$ and $\rho$.

We claim that for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\iff X \hookrightarrow X_e$.

($\Longrightarrow$) Assume that $T_e$ has an infinite path, say $f \in \mathbb{N}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, let $\sigma_n := f \upharpoonright n$. Then for all $m, n \in \mathbb{N}$,

$$d((\sigma_m)_a, (\sigma_n)_a) = d_{e,a}((\sigma_m)_a, (\sigma_n)_a) = |r_{|\sigma_m|} - r_{|\sigma_n|}| = |r_{|f\upharpoonright m|} - r_{|f\upharpoonright n|}| = |r_m - r_n|.$$

Thus, since $(r_n)_{n\in\mathbb{N}}$ converges, $((\sigma_n)_a)_{n\in\mathbb{N}} \subseteq T_{e,a}$ is a Cauchy sequence in $(X_e, d)$. Hence, since $(X_e, d)$ is complete, $((\sigma_n)_a)_{n\in\mathbb{N}}$ converges to a point $c_a \in X_e$. Similarly, $((\sigma_n)_b)_{n\in\mathbb{N}} \subseteq T_{e,b}$ is a Cauchy sequence in $(X_e, d)$ converging to a point $c_b \in X_e$.

Note that for all $n \in \mathbb{N}$, since $\sigma_n \subseteq f$ and $l < \delta_X$, we have that the path $((\sigma_n)_a, \widetilde{f}, (\sigma_n)_b)$ is a shortest path in $\widetilde{G}$ from $(\sigma_n)_a$ to $(\sigma_n)_b$, and so

$$\begin{aligned} d((\sigma_n)_a, (\sigma_n)_b) &= \widetilde{d}((\sigma_n)_a, (\sigma_n)_b) = \widetilde{d}((\sigma_n)_a, \widetilde{f}) + \widetilde{d}(\widetilde{f}, (\sigma_n)_b) \\ &= (l - r_{|\sigma_n|}) + (l - r_{|\sigma_n|}) = 2(l - r_n). \end{aligned}$$

Hence

$$d(c_a, c_b) = \lim_{n\to\infty} d((\sigma_n)_a, (\sigma_n)_b) = \lim_{n\to\infty} 2(l - r_n) = 2(l - l) = 0.$$

So $c_a = c_b$. Therefore, the infinite paths $f_a$ in $T_{e,a}$ and $f_b$ in $T_{e,b}$ give the same limit point at $c_a = c_b$.

Let $\widehat{c} := c_a = c_b$. We will show that $\widehat{c} \in X_e$ is a copy of the point $c \in X$ in the sense that $d(\widehat{c}, x) = d_X(c, x)$ for all $x \in X \setminus \{c\}$.

Note that

$$\widetilde{d}(\widehat{c}, \widetilde{f}) = \lim_{n\to\infty} \widetilde{d}((\sigma_n)_a, \widetilde{f}) = \lim_{n\to\infty} (l - r_n) = l - l = 0.$$

So $\widehat{c} = \widetilde{f}$ in $\widetilde{X}$.

Recall that for each $x \in X \setminus \{a, b, c\}$, there is an edge of weight $d_X(c, x)$ in $\widetilde{G}$ between $\widetilde{f}$ and $x$. So for all $x \in X \setminus \{a, b, c\}$, $d(\widehat{c}, x) = \widetilde{d}(\widehat{c}, x) = \widetilde{d}(\widetilde{f}, x) = d_X(c, x)$.

Note that for all $n \in \mathbb{N}$,

$$d(a, (\sigma_n)_a) = \widetilde{d}(a, (\sigma_n)_a) = \widetilde{d}(a, a_0) + \widetilde{d}(a_0, (\sigma_n)_a) = (d_X(a, c) - l) + r_n.$$

So

$$d(a, \widehat{c}) = d(a, c_a) = \lim_{n\to\infty} d(a, (\sigma_n)_a) = \lim_{n\to\infty} (d_X(a, c) - l + r_n) = d_X(a, c).$$

Similarly, we have $d(b, \widehat{c}) = d_X(b, c)$.

We conclude that $d(\widehat{c}, x) = d_X(c, x)$ for all $x \in X \setminus \{c\}$. It is easy to see that for all $x, y \in X \setminus \{c\}$, $d(x, y) = \widetilde{d}(x, y) = d_X(x, y)$.

Therefore, $X \hookrightarrow X_e$ via the map $\psi := Id_{X\setminus\{c\}} \sqcup \{(c, \widehat{c})\}$, that is,

$$\psi(x) = \begin{cases} \widehat{c} & \text{if } x = c \\ x & \text{if } x \in X \setminus \{c\} \end{cases}$$

($\Longleftarrow$) Assume that $T_e$ has no infinite paths. By (4), there are no $x, y \in Y_{e,a}$ such that $\widetilde{d}(x, y) = l$, and for all $x \in Y_{e,a}$, $\widetilde{d}(a_0, x) < l$. The same is true for $Y_{e,b}$. Let

$$\begin{aligned} L &:= d_X(a, c), \\ E_X &:= \{(x, y) \in X : d_X(x, y) = L\}, \\ E_{X_e} &:= \{(x, y) \in X_e : d(x, y) = L\}. \end{aligned}$$

Note that if $X \hookrightarrow X_e$ via a map $\psi$, then

$$\begin{aligned} \{(\psi(x), \psi(y)) : (x, y) \in E_X\} &\subseteq E_{X_e}, \\ |E_X| = |\{(\psi(x), \psi(y)) : (x, y) \in E_X\}| &\leq |E_{X_e}|. \end{aligned}$$

Therefore, to show that $X \not\hookrightarrow X_e$, it is enough to show that $|E_{X_e}| < |E_X|$.

From the construction of $X_e$, we have $X \setminus \{c\} \subseteq X_e$ and $c \notin X_e$. Thus, since $d_X(a, c) = L$ and $|E_X| < \infty$, we have that if $d(x, y) \neq L$ for all $(x, y) \in (X_e \times X_e) \setminus (X \times X)$, then $|E_{X_e}| < |E_X|$. So it is enough to show that $d(x, y) \neq L$ for all $(x, y) \in (X_e \times X_e) \setminus (X \times X)$.

Note that for any $\sigma, \rho \in \mathbb{N}^{<\mathbb{N}} = T_{e0}$,

$$\begin{aligned} \widetilde{d}(\sigma_a, \rho_a) = \widetilde{d}(\sigma_a, (\tau_{\sigma,\rho})_a) + \widetilde{d}(\tau_{\sigma,\rho})_a, \rho_a) &= (r_{|\sigma|} - r_{|\tau_{\sigma,\rho}|}) + (r_{|\rho|} - r_{|\tau_{\sigma,\rho}|}) \\ &\leq r_{|\sigma|} + r_{|\rho|} < l + l = 2l. \end{aligned}$$

Thus, since $T_{e0,a}$ is dense in $Y_{e0,a}$, we have that $\widetilde{d}(x, y) \leq 2l$ for all $x, y \in Y_{e0,a}$. The same is true of $Y_{e0,b}$.

Recall that for each $f \in \mathbb{N}^{<\mathbb{N}}$, the sequences $((f \upharpoonright n)_a)_{n\in\mathbb{N}}$ and $((f \upharpoonright n)_b)_{n\in\mathbb{N}}$ converge in $\widetilde{X}$ to the same limit point, say $\widehat{c_f}$, in $(\widetilde{X}, \widetilde{d})$, and so $\widehat{c_f} \in Y_{e_0,a} \cap Y_{e_0,b}$. Also recall that $\widehat{c_f} = \widetilde{f}$.

Note that for all $\sigma, \rho \in \mathbb{N}^{<\mathbb{N}}$, if $\sigma, \rho$ are comparable, then there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma \subseteq f$ and $\rho \subseteq f$, and so

$$\begin{aligned}\widetilde{d}(\sigma_a, \rho_b) &= \widetilde{d}(\sigma_a, \widehat{c_f}) + \widetilde{d}(\widehat{c_f}, \rho_b) = (l - r_{|\sigma|}) + (l - r_{|\rho|}) \\ &\leq l + l = 2l.\end{aligned}$$

Note that for all $\sigma, \rho \in \mathbb{N}^{<\mathbb{N}}$, if $\sigma, \rho$ are incomparable, then for any $f \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma \subseteq f$, we have $\tau_{\sigma,\rho} \subseteq \sigma \subseteq f$, and so

$$\begin{aligned}\widetilde{d}(\sigma_a, \rho_b) &= \widetilde{d}(\sigma_a, \widehat{c_f}) + \widetilde{d}(\widehat{c_f}, (\tau_{\sigma,\rho})_b) + \widetilde{d}((\tau_{\sigma,\rho})_b, \rho_b) \\ &= (l - r_{|\sigma|}) + (l - r_{|\tau_{\sigma,\rho}|}) + (r_{|\rho|} - r_{|\tau_{\sigma,\rho}|}) \\ &< l + l + l = 3l.\end{aligned}$$

Thus, since $T_{e_0,a} \cup T_{e_0,b}$ is dense in $Y_{e_0,a} \cup Y_{e_0,b}$, we can conclude that

$$\widetilde{d}(x, y) \leq 3l \text{ for all } x, y \in Y_{e_0,a} \cup Y_{e_0,b}. \tag{$*$}$$

Now we will show that $d(x, y) \neq L$ for all $(x, y) \in (X_e \times X_e) \setminus (X \times X)$.

Let $(x, y) \in (X_e \times X_e) \setminus (X \times X)$. Without loss of generality, assume $y \in X_e \setminus X$. Then $y \in Y_{e,a}$ or $y \in Y_{e,b}$. Without loss of generality, assume $y \in Y_{e,a}$. Then $\widetilde{d}(a_0, y) < l$.

Note that for any $f \in \mathbb{N}^{\mathbb{N}}$, we have $\widehat{c_f} = \widetilde{f}$ and $\widetilde{d}(a_0, \widehat{c_f}) = l$. Since $\widetilde{d}(a_0, y) < l$, we have $y \neq \widehat{c_f}$ for all $f \in \mathbb{N}^{\mathbb{N}}$. If $y \in Y_{e,a} \setminus T_{e,a}$, then $y$ is a limit point of $T_{e,a}$, and so we must have $y = \widehat{c_f}$ for some $f \in \mathbb{N}^{\mathbb{N}}$, which is a contradiction. Therefore, $y \in T_{e,a}$, and so $y = \sigma_a$ for some $\sigma \in \mathbb{N}^{<\mathbb{N}}$.

Case 1. $x = a$: Then

$$\begin{aligned} d(x,y) = \widetilde{d}(a,y) &\leq \widetilde{d}(a,a_0) + \widetilde{d}(a_0,y) = (d_X(a,c) - l) + \widetilde{d}(a_0,y) \\ &< (d_X(a,c) - l) + l = L. \end{aligned}$$

Case 2. $x = b$: Since $y = \sigma_a$, for any $f \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma \subseteq f$, we have $\widetilde{d}(\widetilde{f}, y) = l - r_{|\sigma|} \leq l$, and so

$$\begin{aligned} d(x,y) = \widetilde{d}(b,y) &\leq \widetilde{d}(b,b_0) + \widetilde{d}(b_0,\widetilde{f}) + \widetilde{d}(\widetilde{f},y) \leq \widetilde{d}(b,b_0) + l + l \\ &= (d_X(b,c) - l) + 2l = d_X(b,c) + l. \end{aligned}$$

We also have that

$$\begin{aligned} d(x,y) = \widetilde{d}(b,y) &\geq \widetilde{d}(a,b) - \widetilde{d}(a,a_0) - \widetilde{d}(a_0,y) \\ &= d_X(a,b) - (d_X(a,c) - l) - \widetilde{d}(a_0,y) \\ &> d_X(a,b) - (d_X(a,c) - l) - l \\ &= d_X(a,b) - d_X(a,c) \\ &= d_X(b,c). \end{aligned}$$

So $d_X(b,c) < d(x,y) \leq d_X(b,c) + l$.

If $d_X(a,c) \leq d_X(b,c)$, then $d(x,y) > d_X(b,c) \geq d_X(a,c) = L$.

If $d_X(a,c) > d_X(b,c)$, then $l < \gamma_X \leq d_X(a,c) - d_X(b,c)$, and so

$$\begin{aligned} d(x,y) &\leq d_X(b,c) + l < d_X(b,c) + \gamma_X \\ &\leq d_X(b,c) + (d_X(a,c) - d_X(b,c)) \\ &= d_X(a,c) = L. \end{aligned}$$

From both cases, we have $d(x,y) \neq L$.

<u>Case 3.</u> $x \in Y_{e,a} \cup Y_{e,b}$: Then by $(*)$,

$$d(x,y) = \widetilde{d}(x,y) \leq 3l < 3 \cdot \frac{1}{3}\delta_X = \delta_X \leq d_X(a,c) \leq L.$$

<u>Case 4.</u> $x \in X \setminus \{a,b,c\}$: Then, since $y \in T_{e,a} \subseteq G_e$, we have $x, y \in G_e \subseteq \widetilde{G}$.

<u>Case 4.1.</u> $d_X(a,c) > d_X(x,c)$: Then $l < \gamma_X \leq d_X(a,c) - d_X(x,c)$. Since $y = \sigma_a$, for any $f \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma \subseteq f$, we have $\widetilde{d}(\widetilde{f}, y) = l - r_{|\sigma|} \leq l$, and so

$$\begin{aligned} \widetilde{d}(x,y) &\leq \widetilde{d}(x,\widetilde{f}) + \widetilde{d}(\widetilde{f},y) \leq \widetilde{d}(x,\widetilde{f}) + l = d_X(x,c) + l \\ &< d_X(x,c) + \gamma_X \leq d_X(x,c) + (d_X(a,c) - d_X(x,c)) \\ &= d_X(a,c) = L. \end{aligned}$$

<u>Case 4.2.</u> $d_X(a,c) \leq d_X(x,c)$:

Note that for every $z \in X \setminus \{a,b,c\}$, all vertices in $\widetilde{G}$ that are adjacent to $z$ are in $\{a,b\} \sqcup (X \setminus \{a,b,c,z\}) \sqcup \{\widetilde{f} : f \in \mathbb{N}^{\mathbb{N}}\}$. Recall that for all distinct $u, v \in X \setminus \{c\}$, there is an edge of weight $d_X(u,v)$ in $\widetilde{G}$ between $u$ and $v$. For all $f \in \mathbb{N}^{\mathbb{N}}$ and $u \in X \setminus \{a,b,c\}$, there is an edge of weight $d_X(u,c)$ in $\widetilde{G}$ between $u$ and $\widetilde{f}$.

Let $p$ be a shortest path in $\widetilde{G}$ from $x$ to $y$. Then $p$ must have the form $p = (x, w, v_1, \ldots, v_k, y)$, where $k \in \mathbb{N}$, $x, w, v_1, \ldots, v_k, y$ are all distinct, $v_1, \ldots, v_k \in T_{e,a} \sqcup T_{e,b} \sqcup \{\widetilde{f} : f \in \mathbb{N}^{\mathbb{N}}\}$ and $w \in \{a,b\} \sqcup \{\widetilde{f} : f \in \mathbb{N}^{\mathbb{N}}\}$. Therefore, we have

$$\begin{aligned}\widetilde{d}(x,y) &= \text{the length of the shortest path } p\\ &= \widetilde{d}(x,w) + \widetilde{d}(w,y)\end{aligned}$$

<u>Case 4.2.1.</u> $w = \widetilde{f}$ for some $f \in \mathbb{N}^{\mathbb{N}}$: Then $y \neq \widehat{c_f} = \widetilde{f}$, and so

$$\widetilde{d}(x,y) = \widetilde{d}(x,\widetilde{f}) + \widetilde{d}(\widetilde{f},y) = d_X(x,c) + \widetilde{d}(\widetilde{f},y) > d_X(x,c) \geq d_X(a,c) = L.$$

<u>Case 4.2.2.</u> $w = a$: Then, since $x \neq a$, $2l < \delta_X \leq d_X(x,a)$, and so

$$\begin{aligned}\widetilde{d}(x,y) &= \widetilde{d}(x,a) + \widetilde{d}(a,y)\\ &\geq \widetilde{d}(x,a) + \widetilde{d}(a,a_0) - \widetilde{d}(a_0,y)\\ &= d_X(x,a) + (d_X(a,c) - l) - \widetilde{d}(a_0,y)\\ &> d_X(x,a) + (d_X(a,c) - l) - l\\ &= d_X(x,a) + d_X(a,c) - 2l\\ &> d_X(x,a) + d_X(a,c) - d_X(x,a)\\ &= d_X(a,c) = L.\end{aligned}$$

<u>Case 4.2.3.</u> $w = b$: By Case 2, $d_X(b,c) < d(b,y) = \widetilde{d}(b,y)$. So

$$\begin{aligned}\widetilde{d}(x,y) &= \widetilde{d}(x,b) + \widetilde{d}(b,y) > \widetilde{d}(x,b) + d_X(b,c) \\ &= d_X(x,b) + d_X(b,c) \geq d_X(x,c) \geq d_X(a,c) = L.\end{aligned}$$

From all cases, we conclude that $d(x,y) \neq L$ for all $(x,y) \in (X_e \times X_e) \setminus (X \times X)$. It follows that $|E_{X_e}| < |E_X|$, and so $X \not\hookrightarrow X_e$.

So for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\iff X \hookrightarrow X_e$. Therefore, $\{e : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-hard.

This ends the proof for the case $X$ does not satisfy the strict triangle inequality.

We conclude that $\{e : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-hard for every computably presentable finite metric space $X$ with $|X| > 1$. This completes the proof of Theorem 3.2.5. $\square$

We can use the construction for Case 2 in the proof of Theorem 3.2.5 for infinite metric spaces $X$ that satisfy all properties that are necessary for the proof in Case 2. Note that, by dovetailing, we can construct $Y_{e,a}$ for each $a \in X$ uniformly in $e, a$, and let $X_e = \bigsqcup_{a \in X} Y_{e,a}$. This gives the following.

**Corollary 3.2.6.** *Let $(X, d_X)$ be a computably presentable metric space with $|X| > 1$ ($X$ can be infinite) that satisfies the following conditions:*

(1) $\delta_X := \inf\{d_X(a,b) : a \neq b \in X\} > 0$.

(2) $\varepsilon_X := \inf\{d_X(a,c) + d_X(c,b) - d_X(a,b) : a,b,c \in X \text{ are distinct}\} > 0$.

(3) *There are $x, y \in X$ such that $d_X(x,y) = diam(X) < \infty$.*

*Then the set* $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ *is* $\Sigma^1_1$*-complete.*

Note that (1) implies that $X$ is discrete, (2) implies that $X$ satisfies the strict triangle inequality, and (3) implies that $X$ is bounded.

For example, let $(X, d_X)$ be any bounded computably presentable metric space such that $d_X(a, b)$ is an odd number for all distinct $a, b \in X$. Then $X$ satisfies conditions (1)-(3), and so $\{e : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-complete.

## 3.3 Results on Infinite Metric Spaces

In this section, we find the complexity of the embedding problem $X \hookrightarrow M_e$ for some infinite computably presentable metric spaces $X$.

**Theorem 3.3.1.** *For every unbounded subset* $X$ *of* $\mathbb{R}$ *(equipped with the Euclidean metric), the set* $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ *is* $\Sigma^1_1$*-hard. If* $X$ *is also a computably presentable metric space, then the set* $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ *is* $\Sigma^1_1$*-complete. In particular, the set* $\{e \in \mathbb{N} : \mathbb{R} \hookrightarrow M_e\}$ *is* $\Sigma^1_1$*-complete*

*Proof.* Let $X$ be an unbounded subset of $\mathbb{R}$, equipped with the Euclidean metric. If $X$ is also a computably presentable metric space, then by Proposition 3.1.6, $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ is $\Sigma^1_1$. So we only need to show that $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-hard.

It is enough to build a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\iff X \hookrightarrow X_e$.

The idea to build $(X_e, d)$ is as follows. We attach two copies of the tree $T_e$ together at the root. Then for every $\sigma \in T_e \setminus \{\lambda\}$, we put a copy of all rationals $p$

in $(|\sigma|-1, |\sigma|]$ on the edge between $\sigma$ and $\sigma \restriction (|\sigma|-1)$. Then we let $(X_e, d)$ be the completion of the resulting graph with the shortest path metric $d$.

For each $\sigma \in \mathbb{N}^{<\mathbb{N}} \setminus \{\lambda\}$ and $s \in \mathbb{N}$, let

$$A^+_{\sigma,s} := \{p \in \mathbb{Q}^+ : |\sigma| - 1 < p \leq |\sigma| \text{ and } p = \frac{a}{b} \text{ where } a, b \in \mathbb{N} \setminus \{0\} \text{ and } b \leq s\},$$
$$A^-_{\sigma,s} := -A^+_{\sigma,s} = \{-p : p \in A^+_{\sigma,s}\}.$$

Note that for any infinite path $f \in \mathbb{N}^{\mathbb{N}}$, we have $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} A^+_{f\restriction n,s} = \mathbb{Q}^+$ and $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} A^-_{f\restriction n,s} = \mathbb{Q}^-$, and so $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} (A^+_{f\restriction n,s} \cup A^-_{f\restriction n,s}) = \mathbb{Q} \setminus \{0\}$.

Fix an effective list $(\sigma_i)_{i\in\mathbb{N}}$ of all finite strings in $\mathbb{N}^{<\mathbb{N}}$ with $\sigma_0 = \lambda$. To build $X_e$, we build a computable sequence $G_{e,0} \subseteq G_{e,1} \subseteq \dots$ of weighted (undirected) graphs in stages, and let $G_e = \bigcup_{s\in\mathbb{N}} G_{e,s}$, then let $(X_e, d)$ be the completion of $G_e$ with the shortest path metric $d$. For each $e, s \in \mathbb{N}$, we let $V_{e,s}$ denote the set of all vertices in $G_{e,s}$, and $E_{e,s}$ denote the set of all weighted edges in $G_{e,s}$, where

$$E_{e,s} = \{(\{u, v\}, w) : u, v \text{ are connected by an edge of weight } w \text{ in } G_{e,s}\}.$$

Fix a set $\{v_{0,\lambda}\} \sqcup \{v_{p,\sigma} : \sigma \in \mathbb{N}^{<\mathbb{N}} \setminus \{\lambda\} \wedge p \in \bigcup_{s\in\mathbb{N}} (A^+_{\sigma,s} \cup A^-_{\sigma,s})\}$ of distinct vertices.

**Construction of** $(G_{e,s})_{s\in\mathbb{N}}$

Stage 0: Let $V_{e,0} = \{v_{0,\lambda}\}$ and $E_{e,0} = \emptyset$.

Stage $s+1$: We have built $G_{e,s}$. Since $T_e$ is a computable tree, we can check computably if $\sigma_s \in T_e$. If $\sigma_s \notin T_e$, then let $G_{e,s+1} := G_{e,s}$, and go to the next stage.

If $\sigma_s \in T_e$, then we let $G_{e,s+1} := (V_{e,s+1}, E_{e,s+1})$ where

$$V_{e,s+1} := V_{e,s} \cup \{v_{p,\sigma} : \lambda \neq \sigma \subseteq \sigma_s \wedge p \in A^+_{\sigma,s} \cup A^-_{\sigma,s}\}, \text{ and}$$

$$E_{e,s+1} := E_{e,s} \cup \{(\{v_{p,\sigma}, v_{q,\tau}\}, |p-q|) : \lambda \neq \sigma, \tau \subseteq \sigma_s \wedge p \in A^+_{\sigma,s} \cup A^-_{\sigma,s} \wedge q \in A^+_{\tau,s} \cup A^-_{\tau,s}\}$$
$$\cup \{(\{v_{0,\lambda}, v_{p,\sigma}\}, |p|) : \lambda \neq \sigma \subseteq \sigma_s \wedge p \in A^+_{\sigma,s} \cup A^-_{\sigma,s}\}.$$

This ends the construction.

We claim that for every $e \in \mathbb{N}$, $T_e$ has an infinite path $\iff X \hookrightarrow X_e$.

($\Longrightarrow$) Assume that $T_e$ has an infinite path, say $f \in \mathbb{N}^{\mathbb{N}}$. Then we have $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} (A^+_{f\upharpoonright n,s} \cup A^-_{f\upharpoonright n,s}) = \mathbb{Q} \setminus \{0\}$, and if $\lambda \neq \sigma, \tau \subseteq f$, $p \in A^+_{\sigma,s} \cup A^-_{\sigma,s}$ and $q \in A^+_{\tau,s} \cup A^-_{\tau,s}$, then $v_{p,\sigma}, v_{q,\tau} \in G_e$ and $d(v_{p,\sigma}, v_{q,\tau}) = |p-q| = d_{\mathbb{R}}(p,q)$. So $\mathbb{Q} \setminus \{0\} \hookrightarrow G_e$. Thus, since $\mathbb{Q} \setminus \{0\}$ is dense in $\mathbb{R}$, we have $\mathbb{R} \hookrightarrow X_e$. Therefore, $X \hookrightarrow X_e$.

($\Longleftarrow$) Assume that $T_e$ has no infinite paths. To show that $X \not\hookrightarrow X_e$, suppose for a contradiction that $X \hookrightarrow X_e$ via an isometric embedding $g$. Since $X$ is an unbounded subset of $\mathbb{R}$, there is a strictly monotone sequence $(x_i)_{i\in\mathbb{N}} \subseteq X$ such that $\lim_{i\to\infty} |x_i| = \infty$. Then, since $g : X \to X_e$ is an isometric embedding, $d(g(x_i), g(x_j)) = |x_i - x_j|$ for all $i, j \in \mathbb{N}$ and $\lim_{i\to\infty} d(g(x_0), g(x_i)) = \lim_{i\to\infty} |x_0 - x_i| = \infty$. So the sequence $(g(x_i))_{i\in\mathbb{N}}$ must form a path of infinite length in $X_e$ without tracing back. However, if we start from a point in $G_e$, and try to walk along edges in $G_e$, without tracing back, as long as possible, since $T_e$ has no infinite paths, we will eventually reach a dead end in finitely many steps at some vertex $v_{p,\sigma}$, where $\sigma$ is a leaf of $T_e$ and $p \in \{|\sigma|, -|\sigma|\}$. So the length of the path we walked on is finite, a contradiction. Therefore, $X \not\hookrightarrow X_e$. In particular, $\mathbb{R} \not\hookrightarrow X_e$.

So we have proved the claim. Therefore, $\{e : X \hookrightarrow X_e\}$ is $\Sigma^1_1$-hard. □

**Theorem 3.3.2.** *For every computable real $r > 0$, the set $\{e \in \mathbb{N} : [0, r] \hookrightarrow M_e\}$ is $\Sigma^1_1$-complete, where $[0, r]$ is equipped with the Euclidean metric.*

*Proof.* Let $r' > 0$ be a computable real. Let $r := \frac{r'}{2} > 0$. Then $r$ is a computable real and $[-r, r] \cong [0, r']$ via the isometry $\psi : x \mapsto x + r$.

Since $r' > 0$ is a computable real, there is a computable strictly increasing sequence $(r'_n)_{n\in\mathbb{N}}$ converging to $r'$. Then an effective list $(q_i)_{n\in\mathbb{N}}$ of all rationals $q_i$ such that $0 \leq q_i \leq r'_n$ for some $n \in \mathbb{N}$ forms a computable presentation of $[0, r']$. Therefore, $[0, r']$ is computably presentable. Thus, by Proposition 3.1.6, $\{e : [0, r'] \hookrightarrow M_e\}$ is $\Sigma^1_1$.

To show that $\{e : [0, r] \hookrightarrow M_e\}$ is $\Sigma^1_1$-hard, it is enough to build a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$,

$$T_e \text{ has an infinite path } \iff [-r, r] \hookrightarrow X_e.$$

Our construction will be a combination of the constructions for Theorem 3.2.4, and Theorem 3.3.1.

Since $r > 0$ is a computable real, there is a computable strictly increasing sequence $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ converging to $r$ such that $r_0 = 0$ and $r_{n+1} > r_n + \frac{1}{2}(r - r_n)$ for all $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $w_i := r_{i+1} - r_i > 0$. Then $w_{n+1} < w_n$ and $\sum_{i=0}^{n} w_i = r_{n+1}$ for all $n \in \mathbb{N}$, and $\sum_{i=0}^{\infty} w_i = \lim_{n\to\infty} r_n = r$.

The idea to build $(X_e, d)$ is as follows. We attach two copies of the weighted tree $T_e$ constructed in the proof of Theorem 3.2.4 by putting weight $w_n$ on each edge between level $n$ and level $n+1$ together at the root. So the height of the weighted tree $T_e$ is at most $\sum_{i=0}^{\infty} w_i = r$. Then for every $\sigma \in T_e \setminus \{\lambda\}$, we put a copy of all rationals $p$ in $(r_{|\sigma|-1}, r_{|\sigma|}]$ on the edge between $\sigma$ and $\sigma \upharpoonright (|\sigma| - 1)$. Then we let $(X_e, d)$ be the completion of the resulting weighted graph with the shortest path metric $d$.

For each $\sigma \in \mathbb{N}^{<\mathbb{N}} \setminus \{\lambda\}$ and $s \in \mathbb{N}$, let

$$A^+_{\sigma,s} := \{p \in \mathbb{Q}^+ : r_{|\sigma|-1} < p \leq r_{|\sigma|} \text{ and } p = \frac{a}{b} \text{ where } a, b \in \mathbb{N} \setminus \{0\} \text{ and } b \leq s\}, \text{ and}$$

$$A^-_{\sigma,s} := -A^+_{\sigma,s} = \{-p : p \in A^+_{\sigma,s}\}.$$

Note that for any infinite path $f \in \mathbb{N}^{\mathbb{N}}$, since $\lim_{n\to\infty} r_n = r$, we have $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} A^+_{f\upharpoonright n,s} = [0, r] \cap \mathbb{Q}^+$ and $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} A^-_{f\upharpoonright n,s} = [-r, 0] \cap \mathbb{Q}^-$, and so $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} (A^+_{f\upharpoonright n,s} \cup A^-_{f\upharpoonright n,s}) = ([-r, r] \cap \mathbb{Q}) \setminus \{0\}$.

Fix an effective list $(\sigma_i)_{i\in\mathbb{N}}$ of all finite strings in $\mathbb{N}^{<\mathbb{N}}$ with $\sigma_0 = \lambda$. To build $X_e$, we build a computable sequence $G_{e,0} \subseteq G_{e,1} \subseteq \dots$ of weighted (undirected) graphs in stages, and let $G_e = \bigcup_{s\in\mathbb{N}} G_{e,s}$, then let $(X_e, d)$ be the completion of $G_e$ with the shortest path metric $d$. For each $e, s \in \mathbb{N}$, we let $V_{e,s}$ denote the set of all vertices in $G_{e,s}$, and $E_{e,s}$ denote the set of all weighted edges in $G_{e,s}$, where

$$E_{e,s} = \{(\{u, v\}, w) : u, v \text{ are connected by an edge of weight } w \text{ in } G_{e,s}\}.$$

Fix a set $\{v_{0,\lambda}\} \sqcup \{v_{p,\sigma} : \sigma \in \mathbb{N}^{<\mathbb{N}} \setminus \{\lambda\} \wedge p \in \bigcup_{s\in\mathbb{N}} (A^+_{\sigma,s} \cup A^-_{\sigma,s})\}$ of distinct vertices.

**Construction of** $(G_{e,s})_{s\in\mathbb{N}}$

Stage 0: Let $V_{e,0} = \{v_{0,\lambda}\}$ and $E_{e,0} = \emptyset$.

Stage $s+1$: We have built $G_{e,s}$. Since $T_e$ is a computable tree, we can check computably if $\sigma_s \in T_e$. If $\sigma_s \notin T_e$, then let $G_{e,s+1} := G_{e,s}$, and go to the next stage. If $\sigma_s \in T_e$, then we let $G_{e,s+1} := (V_{e,s+1}, E_{e,s+1})$ where

$$V_{e,s+1} := V_{e,s} \cup \{v_{p,\sigma} : \lambda \neq \sigma \subseteq \sigma_s \wedge p \in A^+_{\sigma,s} \cup A^-_{\sigma,s}\}, \text{ and}$$

$$E_{e,s+1} := E_{e,s} \cup \{(\{v_{p,\sigma}, v_{q,\tau}\}, |p-q|) : \lambda \neq \sigma, \tau \subseteq \sigma_s \wedge p \in A^+_{\sigma,s} \cup A^-_{\sigma,s} \wedge q \in A^+_{\tau,s} \cup A^-_{\tau,s}\}$$
$$\cup \{(\{v_{0,\lambda}, v_{p,\sigma}\}, |p|) : \lambda \neq \sigma \subseteq \sigma_s \wedge p \in A^+_{\sigma,s} \cup A^-_{\sigma,s}\}.$$

This ends the construction.

We claim that for every $e \in \mathbb{N}$,

$$T_e \text{ has an infinite path} \iff [-r,r] \hookrightarrow X_e.$$

($\Longrightarrow$) Assume that $T_e$ has an infinite path, say $f \in \mathbb{N}^{\mathbb{N}}$. Then we have $\bigcup_{s\in\mathbb{N}} \bigcup_{n\in\mathbb{N}\setminus\{0\}} (A^+_{f\restriction n,s} \cup A^-_{f\restriction n,s}) = ([-r,r] \cap \mathbb{Q}) \setminus \{0\}$, and if $\lambda \neq \sigma, \tau \subseteq f$, $p \in A^+_{\sigma,s} \cup A^-_{\sigma,s}$ and $q \in A^+_{\tau,s} \cup A^-_{\tau,s}$, then $v_{p,\sigma}, v_{q,\tau} \in G_e$ and $d(v_{p,\sigma}, v_{q,\tau}) = |p-q| = d_{\mathbb{R}}(p,q)$. So $((-r,r) \cap \mathbb{Q}) \setminus \{0\} \hookrightarrow G_e$. Thus, since $((-r,r) \cap \mathbb{Q}) \setminus \{0\}$ is dense in $[-r,r]$, we have $[-r,r] \hookrightarrow X_e$.

($\Longleftarrow$) Assume that $T_e$ has no infinite paths. To show that $[-r,r] \not\hookrightarrow X_e$, suppose for a contradiction that $[-r,r] \hookrightarrow X_e$ via an isometric embedding $g$.

From the construction, we have that the set of all vertices in the graph $G_e$ is

$$V_e = \{v_{0,\lambda}\} \cup \bigcup_{s\in\mathbb{N}} \{v_{p,\sigma} : \lambda \neq \sigma \subseteq \sigma_s \wedge p \in A^+_{\sigma,s} \cup A^-_{\sigma,s}\}.$$

Let

$$V_e^+ := \{v_{0,\lambda}\} \cup \bigcup_{s\in\mathbb{N}} \{v_{p,\sigma} : \lambda \neq \sigma \subseteq \sigma_s \wedge p \in A_{\sigma,s}^+\},$$
$$V_e^- := \{v_{0,\lambda}\} \cup \bigcup_{s\in\mathbb{N}} \{v_{p,\sigma} : \lambda \neq \sigma \subseteq \sigma_s \wedge p \in A_{\sigma,s}^-\}.$$

Let $G_e^+$ be the subgraph of $G_e$ induced from $V_e^+$, and let $X_e^+$ be the completion of $G_e^+$ with the shortest path metric $d$. We define $G_e^-$ and $X_e^-$ in the same way.

By the proof of Theorem 3.2.4, we have that for all $x \in X_e^+$, $d(v_{0,\lambda}, x) < r$. The same holds for $X_e^-$. Thus, since $g(r), g(-r) \in X = X_e^+ \cup X_e^-$, we have

$$d(g(-r), g(r)) \leq d(g(-r), v_{0,\lambda}) + d(v_{0,\lambda}, g(r)) < r + r = 2r.$$

But since $g : [-r, r] \hookrightarrow X_e$ is an isometric embedding, we must have

$$d(g(-r), g(r)) = d_{\mathbb{R}}(-r, r) = 2r,$$

which is a contradiction. Therefore, $[-r, r] \not\hookrightarrow X_e$.

So we have proved the claim. Therefore, $\{e : [0, r'] \hookrightarrow X_e\}$ is $\Sigma_1^1$-hard, and so it is $\Sigma_1^1$-complete. □

Corollary 3.2.6, Theorem 3.3.1 and Theorem 3.3.2 give some examples of infinite computable metric spaces $X$ such that the embedding problem $X \hookrightarrow M_e$ is $\Sigma_1^1$-complete. In Chapter 6, we will see that the embedding problems for the Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$ are also $\Sigma_1^1$-complete. We strongly believe that the same holds for all infinite computable metric spaces. Since we used different techniques for each $X$ depending on the structure of $X$ (e.g. $X$ has a straight line structure, $X$ has a tree-like structure, etc), it would be surprising if one can give a single proof or technique that covers all infinite computable metric spaces.

**Conjecture.** For every infinite computably presentable metric space $X$, the set $\{e \in \mathbb{N} : X \hookrightarrow M_e\}$ is $\Sigma^1_1$-complete.

# Chapter 4

# Topological Properties

Given a topological property, one can investigate how hard it is to determine whether a computable metric space $M_e$ has that property. For example, Melnikov and Nies [13] showed that the index set of compact computable metric spaces is $\Pi^0_3$-complete. In this chapter, we compute the complexity of the index set of perfect computable metric spaces and the index set of discrete computable metric spaces.

**Theorem 4.0.1.** *The set* $\{e \in \mathbb{N} : M_e$ *is perfect*$\}$ *is* $\Pi^0_2$*-complete within PolSp, and so it is* $\Pi^0_2$*-complete.*

*Proof.* Recall that a topological space is perfect if and only if it has no isolated points. Since $(\mathbb{N}, d_e)$ is dense in $M_e$, a point $x$ is an isolated point of $M_e$ if and only if $x$ is an isolated point of $(\mathbb{N}, d_e)$. So we have

$$M_e \text{ is perfect} \iff (\mathbb{N}, d_e) \text{ has no isolated points}$$

$$\iff (\forall x \in \mathbb{N})(\forall r \in \mathbb{Q}^+)(\exists y \in \mathbb{N})(y \in B(x,r) \setminus \{x\})$$
$$\iff (\forall x \in \mathbb{N})(\forall r \in \mathbb{Q}^+)(\exists y \in \mathbb{N})(d(x,y) < r \wedge d(x,y) > 0),$$

where $B(x,r)$ denotes the open ball around $x$ of radius $r$. Thus, since "$d(x,y) < r$" and "$d(x,y) > 0$" are $\Sigma^0_1$ statements, we have that $\{e : M_e \text{ is perfect}\}$ is $\Pi^0_2$ within $PolSp$.

To show that $\{e : M_e \text{ is perfect}\}$ is $\Pi^0_2$-hard within $PolSp$, it is enough to construct a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$,

$$e \in Tot \iff X_e \text{ is perfect}.$$

Let $(q_i)_{i\in\mathbb{N}}$ be an effective list of all elements in $\mathbb{Q} \cap [0,1]$ with $q_0 = 0$. The idea to build $(X_e)_{e\in\mathbb{N}}$ is that, whenever we see that $\varphi_e(i)\downarrow$ for all $i \leq n$, we will put a copy of the rational $q_{n+1}$ into $X_e$.

For each $e \in \mathbb{N}$, we effectively construct a Polish metric space $(X_e, d)$ uniformly in $e$ as follows.

Stage 0: Let $s_0 = 0$ and $d(s_0, s_0) = 0$.

Stage $n+1$: For each $s > s_n$, starting from $s = s_n + 1$, we check if $\varphi_{e,s}(n)\downarrow$ until we find (if ever) the least $s > s_n$ such that $\varphi_{e,s}(n)\downarrow$.

For each $s$ such that $\varphi_{e,s}(n)\uparrow$, we let

- $d(s,j) = 0$ for all $j \in \{s_n, \ldots, s\}$.
- $d(s,j) = |q_n - q_i|$ for all $j \in \{s_i, \ldots, s_{i+1} - 1\}$ and $i \in \{0, \ldots, n-1\}$.

Whenever we find (if ever) the least $s > s_n$ such that $\varphi_{e,s}(n)\downarrow$, we let

- $s_{n+1} = \min\{s' > s_n : \varphi_{e,s'}(n)\downarrow\} = s$.
- $d(s_{n+1}, s_{n+1}) = 0$.
- $d(s_{n+1}, j) = |q_{n+1} - q_i|$ for all $j \in \{s_i, \dots, s_{i+1} - 1\}$ and $i \in \{0, \dots, n\}$.

Then we go to stage $n + 2$.

This ends the construction.

Let $(X_e, d)$ be the completion of the resulting pseudometric space $(\mathbb{N}, d)$. The key point is that, at stage $n + 1$, whenever we find (if ever) an $s > s_n$ such that $\varphi_{e,s}(n)\downarrow$, we put a new element $s_{n+1}$, which is a copy of the rational $q_{n+1}$, into $X_e$, and then we go to the next stage.

If $\varphi_e$ is total, then we will put a copy of the rational $q_n$ into $X_e$ for all $n \in \mathbb{N}$, and so $X_e \cong [0, 1]$ because $(q_n)_{n\in\mathbb{N}}$ is dense in $[0, 1]$. Since $[0, 1]$ is perfect, $X_e$ is also perfect.

If $\varphi_e$ is not total, say $n$ is the least such that $\varphi_e(n)\uparrow$, then we will put a copy of $q_0, \dots, q_n$ into $X_e$, and then we will be at stage $n + 1$ forever. It follows that $X_e \cong \{q_0, \dots, q_n\}$. So every point in $X_e$ is an isolated point. In particular, $X_e$ is not perfect.

We conclude that $e \in Tot \iff X_e$ is perfect. Thus, since $Tot$ is $\Pi^0_2$-complete, $\{e : M_e \text{ is perfect}\}$ is $\Pi^0_2$-hard within $PolSp$, and so it is $\Pi^0_2$-complete within $PolSp$.

□

**Theorem 4.0.2.** *The set* $\{e \in \mathbb{N} : M_e$ *is discrete*$\}$ *is* $\Pi^1_1$*-complete.*

*Proof.* Recall that a topological space is discrete if and only if every point is an isolated point. Note that for every $x \in M_e$, $x$ has a Cauchy name $f_x : \mathbb{N} \to \mathbb{N}$ in $M_e$. On the other hand, we can think of each Cauchy name $f : \mathbb{N} \to \mathbb{N}$ in $M_e$ as a point $x_f$ in $M_e$, where $x_f$ is the limit of the sequence $(f(k))_{k\in\mathbb{N}}$ in $M_e$. So we have

$$\begin{aligned} M_e \text{ is discrete} &\iff (\forall x \in M_e)(x \text{ is an isolated point}) \\ &\iff (\forall f : \mathbb{N} \to \mathbb{N})(f \text{ is a Cauchy name in } M_e \to x_f \text{ is an isolated point}). \end{aligned}$$

Note that $f$ is a Cauchy name in $M_e \iff (\forall k, l \in \mathbb{N})(d_e(f(k), f(k+l)) \leq 2^{-k})$, and since $(\mathbb{N}, d_e)$ is dense in $M_e$, we have

$$\begin{aligned} x_f \text{ is an isolated point} &\iff (\exists r \in \mathbb{Q}^+)(B(x_f, r) \setminus \{x_f\} = \emptyset) \\ &\iff (\exists r \in \mathbb{Q}^+)(\forall y \in M_e)(y = x_f \vee d_e(x_f, y) \geq r) \\ &\iff (\exists r \in \mathbb{Q}^+)(\forall y \in \mathbb{N})(y = x_f \vee d_e(x_f, y) \geq r) \\ &\iff (\exists r \in \mathbb{Q}^+)(\forall y \in \mathbb{N})(d_e(y, x_f) = 0 \vee d_e(x_f, y) \geq r) \end{aligned}$$

So "$f$ is a Cauchy name in $M_e$" is a $\Pi^0_1$ statement and "$x_f$ is an isolated point" is a $\Sigma^0_2$ statement. Therefore, we can conclude that $\{e : M_e \text{ is discrete}\}$ is $\Pi^1_1$.

It remains to show that $\{e : M_e \text{ is discrete}\}$ is $\Pi^1_1$-hard. Fix any computable real $r > 0$. We use the same construction as in the proof of Theorem 3.2.4.

Since $r > 0$ is a computable real, there is a computable strictly increasing sequence $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ converging to $r$ such that $r_0 = 0$ and $r_{n+1} > r_n + \frac{1}{2}(r - r_n)$ for all $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $w_i := r_{i+1} - r_i > 0$. Then $w_{n+1} < w_n$ and $\sum_{i=0}^{n} w_i = r_{n+1}$ for all $n \in \mathbb{N}$ and $\sum_{i=0}^{\infty} w_i = \lim_{n\to\infty} r_n = r$. So the height of each tree $T_e$ is at most $\sum_{i=0}^{\infty} w_i = r$.

Fix an effective list $(\sigma_i)_{i\in\mathbb{N}}$ of all finite strings in $\mathbb{N}^{<\mathbb{N}}$ with $\sigma_0 = \lambda$. For each $e \in \mathbb{N}$, we put weight $w_n$ on each edge between level $n$ and level $n+1$ in the tree $T_e$. Then we define a pseudometric $d$ (depending on $e$) on $\mathbb{N}$ by

- if $\sigma_i \notin T_e$, let $d(i,0) := 0$ and $d(i,j) := d(0,j)$ for all $j \in \mathbb{N}$,
- if $\sigma_i, \sigma_j \in T_e$, let $d(i,j) :=$ the length of the shortest path in $T_e$ from $\sigma_i$ to $\sigma_j$.

Let $(X_e, d)$ be the completion of the pseudometric space $(\mathbb{N}, d)$. Therefore, $X_e$ is the completion of the tree $T_e$ with the shortest path metric $d$, where we put weight $w_n$ on each edge between level $n$ and level $n+1$.

By the proof of Theorem 3.2.4, $(X_e)_{e\in\mathbb{N}}$ is a computable sequence of Polish metric spaces such that for every $e \in \mathbb{N}$, the following conditions hold:

(1) $d(\lambda, x) \leq r$ for all $x \in X_e$.

(2) If $T_e$ has an infinite path, then there is an $\widehat{x} \in X_e$ such that $d(\lambda, \widehat{x}) = r$.

(3) If $T_e$ has no infinite paths, then there are no $x, y \in X_e$ such that $d(x,y) = r$, and for all $x \in X_e$, $d(\lambda, x) < r$.

We will show that for every $e \in \mathbb{N}$, $T_e$ has an infinite path $\Longleftrightarrow$ $X_e$ is not discrete.

($\Longrightarrow$) Assume that $T_e$ has an infinite path, say $g \in \mathbb{N}^{\mathbb{N}}$. Then for each $n \in \mathbb{N}$, there is a unique $i_n \in \mathbb{N}$ such that $\sigma_{i_n} = g \upharpoonright n$. So $i_0 = 0$ and for all $m, n \in \mathbb{N}$,

$$\begin{aligned} d(i_m, i_n) = &\text{ the length of the shortest path in } T_e \text{ between } \sigma_{i_m} \text{ and } \sigma_{i_n} \\ &= |r_{|\sigma_{i_m}|} - r_{|\sigma_{i_n}|}| = |r_{|g\upharpoonright m|} - r_{|g\upharpoonright n|}| = |r_m - r_n|. \end{aligned}$$

Thus, since $(r_n)_{n\in\mathbb{N}}$ converges, $(i_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X_e, d)$. Hence, since

$(X_e, d)$ is complete, $(i_n)_{n\in\mathbb{N}}$ converges to a point $i \in X_e$. So for all $n \in \mathbb{N}$,

$$d(i, i_n) = \lim_{m\to\infty} d(i_m, i_n) = \lim_{m\to\infty} |r_m - r_n| = |r - r_n| = r - r_n > 0.$$

Hence $i_n \neq i$ for all $n \in \mathbb{N}$, and so $i$ is not an isolated point of $X_e$. Therefore, $X_e$ is not discrete.

($\Longleftarrow$) Assume that $X_e$ is not discrete. Then there is an $x \in X_e$ such that $x$ is not an isolated point of $X_e$. So there is a sequence in $X_e$ converging to $x$. Thus, since $(\mathbb{N}, d)$ is dense in $X_e$, there is a Cauchy sequence $(i_n)_{n\in\mathbb{N}}$ in $(\mathbb{N}, d)$ such that $(i_n)_{n\in\mathbb{N}}$ converges to $x$ in $X_e$. Without loss of generality, we can assume that $\sigma_{i_n} \in T_e$ for all $n \in \mathbb{N}$.

**Claim 1.** $\lim_{n\to\infty} |\sigma_{i_n}| = \infty$.

Suppose for a contradiction that there is an $M \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, there is an $n_k \geq k$ such that $|\sigma_{i_{n_k}}| < M$. We can choose $n_k$ so that $n_0 < n_1 < n_2 < \dots$. So $(i_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(i_n)_{n\in\mathbb{N}}$ and $|\sigma_{i_{n_k}}| < M$ for all $k \in \mathbb{N}$. Thus, since $(i_n)_{n\in\mathbb{N}}$ converges to $x$ in $X_e$, $(i_{n_k})_{k\in\mathbb{N}}$ also converges to $x$ in $X_e$. We will use the following claims.

**Claim 1.1.** For every $i \in (\mathbb{N}, d)$, $i$ is an isolated point of $X_e$.

Let $i \in (\mathbb{N}, d)$. Without loss of generality, assume $\sigma_i \in T_e$. Let $n := |\sigma_i|$. Then for every $j \in (\mathbb{N}, d)$ with $d(i, j) > 0$, we have $\sigma_i \neq \sigma_j$, and so it follows from the definition of $d$ that $d(i, j) \geq w_n$. So $(\forall j \in (\mathbb{N}, d))(d(i, j) > 0 \Longrightarrow d(i, j) \geq w_n)$. Thus, since $(\mathbb{N}, d)$ is dense in $X_e$, we have $(\forall y \in X_e)(d(i, y) > 0 \Longrightarrow d(i, y) \geq w_n)$. Therefore, $i$ is an isolated point of $X_e$.

**Claim 1.2.** $(\forall N \in \mathbb{N})(\exists k, l \geq N)(d(i_{n_k}, i_{n_l}) \neq 0)$.

Suppose for a contradiction that there is an $N \in \mathbb{N}$ such that for all $k, l \geq N$, $d(i_{n_k}, i_{n_l}) = 0$. So $d(i_{n_k}, i_{n_N}) = 0$ for all $k \geq N$. Thus, since $(i_{n_k})_{k\in\mathbb{N}}$ converges to $x$ in $X_e$, we have $d(x, i_{n_N}) = 0$, and so $x = i_{n_N}$ (in $X_e$). Since $x = i_{n_N} \in (\mathbb{N}, d)$, by Claim 1.1, $x$ is an isolated point of $X_e$, but $x$ is not an isolated point of $X_e$, a contradiction.

**Claim 1.3.** $(\forall N \in \mathbb{N})(\exists k, l \geq N)(d(i_{n_k}, i_{n_l}) \geq w_M)$.

Let $N \in \mathbb{N}$. By Claim 1.2, there are $k, l \in \mathbb{N}$ such that $d(i_{n_k}, i_{n_l}) \neq 0$. Then $\sigma_{i_{n_k}} \neq \sigma_{i_{n_l}}$. Let $\tau_{k,l}$ be the longest common initial segment of $\sigma_{i_{n_k}}$ and $\sigma_{i_{n_l}}$, i.e. the longest string such that $\tau_{k,l} \subseteq \sigma_{i_{n_k}}, \sigma_{i_{n_l}}$. Then $\tau_{k,l} \subsetneq \sigma_{i_{n_k}}$ or $\tau_{k,l} \subsetneq \sigma_{i_{n_l}}$. Hence $|\tau_{k,l}| \leq |\sigma_{i_{n_k}}| - 1$ or $|\tau_{k,l}| \leq |\sigma_{i_{n_l}}| - 1$. Also, since $|\tau_{k,l}| \leq |\sigma_{i_{n_k}}| < M$, $w_{|\tau_{k,l}|} > w_M$. So

$$d(i_{n_k}, i_{n_l}) = \sum_{i=|\tau_{k,l}|}^{|\sigma_{i_{n_k}}|-1} w_i + \sum_{i=|\tau_{k,l}|}^{|\sigma_{i_{n_l}}|-1} w_i \geq w_{|\tau_{k,l}|} > w_M.$$

Claim 1.3 implies that $(i_{n_k})_{k\in\mathbb{N}}$ is not Cauchy, but $(i_{n_k})_{k\in\mathbb{N}}$ converges to $x$ in $X_e$, a contradiction. Therefore, $\lim_{n\to\infty} |\sigma_{i_n}| = \infty$, and we have proved Claim 1.

Recall that $(i_n)_{n\in\mathbb{N}}$ converges to $x$ in $X_e$. By Claim 1,

$$d(0, x) = \lim_{n\to\infty} d(0, i_n) = \lim_{n\to\infty} |r_0 - r_{|\sigma_{i_n}|}| = \lim_{n\to\infty} |r_0 - r_n| = \lim_{n\to\infty} r_n = r.$$

Hence $0, x$ are points in $X_e$ with $d(0, x) = r$, and so, by condition (3), $T_e$ must have an infinite path.

Therefore, we can conclude that for every $e \in \mathbb{N}$, $T_e$ has an infinite path $\Longleftrightarrow$ $X_e$ is not discrete. Thus, since $\{e : T_e \text{ has an infinite path}\}$ is $\Sigma^1_1$-hard, $\{e : M_e \text{ is discrete}\}$ is $\Pi^1_1$-hard. $\square$

# Chapter 5

# The Urysohn Space

## 5.1 Fraïssé Limits

Urysohn spaces are closely related to Fraïssé limits. We will see later that they can be built from the Fraïssé limits of some classes of metric spaces. In this section, we give some background and classical results on Fraïssé limits, which can be found in [7].

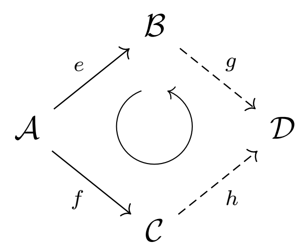
**Definition 5.1.1.** Let $\mathcal{L}$ be a language and $\mathcal{D}$ be an $\mathcal{L}$-structure. The *age* of $\mathcal{D}$ is the class of all finitely generated structures that can be embedded in $\mathcal{D}$. A class $\mathbb{K}$ of finitely generated structures is called an *age* of $\mathcal{D}$ if the structures in $\mathbb{K}$ are, up to isomorphism, exactly the finitely generated substructures of $\mathcal{D}$. A class is called an *age* if it is the age of some structure.

**Definition 5.1.2.** Let $\mathbb{K}$ be a class of finitely generated structures.

- $\mathbb{K}$ has the *hereditary property* (HP) if, whenever $\mathcal{A} \in \mathbb{K}$ and $\mathcal{B}$ is a finitely generated substructure of $\mathcal{A}$, we have that $\mathcal{B}$ is isomorphic to a structure in $\mathbb{K}$.

- $\mathbb{K}$ has the *joint embedding property* (JEP) if, for every $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, there is a $\mathcal{C} \in \mathbb{K}$ such that both $\mathcal{A}$ and $\mathcal{B}$ can be embedded into $\mathcal{C}$.

- $\mathbb{K}$ has the *amalgamation property* (AP) if, whenever $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, and $e : \mathcal{A} \hookrightarrow \mathcal{B}$ and $f : \mathcal{A} \hookrightarrow \mathcal{C}$ are embeddings, we have that there exist a $\mathcal{D} \in \mathbb{K}$ and embeddings $g : \mathcal{B} \hookrightarrow \mathcal{D}$ and $h : \mathcal{C} \hookrightarrow \mathcal{D}$ such that $g \circ e = h \circ f$.




**Theorem 5.1.3** (Fraïssé, see [7]). *A class of finitely generated structures $\mathbb{K}$ is an age if and only if $\mathbb{K}$ satisfies HP and JEP.*

**Definition 5.1.4.** An $\mathcal{L}$-structure $\mathcal{D}$ is *homogeneous* if every isomorphism between finitely generated substructures of $\mathcal{D}$ extends to an automorphism of $\mathcal{D}$.

**Definition 5.1.5.** Let $\mathbb{K}$ be a class of finitely generated structures. A structure $\mathcal{D}$ is the *Fraïssé limit* of $\mathbb{K}$ if $\mathcal{D}$ is countable, homogeneous and has age $\mathbb{K}$.

**Theorem 5.1.6** (Fraïssé, see [7]). *The Fraïssé limit of a class of finitely generated structures is unique up to isomorphism.*

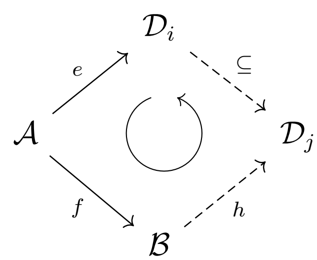
**Theorem 5.1.7** (Fraïssé, see [7]). *A class $\mathbb{K}$ of finitely generated structures has a Fraïssé limit if and only if $\mathbb{K}$ satisfies HP, JEP and AP.*

**Definition 5.1.8.** We say that a countable structure $\mathcal{D}$ of age $\mathbb{K}$ is *universal* (*for* $\mathbb{K}$) if every countable structure of age $\subseteq \mathbb{K}$ is embeddable in $\mathcal{D}$.

**Theorem 5.1.9** (Fraïssé, see [7])**.** *If $\mathcal{D}$ is the Fraïssé limit of a class $\mathbb{K}$ of finitely generated structures, then $\mathcal{D}$ is universal for $\mathbb{K}$.*

The following gives us a way to construct Fraïssé limits.

**Theorem 5.1.10** (see [7])**.** *Let $\mathbb{K}$ be a class of finitely generated structures that satisfies HP, JEP and AP. Let $(D_i)_{i\in\mathbb{N}}$ be a chain of structures in $\mathbb{K}$ with the property that for every $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ and $i \in \mathbb{N}$, if $f : \mathcal{A} \hookrightarrow \mathcal{B}$ and $e : \mathcal{A} \hookrightarrow \mathcal{D}_i$ are embeddings, then there exist a $j > i$ and an embedding $h : \mathcal{B} \hookrightarrow \mathcal{D}_j$ which extends $f$. Then $\mathcal{D} := \bigcup_{i\in\mathbb{N}} \mathcal{D}_i$ is the Fraïssé limit of $\mathbb{K}$.*




## 5.2 Fraïssé Limits for Metric Spaces

Consider a language $\mathcal{L} := \{R_q : q \in \mathbb{Q}_0^+\}$ where $\mathbb{Q}_0^+ := \mathbb{Q}^+ \cup \{0\}$ and $R_q$'s are binary relation symbols. We can think of a rational-valued metric space $(X, d)$ as an $\mathcal{L}$-structure where, for every $x, y \in X$ and $q \in \mathbb{Q}_0^+$,

$$R_q(x, y) \text{ is true in } (X, d) \iff d(x, y) = q.$$

Since $\mathcal{L}$ has no function symbols, the finitely generated $\mathcal{L}$-structures are exactly the finite $\mathcal{L}$-structures. The embeddings from $X$ into $Y$ are exactly the distance-preserving maps from $X$ into $Y$. The isomorphisms between metric spaces are exactly the isometries. We will simply write $f : X \hookrightarrow Y$ to denote that $f$ is an (isometric) embedding from $X$ into $Y$.

Henceforth, we let $\mathbb{K}$ be the class of all finite rational metric spaces, and for each $r \in \mathbb{R}^+ \cup \{\infty\}$, we let $\mathbb{K}_{<r}$ be the class of all finite rational metric spaces with diameter less than $r$, and let $\mathbb{K}_{\leq r}$ be the class of all finite rational metric spaces with diameter less than or equal to $r$. Note that $\mathbb{K}_{<\infty} = \mathbb{K}_{\leq\infty} = \mathbb{K}$.

**Proposition 5.2.1.** *For all $r \in \mathbb{R}^+ \cup \{\infty\}$, $\mathbb{K}_{<r}$ and $\mathbb{K}_{\leq r}$ satisfies HP, JEP and AP, and so they have Fraïssé limits.*

*Proof.* Let $r \in \mathbb{R}^+ \cup \{\infty\}$. First, we show that $\mathbb{K}_{<r}$ satisfies HP. Let $\mathcal{A} \in \mathbb{K}_{<r}$ and $\mathcal{B} \subseteq \mathcal{A}$. Then $diam(B) \leq diam(A) < r$, and so $\mathcal{B} \in \mathbb{K}_{<r}$. Therefore, $\mathbb{K}_{<r}$ satisfies HP.

Next, we show that $\mathbb{K}_{<r}$ satisfies AP. Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}_{<r}$, and $e : \mathcal{A} \hookrightarrow \mathcal{B}$ and $f : \mathcal{A} \hookrightarrow \mathcal{C}$ are embeddings.

We write $\mathcal{A} = \{a_0, \ldots, a_k\}$, $\mathcal{B} \setminus e(\mathcal{A}) = \{b_0, \ldots, b_{n-1}\}$, and $\mathcal{C} \setminus f(\mathcal{A}) = \{c_0, \ldots, c_{m-1}\}$ where $k, n, m \in \mathbb{N}$. Let $d_{\mathcal{A}}, d_{\mathcal{B}}$ and $d_{\mathcal{C}}$ be the metrics associated with $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively

Define a metric space $\mathcal{D} = (D, d_{\mathcal{D}})$ by

$$D := \{a_0, \ldots, a_k\} \sqcup \{\widetilde{b}_0, \ldots, \widetilde{b}_{n-1}\} \sqcup \{\widetilde{c}_0, \ldots, \widetilde{c}_{m-1}\},$$

and $d_{\mathcal{D}}$ is the metric on $\mathcal{D}$ defined by

- $d_{\mathcal{D}}(a_i, a_j) = d_{\mathcal{A}}(a_i, a_j)$, $d_{\mathcal{D}}(\widetilde{b}_i, \widetilde{b}_j) = d_{\mathcal{B}}(b_i, b_j)$, $d_{\mathcal{D}}(\widetilde{c}_i, \widetilde{c}_j) = d_{\mathcal{C}}(c_i, c_j)$,
- $d_{\mathcal{D}}(a_i, \widetilde{b}_j) = d_{\mathcal{B}}(e(a_i), b_j)$, $d_{\mathcal{D}}(a_i, \widetilde{c}_j) = d_{\mathcal{C}}(f(a_i), c_j)$,
- $d_{\mathcal{D}}(\widetilde{b}_i, \widetilde{c}_j) = \max\{|d_{\mathcal{B}}(b_i, e(a_l)) - d_{\mathcal{C}}(f(a_l), c_j)| : l \leq k\}$.

It is straightforward to check that $d_{\mathcal{D}}$ is a metric, and it is clear that $diam(\mathcal{D}) = \max\{diam(\mathcal{B}), diam(\mathcal{C})\} < r$. So $\mathcal{D} \in \mathbb{K}_{<r}$.

Define $g : \mathcal{B} \to \mathcal{D}$ and $h : \mathcal{C} \to \mathcal{D}$ by

$$g : e(a_i) \mapsto a_i, b_i \mapsto \widetilde{b}_i,$$
$$h : f(a_i) \mapsto a_i, c_i \mapsto \widetilde{c}_i.$$

It is easy to see that $g$ and $h$ are embeddings and $g \circ e = h \circ f$. Therefore, $\mathbb{K}_{<r}$ satisfies AP.

Finally, we show that $\mathbb{K}_{<r}$ satisfies JEP. Let $\mathcal{B}, \mathcal{C} \in \mathbb{K}_{<r}$. Let $\mathcal{A} = \{0\}$ be the one-point metric space. Then $\mathcal{A} \in \mathbb{K}_{<r}$, $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{A} \hookrightarrow \mathcal{C}$. Thus, since $\mathbb{K}_{<r}$ satisfies AP, there is a $\mathcal{D} \in \mathbb{K}_{<r}$ such that $\mathcal{B} \hookrightarrow \mathcal{D}$ and $\mathcal{C} \hookrightarrow \mathcal{D}$. Therefore, $\mathbb{K}_{<r}$ satisfies JEP.

Since $\mathcal{K}_{<r}$ satisfies HP, JEP and AP, by Theorem 5.1.7, it has a Fraïssé limit.

We can use the same argument for $\mathbb{K}_{\leq r}$. □

**Definition 5.2.2.**

- The *rational Urysohn space*, denoted by $\mathbb{U}_{\mathbb{Q}}$, is the Fraïssé limit of $\mathbb{K}$.
- The *Urysohn space*, denoted by $\mathbb{U}$, is the completion of $\mathbb{U}_{\mathbb{Q}}$.

- For each $r \in \mathbb{R}^+$, the *bounded rational Urysohn space of diameter* $r$, denoted by $\mathbb{U}_{\mathbb{Q},\le r}$, is the Fraïssé limit of $\mathbb{K}_{\le r}$.
- For each $r \in \mathbb{R}^+$, the *bounded Urysohn space of diameter* $r$, denoted by $\mathbb{U}_{\le r}$, is the completion of $\mathbb{U}_{\mathbb{Q},\le r}$.

Similarly, for each $r \in \mathbb{R}^+$, we can define $\mathbb{U}_{\mathbb{Q},<r}$ and $\mathbb{U}_{<r}$ from $\mathbb{K}_{<r}$. However, it can be shown that $\mathbb{U}_{<r} \cong \mathbb{U}_{\le r}$.

Recall that $\mathbb{K}_{<\infty} = \mathbb{K}_{\le\infty} = \mathbb{K}$. Our arguments for $\mathbb{U}, \mathbb{U}_{<r}$ and $\mathbb{U}_{\le r}$ would be the same in most cases. So, for convenience, we will let $\mathbb{U}_{<\infty} = \mathbb{U}_{\le\infty} = \mathbb{U}$.

It follows from Definition 5.1.5 and Theorem 5.1.9 that for every $r \in \mathbb{R}^+ \cup \{\infty\}$, $\mathbb{U}_{\le r}$ is the unique (up to isometry) Polish metric space with diameter $\le r$ that has the following properties:

(1) $\mathbb{U}_{\le r}$ is *universal* for separable metric spaces with diameter $\le r$, that is, every separable metric space $X$ with $diam(X) \le r$ can be embedded into $\mathbb{U}_{\le r}$.

(2) $\mathbb{U}_{\le r}$ is *homogeneous*, that is, every isometry between two finite subsets of $\mathbb{U}_{\le r}$ extends to a self-isometry of $\mathbb{U}_{\le r}$.

Note that $diam(\mathbb{U}_{\le r}) = diam(\mathbb{U}_{<r}) = r$ for all $r \in \mathbb{R}^+ \cup \{\infty\}$.

Note that if $(X, d)$ is a finite rational metric space, say $X = \{x_0, \ldots, x_n\}$, then $(X, d)$ is a finitely generated structure (because $X$ is finite), and it can be represented by the natural number that codes the finite set

$$\{\langle i, j, q\rangle : i, j \in \{0, \ldots, n\}, q \in \mathbb{Q}_0^+, d(x_i, x_j) = q\}.$$

This implies that $(X, d)$ is computably presentable. Also note that for any $n \in \mathbb{N}$, we can effectively determine if $n$ represents a finite rational metric space. So $\mathbb{K}$ is a computable set of finite metric spaces. In particular, we can effectively enumerate all finite rational metric spaces (up to isometry), that is, $\mathbb{K}$ is c.e.

Now suppose $r \in \mathbb{R}^+$ is left-c.e. Then there is an index $j \in \mathbb{N}$ such that $(\varphi_j(n))_{n\in\mathbb{N}} \subseteq \mathbb{Q}^+$ is an increasing sequence converging to $r$. It is easy to see that $\mathbb{K}_{<\varphi_j(n)}$ is computable uniformly in $n$. Therefore, $\mathbb{K}_{<r} = \bigcup_{n\in\mathbb{N}} \mathbb{K}_{<\varphi_j(n)}$ is c.e.

**Definition 5.2.3.** A *computable chain* of finite metric spaces is a chain

$$\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \dots$$

of computable presentations of finite metric spaces, where both the presentations $\mathcal{D}_i$ and the embeddings $\delta_i : \mathcal{D}_i \hookrightarrow \mathcal{D}_{i+1}$ are computable uniformly in $i$.

The following theorem, which is a special case of Lemma 2.9 in [2], allows us to construct a computable metric space from a computable chain of finite computable metric spaces. We will use this theorem together with Theorem 5.1.10 to build a computable presentation of the Urysohn space and the bounded Urysohn spaces.

**Theorem 5.2.4.** *Let $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \dots$ be a computable chain of finite metric spaces. Then there exist a computable presentation $\mathcal{C}$ of the completion of the union $\bigcup_{i\in\mathbb{N}} \mathcal{D}_i$ of the chain over these embeddings, and embeddings $\theta_i : \mathcal{D}_i \hookrightarrow \mathcal{C}$ that are computable uniformly in $i$.*

Theorem 5.2.4 also applies when the chain $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \dots$ stabilizes, that is, there is a $j \in \mathbb{N}$ such that $\mathcal{D}_i \cong \mathcal{D}_j$ for all $i \geq j$. In this case, we have $\mathcal{D} = \bigcup_{i=0}^{j} \mathcal{D}_i$, which is finite, and so the completion of $\mathcal{D}$ is just $\mathcal{D}$ itself. To build a computable presentation of $\mathcal{D}$, we define a computable function $\varphi$ that induces the finite metric space $\mathcal{D}$. We ensure that $\varphi$ is total by adding rational points that are distance 0 from some fixed point in $\mathcal{D}$. For example, if $\mathcal{D} = (\{0, \ldots, n\}, d_{\mathcal{D}})$, then we can identify all rational points $i > n$ with the rational point 0 by defining for all $k \in \mathbb{N}$,

- $\varphi(i, j, k) = d_{\mathcal{D}}(i, j)$ for all $i, j \leq n$,
- $\varphi(i, j, k) = \varphi(j, i, k) = d_{\mathcal{D}}(0, j)$ for all $i > n$ and $j \leq n$,
- $\varphi(i, j, k) = 0$ for all $i, j > n$.

## 5.3 Katětov Maps and Extension Properties

**Definition 5.3.1.** Let $(X, d)$ be a metric space. A map $f : X \to \mathbb{R}$ is a *Katětov map* on $X$ if

$$(\forall x, y \in X)(|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)).$$

Let $E(X)$ denote the set of all Katětov maps on $X$, and $E_{\mathbb{Q}}(X)$ denote the set of all rational-valued Katětov maps on $X$.

For each $r \in \mathbb{R}^+ \cup \{\infty\}$, we write $|f| < r$ to mean that $|f(x)| < r$ for all $x \in X$. For each $r \in \mathbb{R}^+ \cup \{\infty\}$, we let $E_{<r}(X) := \{f \in E(X) : |f| < r\}$ and $E_{\mathbb{Q},<r}(X) := \{f \in E_{\mathbb{Q}}(X) : |f| < r\}$. We define $E_{\leq r}(X)$ and $E_{\mathbb{Q},\leq r}(X)$ similarly. Note that, if $f \in E(X)$, then $|f(x)| < \infty$ for all $x \in X$. So we have $E_{<\infty}(X) = E_{\leq\infty}(X) = E(X)$.

Observe that if $f \in E(A)$, then for all $x \in X$, $0 = d(x, x) \leq f(x) + f(x)$, and so $f(x) \geq 0$. Therefore, Katětov maps are non-negative functions.

The Katětov maps on $(X,d)$ correspond to the one-point metric extensions of $X$ in the sense that, $f$ is a Katětov map on $X$ if and only if, setting $d(x,z) = f(x)$ defines a metric extension to $X \cup \{z\}$ of the metric $d$ on $X$.

**Definition 5.3.2.** Let $(X,d)$ be a metric space.

- $X$ has the *extension property* if

$$(\forall \text{ finite } A \subseteq X)(\forall f \in E(A))(\exists z \in X)(\forall a \in A)(d(z,a) = f(a)).$$

- $X$ has the *approximate extension property* if

$$(\forall \text{ finite } A \subseteq X)(\forall f \in E(A))(\forall \varepsilon > 0)(\exists z \in X)(\forall a \in A)(|d(z,a) - f(a)| \leq \varepsilon).$$

- For a dense subset $D \subseteq X$, we say $X$ has the *rational approximate extension property with respect to $D$* if

$$(\forall \text{ finite } A \subseteq D)(\forall f \in E_{\mathbb{Q}}(A))(\forall \varepsilon \in \mathbb{Q}^+)(\exists z \in D)(\forall a \in A)(|d(z,a) - f(a)| \leq \varepsilon).$$

- $X$ has the *dense approximate extension property* if $X$ has the rational approximate extension property with respect to some dense subset $D$.

Similarly, for each $r \in \mathbb{R}^+ \cup \{\infty\}$, we define the above properties for $E_{\leq r}$ and $E_{<r}$ by replacing $E(A)$ with $E_{\leq r}(A)$ and $E_{<r}(A)$, respectively.

**Remark 5.3.3.**

(1) *If $X$ has the extension property, then $X$ has the approximate extension property.*

(2) *If $X$ has the approximate extension property, then for every dense subset $D \subseteq X$, $X$ has the rational approximate extension property with respect to $D$.*

*The same is true for the properties for $E_{\leq r}$ and $E_{<r}$.*

**Theorem 5.3.4** (see, e.g. Melleray [10], Urysohn [19])**.** *If $X$ is a complete metric space and has the approximate extension property, then $X$ has the extension property.*

By the same argument as the proof of Theorem 5.3.4 (Theorem 3.4 in [10]), we have the following.

**Theorem 5.3.5.** *Let $r \in \mathbb{R}^+ \cup \{\infty\}$ and $X$ be a complete metric space with $diam(X) \leq r$. If $X$ has the approximate extension property for $E_{<r}$, then $X$ has the extension property for $E_{<r}$.*

*Proof.* Let $r \in \mathbb{R}^+ \cup \{\infty\}$. Assume that $X$ is a complete metric space with $diam(X) \leq r$ and $X$ has the approximate extension property for $E_{<r}$. We want to show that $X$ has the extension property for $E_{<r}$.

The case when $r = \infty$ is Theorem 3.4 in [10]. Now assume $r \in \mathbb{R}^+$. Let $A \subseteq X$ be finite and $f \in E_{<r}(A)$. We write $A = \{a_1, \ldots, a_n\}$.

Fix a $\delta$ such that $0 < \delta < r - \max\{f(a) : a \in A\}$. By the same argument as the proof of Theorem 3.4 in [10], we can construct a Cauchy sequence $(z_k)_{k\in\mathbb{N}}$ in $X$ such that for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$,

- $|d(z_k, a_i) - f(a_i)| \leq 2^{-k}\delta$,
- $d(z_k, z_{k+1}) \leq 2^{1-k}\delta$.

It follows that $X$ has the extension property for $E_{<r}$. □

It turns out that the approximate extension property gives a characterization of the Urysohn space $\mathbb{U}$.

**Theorem 5.3.6** (see, e.g. [10], Urysohn [19])**.** *A Polish metric space has the approximate extension property if and only if it is isometric to the Urysohn space* $\mathbb{U}$*.*

By the same argument as the proof of Theorem 5.3.6, we have the same result for bounded Urysohn spaces.

**Corollary 5.3.7.** *Let* $r \in \mathbb{R}^+ \cup \{\infty\}$ *and* $X$ *be a Polish metric space with* $diam(X) \leq r$*. Then* $X$ *has the approximate extension property for* $E_{\leq r}$ *if and only if* $X \cong \mathbb{U}_{\leq r}$*. The same is true for* $E_{<r}$*.*

The following theorem says that if a metric space $X$ has a dense subset $D$, then we can approximate any finite set $A \subseteq X$ and any Katětov map $f \in E_{<r}(A)$ with a finite set $\widetilde{A} \subseteq D$ and a Katětov map $\widetilde{f}_{\mathbb{Q},<r}(\widetilde{A})$ with any degree of accuracy.

**Theorem 5.3.8.** *Let* $r \in \mathbb{R}^+ \cup \{\infty\}$*. If* $(X, d)$ *is a metric space with* $diam(X) \leq r$ *and* $D$ *is a dense subset of* $X$*, then for all finite sets* $A \subseteq X$*,* $f \in E_{<r}(A)$ *and* $\varepsilon > 0$*, there exist an* $\widetilde{A} = \{\widetilde{a} : a \in A\} \subseteq D$ *and an* $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$ *such that for all* $a \in A$*,*

$$d(\widetilde{a}, a) < \varepsilon \quad \text{and} \quad |\widetilde{f}(\widetilde{a}) - f(a)| < \varepsilon.$$

*Proof.* Assume $(X, d)$ is a metric space with $diam(X) \leq r$ and $D$ is a dense subset of $X$. Let $A \subseteq X$ be finite, $f \in E_{<r}(A)$, and $\varepsilon > 0$.

The case when $A = \emptyset$ is trivial. Now assume that $A \neq \emptyset$. Then $f(A) \neq \emptyset$.

Write $f(A) = \{f(x_1) < f(x_2) < \cdots < f(x_m)\}$ where $x_1, \ldots, x_m \in A$ and $m \geq 1$. Then for each $x \in A$, there is a unique $k \in \{1, \ldots, m\}$ such that $f(x) = f(x_k)$. Since

$f \in E_{<r}(A)$, $f(x_m) < r$. So there exists an $\varepsilon' > 0$ such that $\varepsilon' < \min\{\varepsilon, r - f(x_m)\}$. If $r = \infty$, then for all $a \in \mathbb{R}$, we let $r - a = r + a = \infty$. Thus, in this case, if $r = \infty$, then $r - f(x_m) = \infty$, and so $\varepsilon' < \min\{\varepsilon, \infty\} = \varepsilon$.

First, we consider the case when $m = 1$. Then $f(A) = \{f(x_1)\}$, and so for every $x \in A$, $f(x) = f(x_1)$. Since $D$ is dense in $X$, for each $x \in A$, there exists an $\widetilde{x} \in D$ such that $d(\widetilde{x}, x) < \frac{\varepsilon'}{2}$. Let $\widetilde{A} := \{\widetilde{x} : x \in A\} \subseteq D$. Since $A$ is finite, $\widetilde{A}$ is finite.

Define a function $\widetilde{f} : \widetilde{A} \to \mathbb{Q}$ by

$$\widetilde{f}(\widetilde{x}) = q \quad \text{for all } x \in A,$$

where $q \in \mathbb{Q}$ is such that $f(x_1) + \frac{\varepsilon'}{2} < q < f(x_1) + \varepsilon'$. (Note that $\widetilde{f}$ is well-defined because it is a constant function.) Then for all $x \in A$,

$$f(x) + \frac{\varepsilon'}{2} = f(x_1) + \frac{\varepsilon'}{2} < q = \widetilde{f}(\widetilde{x}) < f(x_1) + \varepsilon' = f(x) + \varepsilon',$$

So for all $x, y \in A$,

$$\begin{aligned} d(\widetilde{x}, \widetilde{y}) &\leq d(\widetilde{x}, x) + d(x, y) + d(y, \widetilde{y}) \quad \text{(by the triangle inequality)} \\ &< \frac{\varepsilon'}{2} + d(x, y) + \frac{\varepsilon'}{2} \\ &\leq \frac{\varepsilon'}{2} + f(x) + f(y) + \frac{\varepsilon'}{2} \quad (\because f \in E_{<r}(A)) \\ &= \left(f(x_1) + \frac{\varepsilon'}{2}\right) + \left(f(x_1) + \frac{\varepsilon'}{2}\right) \\ &< q + q \\ &= \widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y}). \end{aligned}$$

Hence for all $x, y \in A$, $|\widetilde{f}(\widetilde{x}) - \widetilde{f}(\widetilde{y})| = |q - q| = 0 \leq d(\widetilde{x}, \widetilde{y}) \leq \widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y})$. So $\widetilde{f}$ is a Katětov map on $\widetilde{A}$, and so $\widetilde{f} \in E_{\mathbb{Q}}(\widetilde{A})$. We also have that for all $x \in A$,

$$\widetilde{f}(\widetilde{x}) < f(x) + \varepsilon' = f(x_1) + \varepsilon' < f(x_1) + (r - f(x_m)) = r.$$

Therefore, $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$.

Note that for all $x \in A$, $0 < \frac{\varepsilon'}{2} < \widetilde{f}(\widetilde{x}) - f(x) < \varepsilon' < \varepsilon$, and so $|\widetilde{f}(\widetilde{x}) - f(x)| < \varepsilon$. Recall that for all $x \in A$, $d(\widetilde{x}, x) < \frac{\varepsilon'}{2} < \varepsilon' < \varepsilon$. Therefore, $\widetilde{A} \subseteq D$ and $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$ have the desired properties.

Now, we consider the case when $m > 1$. Let

$$\begin{aligned}
\delta &:= \min\left(\{f(x_{k+1}) - f(x_k) : 1 \leq k \leq m-1\} \cup \{\frac{\varepsilon'}{4}\}\right) > 0, \\
\delta' &:= \min\{d(x, y) : x, y \in A, x \neq y\}, \\
\gamma &:= \min\{\frac{\delta}{2^m}, \delta'\}.
\end{aligned}$$

Since $m > 1$, we have $x_1, x_2 \in A$ with $x_1 \neq x_2$, so $\delta' > 0$, and so $\gamma > 0$. Since $D$ is dense in $X$, for each $x \in A$, there exists an $\widetilde{x} \in D$ such that $d(\widetilde{x}, x) < \frac{\gamma}{2}$.

Let $\widetilde{A} := \{\widetilde{x} : x \in A\} \subseteq D$. Since $A$ is finite, $\widetilde{A}$ is finite. Note that for all $x, y \in A$, if $x \neq y$, then by the triangle inequality,

$$d(\widetilde{x}, y) \geq d(x, y) - d(x, \widetilde{x}) > d(x, y) - \frac{\gamma}{2} \geq \delta' - \frac{\gamma}{2} \geq \gamma - \frac{\gamma}{2} = \frac{\gamma}{2} > d(\widetilde{y}, y),$$

and so $\widetilde{x} \neq \widetilde{y}$.

Now we can define a function $\widetilde{f} : \widetilde{A} \to \mathbb{Q}$ by

$$\widetilde{f}(\widetilde{x}) = r_k,$$

where $k \in \{1, \ldots, m\}$ is the unique number such that $f(x) = f(x_k)$, and $r_k \in \mathbb{Q}$ is such that

$$f(x_k) + \frac{3\delta}{2^{k+1}} < r_k < f(x_k) + \frac{4\delta}{2^{k+1}}.$$

First note that for all $x \in A$ with $f(x) = f(x_k)$, we have $\widetilde{f}(\widetilde{x}) = r_k = \widetilde{f}(\widetilde{x}_k)$,

$$f(x_k) + \frac{3\delta}{2^{k+1}} < \widetilde{f}(\widetilde{x}_k) < f(x_k) + \frac{4\delta}{2^{k+1}},$$

and so, since $k \geq 1$,

$$0 < \frac{3\delta}{2^{k+1}} < \widetilde{f}(\widetilde{x}) - f(x) < \frac{4\delta}{2^{k+1}} \leq \delta \leq \frac{\varepsilon'}{4}. \tag{1}$$

Note that for all $k \in \{1, \ldots, m-1\}$,

$$\frac{4\delta}{2^{k+1}} - \frac{3\delta}{2^{k+2}} < \frac{4\delta}{2^{k+1}} \leq \delta \leq f(x_{k+1}) - f(x_k),$$

and so

$$\widetilde{f}(\widetilde{x}_k) < f(x_k) + \frac{4\delta}{2^{k+1}} < f(x_{k+1}) + \frac{3\delta}{2^{k+2}} < \widetilde{f}(\widetilde{x}_{k+1}).$$

Hence for all $k, l \in \{1, \ldots, m\}$,

$$k > l \Longrightarrow \widetilde{f}(\widetilde{x}_k) > \widetilde{f}(\widetilde{x}_l). \tag{2}$$

**Claim 1.** $\gamma < (\widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y})) - (f(x) + f(y))$ for all $x, y \in A$.

Let $x, y \in A$, say $f(x) = f(x_k)$ and $f(y) = f(x_l)$ where $k, l \in \{1, \ldots, m\}$. Then

$$\begin{aligned}
(\widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y})) - (f(x) + f(y)) &= (\widetilde{f}(\widetilde{x}_k) + \widetilde{f}(\widetilde{x}_l)) - (f(x_k) + f(x_l)) \\
&= (\widetilde{f}(\widetilde{x}_k) - f(x_k)) + (\widetilde{f}(\widetilde{x}_l) - f(x_l)) \\
&> \frac{3\delta}{2^{k+1}} + \frac{3\delta}{2^{l+1}} \quad \text{(by (1))} \\
&\geq \frac{3\delta}{2^{m+1}} + \frac{3\delta}{2^{m+1}} \quad (\because k, l \leq m) \\
&> \frac{\delta}{2^m} \\
&\geq \gamma.
\end{aligned}$$

**Claim 2.** $\gamma < |f(x) - f(y)| - |\widetilde{f}(\widetilde{x}) - \widetilde{f}(\widetilde{y})|$ for all $x, y \in A$ with $f(x) \neq f(y)$.

Let $x, y \in A$ be such that $f(x) \neq f(y)$, say $f(x) = f(x_k)$ and $f(y) = f(x_l)$ where $k, l \in \{1, \ldots, m\}$. Since $f(x) \neq f(y)$, $k \neq l$. Without loss of generality, assume $k > l$. Then $l + 1 \leq k \leq m$, $f(x_k) > f(x_l)$, and by (2), $\widetilde{f}(\widetilde{x}_k) > \widetilde{f}(\widetilde{x}_l)$. So we have

$$\begin{aligned}
|f(x) - f(y)| - |\widetilde{f}(\widetilde{x}) - \widetilde{f}(\widetilde{y})| &= |f(x_k) - f(x_l)| - |\widetilde{f}(\widetilde{x}_k) - \widetilde{f}(\widetilde{x}_l)| \\
&= (f(x_k) - f(x_l)) - (\widetilde{f}(\widetilde{x}_k) - \widetilde{f}(\widetilde{x}_l)) \\
&= (\widetilde{f}(\widetilde{x}_l) - f(x_l)) - (\widetilde{f}(\widetilde{x}_k) - f(x_k)) \\
&> \frac{3\delta}{2^{l+1}} - \frac{4\delta}{2^{k+1}} \quad \text{(by (1))} \\
&\geq \frac{3\delta}{2^{l+1}} - \frac{4\delta}{2^{l+2}} \quad (\because k \geq l + 1) \\
&= \frac{\delta}{2^{l+1}} \\
&\geq \frac{\delta}{2^m} \quad (\because l + 1 \leq m)
\end{aligned}$$

$$\geq \gamma.$$

Now we claim that $\widetilde{f} \in E_{\mathbb{Q}}(\widetilde{A})$. Since $\widetilde{f} : \widetilde{A} \to \mathbb{Q}$ from the definition, it remains to show that $\widetilde{f}$ is a Katětov map on $\widetilde{A}$. Let $x, y \in A$, say $f(x) = f(x_k)$ and $f(y) = f(x_l)$ where $k, l \in \{1, \ldots, m\}$. Then

$$\begin{aligned} d(\widetilde{x}, \widetilde{y}) &\leq d(\widetilde{x}, x) + d(x, y) + d(y, \widetilde{y}) \quad \text{(by the triangle inequality)} \\ &< \frac{\gamma}{2} + d(x, y) + \frac{\gamma}{2} \\ &\leq f(x) + f(y) + \gamma \quad (\because f \in E(A)) \\ &< \widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y}). \quad \text{(by Claim 1)} \end{aligned}$$

If $f(x) = f(y)$, then $k = l$, so $\widetilde{f}(\widetilde{x}) = \widetilde{f}(\widetilde{x}_k) = \widetilde{f}(\widetilde{x}_l) = \widetilde{f}(\widetilde{y})$, and so

$$|\widetilde{f}(\widetilde{x}) - \widetilde{f}(\widetilde{y})| = 0 \leq d(\widetilde{x}, \widetilde{y}).$$

If $f(x) \neq f(y)$, then

$$\begin{aligned} d(\widetilde{x}, \widetilde{y}) &\geq d(x, y) - d(x, \widetilde{x}) - d(\widetilde{y}, y) \quad \text{(by the triangle inequality)} \\ &> d(x, y) - \frac{\gamma}{2} - \frac{\gamma}{2} \\ &\geq |f(x) - f(y)| - \gamma \quad (\because f \in E(A)) \\ &> |\widetilde{f}(\widetilde{x}) - \widetilde{f}(\widetilde{y})|. \quad \text{(by Claim 2)} \end{aligned}$$

Hence for all $x, y \in A$, $|\widetilde{f}(\widetilde{x}) - \widetilde{f}(\widetilde{y})| \leq d(\widetilde{x}, \widetilde{y}) \leq \widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y})$. That is, $\widetilde{f}$ is a Katětov map on $\widetilde{A}$, and so $\widetilde{f} \in E_{\mathbb{Q}}(\widetilde{A})$.

From (1), we have that for all $x \in A$,

$$0 \leq f(x) < \widetilde{f}(\widetilde{x}) < f(x) + \frac{\varepsilon'}{4} < f(x) + \varepsilon' \leq f(x_m) + \varepsilon' < f(x_m) + (r - f(x_m)) = r.$$

Therefore, $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$.

From (1), we also have that for all $x \in A$, $|\widetilde{f}(\widetilde{x}) - f(x)| < \frac{\varepsilon'}{4} < \varepsilon' < \varepsilon$. Also note that for all $x \in A$, $d(\widetilde{x}, x) < \frac{\gamma}{2} < \delta < \varepsilon' < \varepsilon$. Therefore, $\widetilde{A} \subseteq D$ and $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$ have the desired properties.

We conclude that there exist an $\widetilde{A} = \{\widetilde{a} : a \in A\} \subseteq D$ and an $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$ such that for all $a \in A$,

$$d(\widetilde{a}, a) < \varepsilon \quad \text{and} \quad |\widetilde{f}(\widetilde{a}) - f(a)| < \varepsilon. \qquad \square$$

**Theorem 5.3.9.** *Let $r \in \mathbb{R}^+ \cup \{\infty\}$ and $X$ be a Polish metric space with diam$(X) \leq r$. If $X$ has the dense approximate extension property for $E_{<r}$, then $X$ has the approximate extension property for $E_{<r}$.*

*Proof.* Assume that $(X, d)$ has the dense approximate extension property for $E_{<r}(A)$. Then there exists a dense subset $D \subseteq X$ such that

$$(\forall \text{ finite } \widetilde{A} \subseteq D)(\forall \widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A}))(\forall \varepsilon' \in \mathbb{Q}^+)(\exists z \in D)(\forall \widetilde{a} \in A)(|d(z, \widetilde{a}) - \widetilde{f}(\widetilde{a})| \leq \varepsilon').$$

To show that $X$ has the approximate extension property for $E_{<r}$, let $A \subseteq X$ be finite, $f \in E_{<r}(A)$ and $\varepsilon > 0$. We want to show that $(\exists z \in X)(\forall a \in A)(|d(z, a) - f(a)| \leq \varepsilon)$.

By Theorem 5.3.8, there exist an $\widetilde{A} = \{\widetilde{a} : a \in A\} \subseteq D$ and an $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$

such that for all $a \in A$,

$$d(\widetilde{a}, a) < \frac{\varepsilon}{4} \quad \text{and} \quad |\widetilde{f}(\widetilde{a}) - f(a)| < \frac{\varepsilon}{4}.$$

Since $\widetilde{A} \subseteq D$ is finite, $\widetilde{f} \in E_{\mathbb{Q},<r}(\widetilde{A})$ and $X$ has the rational approximate extension property for $E_{<r}$ with respect to $D$, we have that there exists a $z \in D$ such that for all $a \in A$,

$$|d(z, \widetilde{a}) - \widetilde{f}(\widetilde{a})| < \frac{\varepsilon}{2}.$$

Hence for all $a \in A$, $|d(z,a) - d(z,\widetilde{a})| \leq d(\widetilde{a}, a) < \frac{\varepsilon}{4}$, and so

$$\begin{aligned} |d(z,a) - f(a)| &\leq |d(z,a) - d(z,\widetilde{a})| + |d(z,\widetilde{a}) - \widetilde{f}(\widetilde{a})| + |\widetilde{f}(\widetilde{a}) - f(a)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

We conclude that

$$(\forall \text{ finite } A \subseteq X)(\forall f \in E_{<r}(A))(\forall \varepsilon > 0)(\exists z \in X)(\forall a \in A)(|d(z,a) - f(a)| \leq \varepsilon),$$

that is, $X$ has the approximate extension property for $E_{<r}$. □

By Theorem 5.3.9 and Remark 5.3.3, we have the following corollary.

**Corollary 5.3.10.** *Let $r \in \mathbb{R}^+ \cup \{\infty\}$ and $X$ be a Polish metric space with diam$(X) \leq r$. If $X$ has the dense approximate extension property for $E_{<r}$, then for every dense subset $D \subseteq X$, $X$ has the rational approximate extension property for $E_{<r}$ with respect to $D$.*

**Theorem 5.3.11.** *Let $r \in \mathbb{R}^+ \cup \{\infty\}$ and $X$ be a Polish metric space with $diam(X) \leq r$. If $X$ has the approximate extension property for $E_{<r}$, then $X$ has the approximate extension property for $E_{\leq r}$.*

*Proof.* The case when $r = \infty$ is trivial. Assume that $r \in \mathbb{R}^+$ and $(X, d)$ has the approximate extension property for $E_{<r}$. By Theorem 5.3.5, $X$ has the extension property for $E_{<r}$. We want to show that $X$ has the approximate extension property for $E_{\leq r}$, that is,

$$(\forall \text{ finite } A \subseteq X)(\forall f \in E_{\leq r}(A))(\forall \varepsilon > 0)(\exists z \in X)(\forall a \in A)(|d(z,a) - f(a)| \leq \varepsilon),$$

Let $A \subseteq X$ be finite, $f \in E_{\leq r}(A)$ and $\varepsilon > 0$. The case $A = \emptyset$ is trivial. Assume $A \neq \emptyset$.

First, we consider the case when $|A| = 1$, say $A = \{a\}$. Since $f \in E_{\leq r}(A)$, $0 \leq f(a) \leq r$. If $f(a) < r$, then $f \in E_{<r}(A)$, and so, since $X$ has the approximate extension property for $E_{<r}$, we are done.

Assume $f(a) = r$. Then $r \in \mathbb{R}^+$. Let $\delta := \min\{\frac{r}{2}, \frac{\varepsilon}{2}\} > 0$. Define $\widetilde{f} : A \to \mathbb{R}$ by $\widetilde{f}(a) = r - \delta$. Then $\widetilde{f} \in E_{<r}(A)$ and $|\widetilde{f}(a) - f(a)| = \delta \leq \frac{\varepsilon}{2}$. Thus, since $X$ has the approximate extension property for $E_{<r}$, there exists a $z \in X$ such that $|d(z,a) - \widetilde{f}(a)| \leq \frac{\varepsilon}{2}$. Therefore,

$$|d(z,a) - f(a)| \leq |d(z,a) - \widetilde{f}(a)| + |\widetilde{f}(a) - f(a)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It remains to consider the case when $|A| \geq 2$. We write $A = \{a_0, \ldots, a_n\}$ where

$n \geq 1$. Let $r_A := diam(A)$. Since $|A| \geq 2$, $r_A > 0$. Since $A \subseteq X$, $r_A \leq diam(X) \leq r$. Fix a $\delta$ such that $0 < \delta < \min\{\frac{2\varepsilon}{3r}, 1\}$. Since $0 < \delta < 1$ and $r_A > 0$, we have $0 < \delta r_A < r_A \leq r$.

Our plan is to shrink the set $A$ a little bit to get a set $\widetilde{A} = \{\widetilde{a_0}, \ldots, \widetilde{a_n}\} \subseteq X$ such that $diam(\widetilde{A}) < r$, and then apply the extension property for $E_{<r}$ of $X$ to $\widetilde{A}$.

By induction, we construct points $\widetilde{a_0}, \ldots, \widetilde{a_n} \in X$ such that for every $i, j \leq n$,

(1) $d(\widetilde{a}_i, \widetilde{a}_j) = (1-\delta)d(a_i, a_j)$,

(2) $d(a_i, \widetilde{a}_j) = \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_j) = d(a_i, \widetilde{a}_i) + d(\widetilde{a}_i, \widetilde{a}_j)$, in particular, $d(a_i, \widetilde{a}_i) = \frac{1}{2}\delta r_A$.

First, we define $f_0 : A \to \mathbb{R}$ by

$$f_0(a_i) = \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_0).$$

Then for all $i, j \leq n$,

$$f_0(a_i) \leq \frac{1}{2}\delta r + (1-\delta)r < r,$$

$$|f_0(a_i) - f_0(a_j)| = (1-\delta)|d(a_i, a_0) - d(a_j, a_0)| \leq (1-\delta)d(a_i, a_j) \leq d(a_i, a_j),$$

$$\begin{aligned} f_0(a_i) + f_0(a_j) &= \delta r_A + (1-\delta)(d(a_i, a_0) + d(a_j, a_0)) \\ &\geq \delta r_A + (1-\delta)d(a_i, a_j) \\ &\geq \delta d(a_i, a_j) + (1-\delta)d(a_i, a_j) \\ &= d(a_i, a_j). \end{aligned}$$

Therefore, $f_0 \in E_{<r}(A)$, and so, since $X$ has the extension property for $E_{<r}$, there

exists an $\widetilde{a}_0 \in X$ such that for all $x \in A$, $d(\widetilde{a}_0, x) = f_0(x)$. So for all $i \leq n$,

$$d(a_i, \widetilde{a}_0) = d(\widetilde{a}_0, a_i) = f_0(a_i) = \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_0).$$

Note that $d(\widetilde{a}_0, \widetilde{a}_0) = 0 = (1-\delta)d(a_0, a_0)$. Therefore, $\widetilde{a}_0$ satisfies (1) and (2).

Now let $k < n$ and assume by induction that we have constructed points $\widetilde{a}_0, \ldots, \widetilde{a}_k \in X$ that satisfy (1) and (2). Define $f_{k+1} : A \cup \{\widetilde{a}_0, \ldots, \widetilde{a}_k\} \to \mathbb{R}$ by

- $f_{k+1}(\widetilde{a}_i) = (1-\delta)d(a_i, a_{k+1})$,
- $f_{k+1}(a_i) = \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_{k+1})$.

Then for all $i \leq n$ and $j \leq k$,

$$0 \leq f_{k+1}(a_i) \leq \frac{1}{2}\delta r + (1-\delta)r < r,$$
$$0 \leq f_{k+1}(\widetilde{a}_j) \leq (1-\delta)r < r.$$

We claim that $f_{k+1} \in E_{<r}(A \cup \{\widetilde{a}_0, \ldots, \widetilde{a}_k\})$.

Let $A_k := A \cup \{\widetilde{a}_0, \ldots, \widetilde{a}_k\}$. Then the Katětov maps on $A_k$ correspond to the one-point metric extensions of $A_k$, where we add a new (imaginary) point, say $\widetilde{a}_{k+1}$. Define a function $d_k : A_k \cup \{\widetilde{a}_{k+1}\} \to \mathbb{R}$ by

- $d_k(\widetilde{a}_i, \widetilde{a}_j) = (1-\delta)d(a_i, a_j)$ for all $i, j \leq k+1$,
- $d_k(a_i, \widetilde{a}_j) = \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_j)$ for all $i \leq n$ and $j \leq k+1$,
- $d_k(a_i, a_j) = d(a_i, a_j)$ for all $i, j \leq n$,
- $d_k(x, x) = 0$ for all $x \in A_k \cup \{\widetilde{a}_{k+1}\}$,

- $d_k(x, y) = d_k(y, x)$ for all $x, y \in A_k \cup \{\widetilde{a}_{k+1}\}$.

Then $d_k(x, \widetilde{a}_{k+1}) = f_{k+1}(x)$ for all $x \in A_k$. Since $\widetilde{a}_0, \ldots, \widetilde{a}_k$ satisfy (1) and (2), $d_k|_{A_k \times A_k} = d|_{A_k \times A_k}$ is a metric on $A_k$. So $f_{k+1}$ is a Katětov map on $A_k$ if and only if $d_k$ is a metric on $A_k \cup \{\widetilde{a}_{k+1}\}$. We check that $d_k$ satisfies the triangle inequality as follows.

$$
\begin{aligned}
d_k(a_i, a_l) + d_k(a_l, a_j) &= d(a_i, a_l) + d(a_l, a_j) \\
&\geq d(a_i, a_j) \\
&= d_k(a_i, a_j), \\
d_k(a_i, \widetilde{a}_l) + d_k(\widetilde{a}_l, a_j) &= (\frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_l)) + (\frac{1}{2}\delta r_A + (1-\delta)d(a_l, a_j)) \\
&= \delta r_A + (1-\delta)(d(a_i, a_l) + d(a_l, a_j)) \\
&\geq \delta d(a_i, a_j) + (1-\delta)d(a_i, a_j) \\
&= d(a_i, a_j) \\
&= d_k(a_i, a_j), \\
d_k(a_i, a_l) + d_k(a_l, \widetilde{a}_j) &= d(a_i, a_l) + (\frac{1}{2}\delta r_A + (1-\delta)d(a_l, a_j)) \\
&\geq (1-\delta)d(a_i, a_l) + \frac{1}{2}\delta r_A + (1-\delta)d(a_l, a_j) \\
&\geq \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_j) \\
&= d_k(a_i, \widetilde{a}_j), \\
d_k(a_i, \widetilde{a}_l) + d(\widetilde{a}_l, \widetilde{a}_j) &= (\frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_l)) + (1-\delta)d(a_l, a_j) \\
&\geq \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_j) \\
&= d_k(a_i, \widetilde{a}_j), \\
d_k(\widetilde{a}_i, a_l) + d_k(a_l, \widetilde{a}_j) &= (\frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_l)) + (\frac{1}{2}\delta r_A + (1-\delta)d(a_l, a_j))
\end{aligned}
$$

$$
\begin{aligned}
&\geq \delta r_A + (1-\delta)d(a_i, a_j) \\
&\geq (1-\delta)d(a_i, a_j) \\
&= d_k(\widetilde{a}_i, \widetilde{a}_j), \\
d_k(\widetilde{a}_i, \widetilde{a}_l) + d_k(\widetilde{a}_l, \widetilde{a}_j) &= (1-\delta)d(a_i, a_l) + (1-\delta)d(a_l, a_j) \\
&\geq (1-\delta)d(a_i, a_j) \\
&= d_k(\widetilde{a}_i, \widetilde{a}_j).
\end{aligned}
$$

We conclude that $d_k$ is a metric on $A_k \cup \{\widetilde{a}_{k+1}\}$, and so $f_{k+1}$ is a Katětov map on $A_k$. Therefore, $f_{k+1} \in E_{<r}(A_k)$, and so, since $X$ has the extension property for $E_{<r}$, there exists an $\widetilde{a}_{k+1} \in X$ such that for all $x \in A_k$, $d(\widetilde{a}_{k+1}, x) = f_{k+1}(x)$.

So for all $i \leq n$,

$$
d(a_i, \widetilde{a}_{k+1}) = d(\widetilde{a}_{k+1}, a_i) = f_{k+1}(a_i) = \frac{1}{2}\delta r_A + (1-\delta)d(a_i, a_{k+1}),
$$
$$
d(\widetilde{a}_i, \widetilde{a}_{k+1}) = d(\widetilde{a}_{k+1}, \widetilde{a}_i) = f_{k+1}(\widetilde{a}_i) = (1-\delta)d(a_i, a_{k+1}).
$$

Therefore, since $\widetilde{a}_0, \ldots, \widetilde{a}_k$ satisfy (1) and (2), we have that $\widetilde{a}_0, \ldots, \widetilde{a}_{k+1}$ satisfy (1) and (2).

This ends the construction of $\widetilde{a}_0, \ldots, \widetilde{a}_n$.

Let $\widetilde{A} := \{\widetilde{a}_0, \ldots, \widetilde{a}_n\} \subseteq X$. Define $\widetilde{f} : \widetilde{A} \to \mathbb{R}$ by

$$
\widetilde{f}(\widetilde{a}_i) = (1-\delta)f(a_i).
$$

Since $f \in E_{\leq r}(A)$, we have that for all $i, j \leq n$,

$$\widetilde{f}(\widetilde{a}_i) = (1-\delta)f(a_i) \leq (1-\delta)r < r,$$

$$|\widetilde{f}(\widetilde{a}_i) - \widetilde{f}(\widetilde{a}_j)| = (1-\delta)|f(a_i) - f(a_j)| \leq (1-\delta)d(a_i, a_j) = d(\widetilde{a}_i, \widetilde{a}_j),$$

$$\widetilde{f}(\widetilde{a}_i) + \widetilde{f}(\widetilde{a}_j) = (1-\delta)(f(a_i) + f(a_j)) \geq (1-\delta)d(a_i, a_j) = d(\widetilde{a}_i, \widetilde{a}_j).$$

Therefore, $\widetilde{f} \in E_{<r}(\widetilde{A})$, and so, since $X$ has the extension property for $E_{<r}$, there exists a $z \in X$ such that for all $i \leq n$, $d(z, \widetilde{a}_i) = \widetilde{f}(\widetilde{a}_i)$. So for all $i \leq n$,

$$|d(z, a_i) - \widetilde{f}(\widetilde{a}_i)| = |d(z, a_i) - d(z, \widetilde{a}_i)| \leq d(a_i, \widetilde{a}_i) = \frac{1}{2}\delta r_A \leq \frac{1}{2}\delta r,$$

$$|\widetilde{f}(\widetilde{a}_i) - f(a_i)| = |(1-\delta)f(a_i) - f(a_i)| = \delta f(a_i) \leq \delta r,$$

$$\begin{aligned} |d(z, a_i) - f(a_i)| &\leq |d(z, a_i) - \widetilde{f}(\widetilde{a}_i)| + |\widetilde{f}(\widetilde{a}_i) - f(a_i)| \\ &\leq \frac{1}{2}\delta r + \delta r = \frac{3}{2}\delta r < \frac{3}{2}\left(\frac{2\varepsilon}{3r}\right) r = \varepsilon. \end{aligned}$$

We conclude that

$$(\forall \text{ finite } A \subseteq X)(\forall f \in E_{\leq r}(A))(\forall \varepsilon > 0)(\exists z \in X)(\forall a \in A)(|d(z, a) - f(a)| \leq \varepsilon),$$

that is, $X$ has the approximate extension property for $E_{\leq r}$. □

Now we can conclude the relationship among the extension properties in the following theorem.

**Theorem 5.3.12.** *Let $r \in \mathbb{R}^+ \cup \{\infty\}$ and $X$ be a Polish metric space with diam$(X) \leq r$. The following are equivalent:*

(1) *$X$ has the dense approximate extension property for $E_{<r}$.*

(2) *$X$ has the approximate extension property for $E_{<r}$.*

(3) *$X$ has the approximate extension property for $E_{\leq r}$.*

(4) *$X \cong \mathbb{U}_{\leq r}$.*

*Proof.* It is clear that (3) $\implies$ (2) $\implies$ (1). By Theorem 5.3.9, we have (1) $\implies$ (2). By Theorem 5.3.11, we have (2) $\implies$ (3). By Corollary 5.3.7, we have (3) $\iff$ (4). □

By applying Corollary 5.3.7 and Theorem 5.3.12 to $X := \mathbb{U}_{<r}$, we have the following.

**Corollary 5.3.13.** *For all $r \in \mathbb{R}^+ \cup \{\infty\}$, $\mathbb{U}_{<r} \cong \mathbb{U}_{\leq r}$.*

## 5.4 Computable Presentations of $\mathbb{U}$ and $\mathbb{U}_{\leq r}$

**Theorem 5.4.1.** *If $r = \infty$ or $r \in \mathbb{R}^+$ is left-c.e., then $\mathbb{U}_{\leq r}$ is computably presentable.*

*Proof.* Assume $r = \infty$ or $r \in \mathbb{R}^+$ is left-c.e. Then $\mathbb{K}_{<r}$ is c.e., that is, we can effectively list all finite metric spaces in $\mathbb{K}_{<r}$, say $\mathbb{K}_{<r} = \{\mathcal{A}_i : i \in \mathbb{N}\}$.

Recall that $\mathbb{U}_{\leq r} \cong \mathbb{U}_{<r}$ and $\mathbb{U}_{<r}$ is the completion of $\mathbb{U}_{\mathbb{Q},<r}$ where $\mathbb{U}_{\mathbb{Q},<r}$ is the Fraïssé limit of $\mathbb{K}_{<r}$.

To construct a computable presentation of $\mathbb{U}_{\leq r}$, we build a computable chain $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \dots$ of finite metric spaces that satisfies the property in Theorem 5.1.10 for $\mathbb{K}_{<r}$. Then, by Theorem 5.2.4, there is a computable presentation $\mathcal{C}$ of the completion of the union $\mathcal{D} := \bigcup_{s \in \mathbb{N}} \mathcal{D}_s$. By Theorem 5.1.10, $\mathcal{D}$ is the Fraïssé limit of $\mathbb{K}_{<r}$,

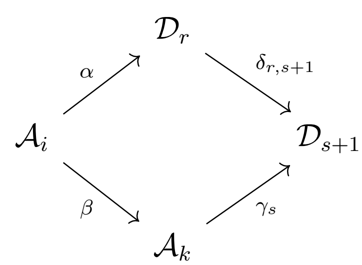
and so $\mathcal{D} \cong \mathbb{U}_{\mathbb{Q},<r}$. Hence their completions are isometric, that is, $\mathcal{C} \cong \mathbb{U}_{<r} \cong \mathbb{U}_{\leq r}$. Therefore, $\mathcal{C}$ is a computable presentation of $\mathbb{U}_{\leq r}$, and so $\mathbb{U}_{\leq r}$ is computably presentable.

It remains to build a computable chain $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \ldots$ of finite metric spaces that satisfies the property in Theorem 5.1.10 for $\mathbb{K}_{<r}$.

We will use a similar argument as the proof of Theorem 3.9 in [2]. For $i < j$, we let $\delta_{i,j}$ denote the embedding $\delta_{j-1} \circ \cdots \circ \delta_i : \mathcal{D}_i \hookrightarrow \mathcal{D}_j$.

For $\mathcal{D}$ to be the Fraïssé limit of $\mathbb{K}_{<r}$, it is enough to satisfy the following requirements:

$R_{\langle i,r,k,\alpha,\beta\rangle}$ : If $\alpha : \mathcal{A}_i \hookrightarrow \mathcal{D}_r$ and $\beta : \mathcal{A}_i \hookrightarrow \mathcal{A}_k$, then there exist an $s \geq r$ and an embedding $\gamma_s : \mathcal{A}_k \hookrightarrow \mathcal{D}_{s+1}$ such that $\delta_{r,s+1} \circ \alpha = \gamma_s \circ \beta$.



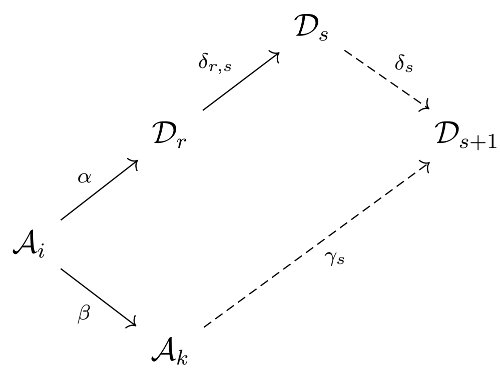

**Construction of $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \ldots$**

Stage 0: Let $\mathcal{D}_0 := \{0\}$ be the one-point metric space.

Stage $s + 1 = \langle i, r, k, \alpha, \beta\rangle + 1$ where $i, r, k, \alpha, \beta \in \mathbb{N}$: We have constructed $\mathcal{D}_t$ for all $t \leq s$ and $\delta_t$ for all $t < s$. Without loss of generality, assume $r \leq s$. We check if

$\alpha : \mathcal{A}_i \hookrightarrow \mathcal{D}_r$ and $\beta : \mathcal{A}_i \hookrightarrow \mathcal{A}_k$. (We decode $\alpha$ and $\beta$ as functions on a finite subset of $\mathbb{N}$.)

If so, then we apply AP (amalgamation property) of $\mathbb{K}_{<r}$ to $\delta_{r,s} \circ \alpha : \mathcal{A}_i \hookrightarrow \mathcal{D}_s$ and $\beta : \mathcal{A}_i \hookrightarrow \mathcal{A}_k$ to get $\mathcal{D}_{s+1} \in \mathbb{K}_{<r}$, $\delta_s : \mathcal{D}_s \hookrightarrow \mathcal{D}_{s+1}$ and $\gamma_s : \mathcal{A}_k \hookrightarrow \mathcal{D}_{s+1}$ such that $\delta_s \circ \delta_{r,s} \circ \alpha = \beta \circ \gamma_s$. Then go to the next stage.




If not, then let $\mathcal{D}_{s+1} := \mathcal{D}_s$ and $\delta_s := Id_{\mathcal{D}_s} : \mathcal{D}_s \hookrightarrow \mathcal{D}_{s+1}$, and go to the next stage.

This ends the construction.

Note that the construction is effective. So $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \ldots$ is a computable chain of finite metric spaces. It is clear from the construction that the chain $(\mathcal{D}_s)_{s\in\mathbb{N}}$ satisfies the requirements $R_{\langle i,r,k,\alpha,\beta\rangle}$.

It follows that there is a computable presentation $\mathcal{C}$ of the completion of the union $\mathcal{D} := \bigcup_{s\in\mathbb{N}} \mathcal{D}_s$ and $\mathcal{D}$ is the Fraïssé limit of $\mathbb{K}_{<r}$ Therefore, $\mathcal{C} \cong \mathbb{U}_{<r} \cong \mathbb{U}_{\leq r}$, and so $\mathbb{U}_{\leq r}$ is computably presentable. □

We will use the construction of Fraïssé limits in the proof of Theorem 5.4.1 again to prove some results about $\mathbb{U}_{\leq r}$. Thus, for convenience, we will call the construction in the proof of Theorem 5.4.1 the *"Fraïssé limit construction"*, and we will call the requirements $R_{\langle i,r,k,\alpha,\beta\rangle}$ the *"Fraïssé limit requirements"*.

We can relativize the notions for computable metric spaces in an obvious way. For example, we let $d_e^A$ denote the pseudometric induced by the partial $A$-computable function $\varphi_e^A$. The Polish metric space induced by $\varphi_e^A$ is an *$A$-computable metric space*, denoted by $M_e^A$. A Polish metric space $X$ is *$A$-computably presentable* if it has an $A$-computable presentation, that is, $X \cong M_e^A$ for some $e \in \mathbb{N}$.

**Theorem 5.4.2.** *Let $A \subseteq \mathbb{N}$. If $X$ is an $A$-computably presentable metric space, then $diam(X) = \infty$ or $diam(X)$ is a left-$A$-c.e. real.*

*Proof.* Assume $X$ is an $A$-computably presentable metric space. Then $X \cong M_e^A$ for some $e \in \mathbb{N}$. Since $\varphi_e^A$ induces the Polish metric space $M_e^A$, $\varphi_e^A$ is total. Assume $diam(X) < \infty$. Let $r := diam(X)$. Then $diam(M_e^A) = diam(X) = r \in \mathbb{R}_0^+$. Define an $A$-computable function $f : \mathbb{N} \to \mathbb{Q}$ inductively by

- $f(0) = \varphi_e^A(i_0, j_0, k_0) - 2^{-k_0}$ where $0 = \langle i_0, j_0, k_0\rangle$,
- $f(n+1) = \begin{cases} \varphi_e^A(i,j,k) - 2^{-k} & \text{if } n+1 = \langle i,j,k\rangle \text{ and } f(n) < \varphi_e^A(i,j,k) - 2^{-k} \\ f(n) & \text{otherwise} \end{cases}$

It is clear that $f$ is an increasing function.

We will show that $\lim_{n\to\infty} f(n) = r$.

**Claim 1.** $f(n) \leq r$ for all $n \in \mathbb{N}$.

Note that for all $i, j, k \in \mathbb{N}$,

$$\varphi_e^A(i, j, k) - 2^{-k} \leq d_e^A(i, j) \leq diam(M_e^A) = r.$$

So, by induction on $n$, $f(n) \leq r$ for all $n \in \mathbb{N}$.

**Claim 2.** $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(f(n) > r - \varepsilon)$.

Let $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $2^{-k} < \frac{\varepsilon}{3}$. Since $r - 2^{-k} < r = diam(M_e) = \sup\{d_e^A(i, j) : i, j \in \mathbb{N}\}$, there exist $i, j \in \mathbb{N}$ such that $r - 2^{-k} < d_e^A(i, j)$. So

$$\varphi_e^A(i, j, k) - 2^{-k} \geq (d_e^A(i, j) - 2^{-k}) - 2^{-k} > r - 2^{-k} - 2^{-k} - 2^{-k} = r - 3 \cdot 2^{-k} > r - \varepsilon.$$

Let $n := \langle i, j, k \rangle$. If $f(n) = \varphi_e^A(i, j, k) - 2^{-k}$, then $f(n) > r - \varepsilon$. Otherwise, we must have $n > 0$ and $f(n) = f(n-1)$ where $f(n-1) \geq \varphi_e^A(i, j, k) - 2^{-k}$, so $f(n) \geq \varphi_e^A(i, j, k) - 2^{-k} > r - \varepsilon$. In both cases, we have $f(n) > r - \varepsilon$.

Since $f$ is increasing, by Claim 1 and Claim 2, we have $\lim\limits_{n \to \infty} f(n) = \sup\limits_{n \in \mathbb{N}} f(n) = r$. Therefore, since $(f(n))_{n \in \mathbb{N}}$ is an $A$-computable increasing sequence of rationals converging to $r = diam(X)$, we have that $diam(X)$ is a left-$A$-c.e. real. □

By Theorem 5.4.2 and the relativized version of Theorem 5.4.1, we have the following.

**Theorem 5.4.3.** *Let $r \in \mathbb{R}^+$ and $A \subseteq \mathbb{N}$. Then $\mathbb{U}_{\leq r}$ is $A$-computably presentable if and only if $r$ is a left-$A$-c.e. real.*

We have proved that if $r = \infty$ or $r \in \mathbb{R}^+$ is left-c.e., then the bounded Urysohn space $\mathbb{U}_{\leq r}$ has a computable presentation. It turns out that $\mathbb{U}_{\leq r}$ has a unique computable presentation (up to isometry).

**Theorem 5.4.4** (Melnikov [12])**.** *The Urysohn space* $\mathbb{U}$ *is computably categorical.*

The proof of Theorem 5.4.4 (see Theorem 7.3 in [12]) also works for the bounded Urysohn spaces. So we have the following corollary.

**Corollary 5.4.5.** *For every left-c.e. real* $r \in \mathbb{R}^+$, $\mathbb{U}_{\leq r}$ *is computably categorical.*

## 5.5 Index Set Results on $\mathbb{U}$ and $\mathbb{U}_{\leq r}$

**Proposition 5.5.1.** *If* $r = \infty$ *or* $r \in \mathbb{R}^+$ *is left-c.e., then the set*

$$\{e \in \mathbb{N} : M_e \textit{ has the rational approximate extension property for } E_{<r} \textit{ w.r.t. } (\mathbb{N}, d_e)\}$$

*is* $\Pi^0_2$.

*Proof.* Assume $r = \infty$ or $r \in \mathbb{R}^+$ is left-c.e. Then there is a computable increasing sequence $(r_n)_{n\in\mathbb{N}}$ of rationals such that $\lim\limits_{n\to\infty} r_n = r$. For the case when $r = \infty$, we can choose $r_n = n$ for all $n \in \mathbb{N}$.

Let $e \in PolSp$. Then $M_e$ is a computable Polish metric space with a dense subset $D := (\mathbb{N}, d_e)$

Note that

$$\begin{aligned}
&M_e \text{ has the rational approximate extension property for } E_{<r} \text{ w.r.t. } (\mathbb{N}, d_e) \\
&\iff (\forall \text{ finite } A \subseteq D)(\forall \varepsilon \in \mathbb{Q}^+)(\forall f \in E_{\mathbb{Q},<r}(A))(\exists z \in D)(\forall a \in A)(|d_e(z, a) - f(a)| < \varepsilon)
\end{aligned}$$

$$\iff (\forall \text{ finite } A \subseteq D)(\forall \varepsilon \in \mathbb{Q}^+)(\forall f \in \mathbb{Q}^{<\mathbb{N}})$$

$$[f \notin E_{\mathbb{Q},<r}(A) \vee (\exists z \in D)(\forall a \in A)(|d_e(z,a) - f(a)| < \varepsilon)].$$

Now note that

$$f \in E_{\mathbb{Q}}(A) \iff f : A \to \mathbb{Q} \wedge (\forall x, y \in A)(|f(x) - f(y)| \leq d_e(x,y) \leq f(x) + f(y)).$$

Thus, since $A$ is finite, "$f \in E_{\mathbb{Q}}(A)$" is a $\Pi^0_1$ statement.

Since $A$ is finite and $(r_n)_{n\in\mathbb{N}}$ is a computable increasing sequence of rationals such that $\lim_{n\to\infty} r_n = r$, we have

$$\begin{aligned}
|f| < r &\iff (\forall a \in A)(|f(a)| < r) \\
&\iff (\forall a \in A)(\exists n \in \mathbb{N})(|f(a)| < r_n) \\
&\iff (\exists m \in \mathbb{N})(\forall a \in A)(|f(a)| < r_m),
\end{aligned}$$

and so "$|f| < r$" is a $\Sigma^0_1$ statement.

Hence we have

$$\begin{aligned}
f \in E_{\mathbb{Q},<r}(A) &\iff f \in E_{\mathbb{Q}}(A) \wedge |f| < r \\
&\iff (\forall k \in \mathbb{N})R(e, A, f, k) \wedge (\exists m \in \mathbb{N})Q(A, f, m),
\end{aligned}$$

where $R$ and $Q$ are computable relations.

Since $A$ is finite and "$|d_e(z,a) - f(a)| < \varepsilon$" is a $\Sigma^0_1$ statement, we have

$$(\forall a \in A)(|d_e(z,a) - f(a)| < \varepsilon) \Longleftrightarrow (\exists l \in \mathbb{N})P(e, A, \varepsilon, f, z, l),$$

where $P$ is a computable relation.

From all of the above, we have

$$\begin{aligned}
&M_e \text{ has the rational approximate extension property for } E_{<r} \text{ w.r.t. } (\mathbb{N}, d_e)\\
&\Longleftrightarrow (\forall \text{ finite } A \subseteq D)(\forall \varepsilon \in \mathbb{Q}^+)(\forall f \in \mathbb{Q}^{<\mathbb{N}})\\
&\qquad\qquad [f \notin E_{\mathbb{Q},<r}(A) \vee (\exists z \in D)(\forall a \in A)(|d_e(z,a) - f(a)| < \varepsilon)]\\
&\Longleftrightarrow (\forall \text{ finite } A \subseteq D)(\forall \varepsilon \in \mathbb{Q}^+)(\forall f \in \mathbb{Q}^{<\mathbb{N}})\\
&\qquad\qquad [(\exists k \in \mathbb{N})\neg R(e, A, f, k) \vee (\forall m \in \mathbb{N})\neg Q(A, f, m)\\
&\qquad\qquad \vee (\exists z \in D)(\exists l \in \mathbb{N})P(e, A, \varepsilon, f, z, l)]\\
&\Longleftrightarrow (\forall \text{ finite } A \subseteq D)(\forall \varepsilon \in \mathbb{Q}^+)(\forall f \in \mathbb{Q}^{<\mathbb{N}})(\forall m \in \mathbb{N})(\exists k \in \mathbb{N})(\exists z \in D)(\exists l \in \mathbb{N})\\
&\qquad\qquad [\neg R(e, A, f, k) \vee \neg Q(A, f, m) \vee P(e, A, \varepsilon, f, z, l)].
\end{aligned}$$

Therefore, the set

$$\{e \in \mathbb{N} : M_e \text{ has the rational approximate extension property for } E_{<r} \text{ w.r.t. } (\mathbb{N}, d_e)\}$$

is $\Pi^0_2$. □

The following theorem is our main tool to find the complexity of several index sets involving the spaces $\mathbb{U}_{\leq r}$.

**Theorem 5.5.2.** *Assume $r = \infty$ or $r \in \mathbb{R}^+$ is left-c.e. Let $(r_n)_{n\in\mathbb{N}}$ be a computable* *strictly increasing sequence of rationals such that* $\lim_{n\to\infty} r_n = r$*. Let* $A$ *be a* $\Sigma^0_2$ *set and* $B$ *be a* $\Pi^0_2$ *set. Then there exist computable functions* $f, g : \mathbb{N} \to \mathbb{N}$ *and a uniformly c.e. sequence* $(\mathbb{K}_e)_{e\in\mathbb{N}}$ *of classes of finite metric spaces such that for all* $e \in \mathbb{N}$*, the following conditions hold:*

(1) $f(e) \in PolSp$*, and so* $M_{f(e)}$ *is a computable Polish metric space.*

(2) $e \in A \Longrightarrow \mathbb{K}_{g(e)} = \mathbb{K}_{<r_n}$ *for some* $n \in \mathbb{N}$*,*

(3) $e \notin A \Longrightarrow \mathbb{K}_{g(e)} = \mathbb{K}_{<r}$*,*

(4) $e \in B \Longrightarrow (\mathbb{N}, d_{f(e)})$ *is the Fraïssé limit of* $\mathbb{K}_{g(e)}$*.*

(5) $e \notin B \Longrightarrow M_{f(e)}$ *is finite* $\Longrightarrow M_{f(e)} \ncong \mathbb{U}_{\leq r'}$ *for all* $r' \in \mathbb{R}^+ \cup \{\infty\}$*.*

(6) $e \in A \cap B \Longrightarrow M_{f(e)} \cong \mathbb{U}_{\leq r_n}$ *for some* $n \in \mathbb{N}$*.*

(7) $e \in B \setminus A \Longrightarrow M_{f(e)} \cong \mathbb{U}_{\leq r}$*.*

*Proof.* Recall that $Fin := \{e : dom(\varphi_e) \text{ is finite}\}$ is $\Sigma^0_2$-complete and $Tot := \{e : dom(\varphi_e) = \mathbb{N}\}$ is $\Pi^0_2$-complete.

Thus, since $A$ is $\Sigma^0_2$ and $B$ is $\Pi^0_2$, there are computable functions $g$ and $h$ such that for all $e \in \mathbb{N}$,

$$e \in A \Longleftrightarrow g(e) \in Fin,$$

$$e \in B \Longleftrightarrow h(e) \in Tot.$$

Recall that $\mathbb{K}$ is the class of all finite rational metric spaces and $\mathbb{K}$ is c.e. So we can effectively list all finite metric spaces in $\mathbb{K}$ as

$$\mathbb{K} = \{\mathcal{A}_i : i \in \mathbb{N}\}.$$

For each $e, s \in \mathbb{N}$, let $n_{e,s} := |dom(\varphi_{e,s})|$. Then $n_{e,0} \leq n_{e,1} \leq \dots$ and we can compute $n_{e,s}$ uniformly in $e, s$. Let

$$n_e := \lim_{s \to \infty} n_{e,s} = |dom(\varphi_e)| = \begin{cases} |dom(\varphi_e)| \in \mathbb{N} & \text{if } dom(\varphi_e) \text{ is finite} \\ \infty & \text{if } dom(\varphi_e) \text{ is infinite} \end{cases}$$

Without loss of generality, assume that $r_n > 0$ for all $n \in \mathbb{N}$. Then for all $e \in \mathbb{N}$,

$$r_{n_e} = \lim_{s \to \infty} r_{n_{e,s}},$$

where $r_\infty = \lim_{n \to \infty} r_n = r$.

For each $e \in \mathbb{N}$, let

$$\begin{aligned} \mathbb{K}_e := \mathbb{K}_{<r_{n_e}} &= \bigcup_{s \in \mathbb{N}} \mathbb{K}_{<r_{n_{e,s}}} \\ &= \{\mathcal{A}_i : (\exists s \in \mathbb{N})(diam(\mathcal{A}_i) < r_{n_{e,s}}), i \in \mathbb{N}\} \subseteq \mathbb{K}. \end{aligned}$$

Then, since $(r_n)_{n \in \mathbb{N}}$ is a computable sequence of rationals and $n_{e,s}$ are computable uniformly in $e, s$, we have that $(\mathbb{K}_e)_{e \in \mathbb{N}}$ is a uniformly c.e. sequence of nonempty classes of finite metric spaces. So we can effectively enumerate all metric spaces in

$\mathbb{K}_e$ uniformly in $e$ as

$$\mathbb{K}_e = \{\mathcal{A}_{e,i} : i \in \mathbb{N}\}.$$

So for all $e \in \mathbb{N}$, since $\lim_{n\to\infty} r_n = r$, we have

$$\begin{aligned} e \in A &\Longrightarrow g(e) \in Fin \Longrightarrow n_{g(e)} = |dom(\varphi_{g(e)})| \in \mathbb{N} \\ &\Longrightarrow \mathbb{K}_{g(e)} = \mathbb{K}_{<r_{n_{g(e)}}} \text{ and } r_{n_{g(e)}} < r, \\ e \notin A &\Longrightarrow g(e) \notin Fin \Longrightarrow n_{g(e)} = |dom(\varphi_{g(e)})| = \infty \Longrightarrow r_{n_{g(e)}} = r_\infty = r \\ &\Longrightarrow \mathbb{K}_{g(e)} = \mathbb{K}_{<r_{n_{g(e)}}} = \mathbb{K}_{<r}. \end{aligned}$$

Hence conditions (2) and (3) are satisfied. Therefore, for all $e \in \mathbb{N}$, $\mathbb{K}_{g(e)}$ satisfies HP, JEP and AP, and so $\mathbb{K}_{g(e)}$ has a Fraïssé limit.

For each $e \in \mathbb{N}$, we construct a computable Polish metric space $X_e$ uniformly in $e$ as follows.

**Construction of** $(X_e)_{e\in\mathbb{N}}$

To construct a computable Polish space $X_e$, we build a computable chain $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \dots$ of finite metric spaces such that for all $e \in \mathbb{N}$,

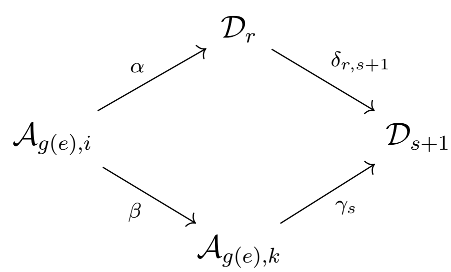
$$\begin{aligned} e \in B &\Longrightarrow (\mathcal{D}_s)_{s\in\mathbb{N}} \text{ satisfies the Fraïssé limit requirements for } \mathbb{K}_{g(e)} = \mathbb{K}_{<r_{n_{g(e)}}}, \\ e \notin B &\Longrightarrow \text{ the chain } (\mathcal{D}_s)_{s\in\mathbb{N}} \text{ is eventually stable.} \end{aligned}$$

Then we let $\mathcal{D} := \bigcup_{s\in\mathbb{N}} \mathcal{D}_s$, and let $X_e$ be the completion of $\mathcal{D}$.

For $i < j$, we let $\delta_{i,j}$ denote the embedding $\delta_{j-1} \circ \cdots \circ \delta_i : \mathcal{D}_i \hookrightarrow \mathcal{D}_j$.

For the union $\mathcal{D} = \bigcup_{s \in \mathbb{N}} \mathcal{D}_s$ to be the Fraïssé limit of $\mathbb{K}_{g(e)}$, it is enough to satisfy the following Fraïssé limit requirements:

$R_{\langle i,r,k,\alpha,\beta\rangle}$ : If $\alpha : \mathcal{A}_{g(e),i} \hookrightarrow \mathcal{D}_r$ and $\beta : \mathcal{A}_{g(e),i} \hookrightarrow \mathcal{A}_{g(e),k}$, then there exist an $s \geq r$ and an embedding $\gamma_s : \mathcal{A}_{g(e),k} \hookrightarrow \mathcal{D}_{s+1}$ such that $\delta_{r,s+1} \circ \alpha = \gamma_s \circ \beta$.




**Construction of $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \dots$**

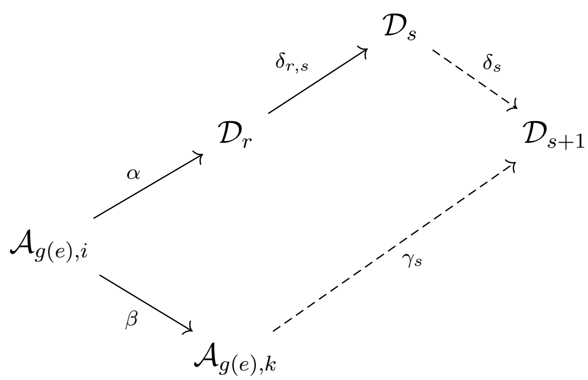
Stage 0: Let $\mathcal{D}_0 := \{0\}$ be the one-point metric space.

Stage $s + 1 = \langle i, r, k, \alpha, \beta\rangle + 1$ where $i, r, k, \alpha, \beta \in \mathbb{N}$: We have constructed $\mathcal{D}_t$ for all $t \leq s$ and $\delta_t$ for all $t < s$. Without loss of generality, assume $r \leq s$. We have 2 steps.

Step 1: For each $t \in \mathbb{N}$, starting from $t = 0$, we check if $\varphi_{h(e),t}(s)\downarrow$. Whenever we find (if ever) the least $t_0$ such that $\varphi_{h(e),t_0}(s)\downarrow$, we go to Step 2.

Step 2: We do the Fraïssé limit construction for $\mathcal{A}_{g(e),i} \in \mathbb{K}_{g(e)}$. That is, we check if $\alpha : \mathcal{A}_{g(e),i} \hookrightarrow \mathcal{D}_r$ and $\beta : \mathcal{A}_{g(e),i} \hookrightarrow \mathcal{A}_{g(e),k}$.

If so, then we apply AP (amalgamation property) of $\mathbb{K}_{g(e)}$ to $\delta_{r,s} \circ \alpha : \mathcal{A}_{g(e),i} \hookrightarrow \mathcal{D}_s$ and $\beta : \mathcal{A}_{g(e),i} \hookrightarrow \mathcal{A}_{g(e),k}$ to get $\mathcal{D}_{s+1} \in \mathbb{K}_{g(e)}$, $\delta_s : \mathcal{D}_s \hookrightarrow \mathcal{D}_{s+1}$ and $\gamma_s : \mathcal{A}_{g(e),k} \hookrightarrow \mathcal{D}_{s+1}$ such that $\delta_s \circ \delta_{r,s} \circ \alpha = \gamma_s \circ \beta$. Then go to the next stage.




If not, then let $\mathcal{D}_{s+1} := \mathcal{D}_s$ and $\delta_s := Id_{\mathcal{D}_s} : \mathcal{D}_s \hookrightarrow \mathcal{D}_{s+1}$, and go to the next stage.

This ends the construction.

Note that the construction is effective uniformly in $e$. So $\mathcal{D}_0 \overset{\delta_0}{\hookrightarrow} \mathcal{D}_1 \overset{\delta_1}{\hookrightarrow} \dots$ is a computable chain of finite metric spaces. Then, by Theorem 5.2.4, there is a computable presentation $X_e$, computable uniformly in $e$, of the completion of the union $\mathcal{D} := \bigcup_{s\in\mathbb{N}} \mathcal{D}_s$. By the *s-m-n* Theorem, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(e) \in PolSp$ and $X_e \cong M_{f(e)}$ for all $e \in \mathbb{N}$.

Next, we consider the following cases.

<u>Case $e \in B$:</u> Then $h(e) \in Tot$, and so $\varphi_{h(e)}$ is total. Hence, at every stage $s+1$, we will always find the least $t_0$ such that $\varphi_{h(e),t_0}(s) \downarrow$ in Step 1, and then we will go to Step 2 and do the Fraïssé limit construction. So it is clear from the construction that the chain $(\mathcal{D}_s)_{s\in\mathbb{N}}$ satisfies the Fraïssé limit requirements $R_{\langle i,r,k,\alpha,\beta\rangle}$. Thus, by Theorem 5.1.10, $\mathcal{D} := \bigcup_{s\in\mathbb{N}} \mathcal{D}_s$ is the Fraïssé limit of $\mathbb{K}_{g(e)} = \mathbb{K}_{<r_{n_{g(e)}}}$, and so $\mathcal{D} \cong \mathbb{U}_{\mathbb{Q},<r_{n_{g(e)}}}$. Note that $(\mathbb{N}, d_{f(e)}) \cong \mathcal{D}$, and so $(\mathbb{N}, d_{f(e)})$ is the Fraïssé limit of

$\mathbb{K}_{g(e)}$. Thus, the completion $X_e$ of $\mathcal{D}$ is isometric to the completion of $\mathbb{U}_{\mathbb{Q},<r_{n_{g(e)}}}$, that is, $X_e \cong \mathbb{U}_{<r_{n_{g(e)}}} \cong \mathbb{U}_{\leq r_{n_{g(e)}}}$.

Case $e \notin B$: Then $h(e) \notin Tot$, and so there is the least $s_0$ such that $\varphi_{h(e)}(s_0) \uparrow$. So we will do the construction until stage $s_0 + 1$, and we will never find a least $t_0$ such that $\varphi_{h(e),t_0}(s_0) \downarrow$ in Step 1. Hence we will never go to Step 2. It follows that the resulting chain is

$$\mathcal{D}_0 \hookrightarrow \mathcal{D}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{D}_{s_0}.$$

(We never defined $\mathcal{D}_s$ for all $s > s_0$.) So we have $\mathcal{D} = \mathcal{D}_{s_0}$, which is a finite metric space. Hence $X_e = \mathcal{D}$ is a finite metric space. Therefore, for all $r' \in \mathbb{R}^+ \cup \{\infty\}$, $X_e$ is not universal for $\mathbb{K}_{\leq r'}$, and so $X_e \ncong \mathbb{U}_{\leq r}$.

Case $e \in A \cap B$: Since $e \in A$, we have $n_{g(e)} = |dom(\varphi_{g(e)})| \in \mathbb{N}$, $\mathbb{K}_{g(e)} = \mathbb{K}_{<r_{n_{g(e)}}}$ and $r_{n_{g(e)}} < r$. Since $e \in B$, we have $X_e \cong \mathbb{U}_{\leq r_{n_{g(e)}}}$.

Case $e \in B \setminus A$: Since $e \notin A$, we have $n_{g(e)} = |dom(\varphi_{g(e)})| = \infty$, $r_{n_{g(e)}} = r$ and $\mathbb{K}_{g(e)} = \mathbb{K}_{<r_{n_{g(e)}}} = \mathbb{K}_{<r}$. Thus, since $e \in B$, we have $X_e \cong \mathbb{U}_{\leq r_{n_{g(e)}}} = \mathbb{U}_{\leq r}$.

From all of the above, we conclude that conditions (1)-(7) are satisfied. □

**Theorem 5.5.3.** *If $r = \infty$ or $r \in \mathbb{R}^+$ is left-c.e., then the set $\{e \in \mathbb{N} : M_e \cong \mathbb{U}_{\leq r}\}$ is $\Pi^0_2$-complete within PolSp, and so it is $\Pi^0_2$-complete.*

*Proof.* Assume $r = \infty$ or $r \in \mathbb{R}^+$ is left-c.e. Then there is a computable strictly increasing sequence $(r_n)_{n \in \mathbb{N}}$ of rationals such that $\lim_{n\to\infty} r_n = r$. For the case when $r = \infty$, we can choose $r_n = n$ for all $n \in \mathbb{N}$.

By Theorem 5.3.12 and Remark 5.3.3, we have that for all $e \in PolSp$, $M_e \cong U_{\leq r}$

if and only if $diam(M_e) \leq r$ and $M_e$ has the rational approximate extension property for $E_{<r}$ w.r.t. the dense set $(\mathbb{N}, d_e)$.

Since $(r_n)_{n\in\mathbb{N}}$ is a computable increasing sequence of rationals such that $\lim_{n\to\infty} r_n = r$, we have

$$\begin{aligned} diam(M_e) \leq r &\iff (\forall i, j \in \mathbb{N})(d_e(i,j) \leq r) \\ &\iff (\forall i, j \in \mathbb{N})(\forall \varepsilon \in \mathbb{Q}^+)(\exists n \in \mathbb{N})(d_e(i,j) - \varepsilon < \varphi_e(n)). \end{aligned}$$

Hence "$diam(M_e) \leq r$" is a $\Pi^0_2$ statement. Therefore, by Proposition 5.5.1, $\{e : M_e \cong \mathbb{U}_{\leq r}\}$ is $\Pi^0_2$.

To show that $\{e : M_e \cong \mathbb{U}_{\leq r}\}$ is $\Pi^0_2$-hard within *PolSp*, we apply Theorem 5.5.2 to $A := \emptyset$ and $B := Tot$. So we have that there is a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$,

$$\begin{aligned} &e \in Tot \implies e \in B \setminus A \implies X_e \cong \mathbb{U}_{\leq r}, \\ &e \notin Tot \implies e \notin B \implies X_e \text{ is finite } \implies X_e \ncong \mathbb{U}_{\leq r}. \end{aligned}$$

Therefore, $\{e : M_e \cong \mathbb{U}_{\leq r}\}$ is $\Pi^0_2$-hard within *PolSp*. □

**Theorem 5.5.4.** *The set*

$$\{e \in \mathbb{N} : (\exists r \in \mathbb{R}^+ \cup \{\infty\})(M_e \cong \mathbb{U}_{\leq r})\} = \{e \in \mathbb{N} : M_e \cong \mathbb{U}_{\leq diam(M_e)}\}$$

*is* $\Pi^0_2$*-complete within PolSp, and so it is* $\Pi^0_2$*-complete.*

*Proof.* It is clear that

$$\{e : (\exists r \in \mathbb{R}^+ \cup \{\infty\})(M_e \cong \mathbb{U}_{\leq r})\} = \{e : M_e \cong \mathbb{U}_{\leq diam(M_e)}\}.$$

By Theorem 5.3.12 and Remark 5.3.3, we have that for all $e \in PolSp$, $M_e \cong U_{\leq diam(M_e)}$ if and only if $M_e$ has the rational approximate extension property for $E_{<diam(M_e)}$ w.r.t. the dense set $(\mathbb{N}, d_e)$.

For each $q \in \mathbb{Q}^+$, let

$$I_q := \{e : M_e \text{ has the rational approximate extension property for } E_{<q} \text{ w.r.t. } (\mathbb{N}, d_e)\}.$$

We can use a similar argument as in the proof of Proposition 5.5.1. (We will have "$|f| < q$" is a $\Delta^0_1$ statement, and so "$f \in E_{\mathbb{Q},<q}(A)$" is a $\Pi^0_1$ statement.) It follows that $I_q$ is $\Pi^0_2$ uniformly in $q \in \mathbb{Q}^+$, that is, there is a computable relation $R(n, m, q, e)$ such that for all $q \in \mathbb{Q}^+$,

$$e \in I_q \iff \forall n \exists m R(n, m, q, e).$$

Then for all $e \in PolSp$,

$$\begin{aligned} M_e \cong \mathbb{U}_{\leq diam(M_e)} &\iff (\forall q \in \mathbb{Q}^+)(q < diam(M_e) \implies e \in I_q) \\ &\iff (\forall q \in \mathbb{Q}^+)((\forall i, j \in \mathbb{N})(d_e(i, j) \leq q) \vee e \in I_q). \end{aligned}$$

Therefore, $\{e : (\exists r \in \mathbb{R}^+ \cup \{\infty\})(M_e \cong \mathbb{U}_{\leq r})\}$ is $\Pi^0_2$ within $PolSp$.

To show that $\{e : (\exists r \in \mathbb{R}^+ \cup \{\infty\})(M_e \cong \mathbb{U}_{\leq r})\}$ is $\Pi^0_2$-hard within $PolSp$, we apply Theorem 5.5.2 to $A := \emptyset$, $B := Tot$, $r := \infty$ and $r_n := n$ for all $n \in \mathbb{N}$. So we have that there is a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$,

$$e \in Tot \Longrightarrow e \in B \setminus A \Longrightarrow X_e \cong \mathbb{U}_{\leq r} = \mathbb{U}_{\leq \infty},$$
$$e \notin Tot \Longrightarrow e \notin B \Longrightarrow X_e \text{ is finite } \Longrightarrow X_e \ncong \mathbb{U}_{\leq r'} \text{ for all } r' \in \mathbb{R}^+ \cup \{\infty\}.$$

Therefore, $\{e : (\exists r \in \mathbb{R}^+ \cup \{\infty\})(M_e \cong \mathbb{U}_{\leq r})\}$ is $\Pi^0_2$-hard within *PolSp*. □

**Theorem 5.5.5.** *The set*

$$\{e \in \mathbb{N} : (\exists r \in \mathbb{R}^+)(M_e \cong \mathbb{U}_{\leq r})\} = \{e \in \mathbb{N} : M_e \cong \mathbb{U}_{\leq diam(M_e)} \textit{ and } M_e \textit{ is bounded}\}$$

*is d-$\Sigma^0_2$-complete within PolSp, and so it is d-$\Sigma^0_2$-complete.*

*Proof.* It is clear that

$$\{e : (\exists r \in \mathbb{R}^+)(M_e \cong \mathbb{U}_{\leq r})\} = \{e : M_e \cong \mathbb{U}_{\leq diam(M_e)} \text{ and } M_e \text{ is bounded}\},$$

and so, by Theorem 5.5.4 and Theorem 2.0.9, it is $d$-$\Sigma^0_2$ within *PolSp*.

To show that $\{e : (\exists r \in \mathbb{R}^+)(M_e \cong \mathbb{U}_{\leq r})\}$ is $d$-$\Sigma^0_2$-hard within *PolSp*, we let $C$ be a $d$-$\Sigma^0_2$ set, say $C = A \cap B$ where $A$ is $\Sigma^0_2$ and $B$ is $\Pi^0_2$.

Then we apply Theorem 5.5.2 to $A$, $B$, $r := \infty$ and $r_n := n$ for all $n \in \mathbb{N}$. So we have that there is a computable sequence $(X_e)_{e\in\mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$,

$$e \in C = A \cap B \Longrightarrow (\exists n \in \mathbb{N})(X_e \cong \mathbb{U}_{\leq r_n}) \Longrightarrow (\exists n \in \mathbb{N})(X_e \cong \mathbb{U}_{\leq n}),$$

$$e \notin B \Longrightarrow X_e \text{ is finite } \Longrightarrow X_e \ncong \mathbb{U}_{\leq r'} \text{ for all } r' \in \mathbb{R}^+ \cup \{\infty\},$$

$$e \in B \setminus A \Longrightarrow X_e \cong \mathbb{U}_{\leq \infty} = \mathbb{U} \Longrightarrow X_e \ncong \mathbb{U}_{\leq r'} \text{ for all } r' \in \mathbb{R}^+.$$

Therefore, $\{e : (\exists r \in \mathbb{R}^+)(M_e \cong \mathbb{U}_{\leq r})\}$ is $d$-$\Sigma^0_2$-hard within $PolSp$. □

# Chapter 6

# Cantor Space and Baire Space

## 6.1 Cantor Space

We consider the Cantor space $2^{\mathbb{N}}$ equipped with the metric

$$d(X,Y) = 2^{-\min\{n\in\mathbb{N}:X(n)\neq Y(n)\}} \quad \text{for all } X,Y \in 2^{\mathbb{N}},$$

where $d(X,Y) = 0$ if $X = Y$.

The Cantor space is a Polish metric space, and the infinite binary strings that are eventually 0 (i.e. the strings $\sigma^\frown 0^{\mathbb{N}}$ where $\sigma \in 2^{<\mathbb{N}}$) form a computable presentation. In this section, we find the complexity of the embedding problem $2^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically and the embedding problem $2^{\mathbb{N}} \hookrightarrow M_e$ isometrically.

**Theorem 6.1.1.** *The set $\{e \in \mathbb{N} : 2^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically$\}$ is $\Sigma^1_1$.*

*Proof.* To show that $\{e : 2^{\mathbb{N}} \hookrightarrow M_e \text{ homeomorphically}\}$ is $\Sigma^1_1$, we will construct a computable sequence $(S_e)_{e\in\mathbb{N}}$ of trees such that for all $e \in PolSp$,

$$S_e \text{ has an infinite path } \iff 2^{\mathbb{N}} \hookrightarrow M_e \text{ homeomorphically.}$$

Since $(S_e)_{e\in\mathbb{N}}$ is a computable sequence of trees, $\{e : S_e \text{ has an infinite path}\}$ is $\Sigma^1_1$, and so $\{e : 2^{\mathbb{N}} \hookrightarrow M_e \text{ homeomorphically}\}$ is $\Sigma^1_1$.

Recall that for each $e \in PolSp$, $(M_e, d_e)$ is a computable metric space where $(\mathbb{N}, d_e)$ is a countable dense subset consisting of all rational points. For each $p \in \mathbb{N}$ and $r \in \mathbb{Q}^+$, we let $B(p, r)$ denote the rational open ball in $M_e$ around $p$ of radius $r$, that is,

$$B(p, r) := \{x \in M_e : d_e(x, p) < r\}.$$

Then the closure $\overline{B(p, r)} = \{x \in M_e : d_e(x, p) \leq r\}$ is a rational closed ball. It follows from the triangle inequality that

- $d_e(q, p) < r - s \Longrightarrow \overline{B(q, s)} \subseteq B(p, r)$,
- $d_e(p, q) > s + r \Longrightarrow \overline{B(p, r)} \cap \overline{B(q, s)} = \emptyset$.

For any pair $(B(p, r), B(q, s))$ of rational open balls, we consider the following conditions:

(1) $d_e(q, p) < r - s$. (This implies $\overline{B(q, s)} \subseteq B(p, r)$.)

(2) $d_e(p, q) > s + r$. (This implies $\overline{B(p, r)} \cap \overline{B(q, s)} = \emptyset$.)

Since conditions (1) and (2) are $\Sigma^0_1$ statements, $0'$ can determine whether a pair of rational open balls satisfies (1) and (2).

By using a fixed computable coding function, we can code each finite sequence $(B_1, \ldots, B_n)$ of rational open balls by a natural number. Let $\langle B_1, \ldots, B_n \rangle$ denote the code of the sequence $(B_1, \ldots, B_n)$.

We say that a family of nonempty open balls $(B_\sigma)_{\sigma \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$ in $M_e$ is a *Cantor scheme* if it satisfies the following conditions:

- If $\tau$ and $\sigma$ are incompatible, then $\overline{B_\tau} \cap \overline{B_\sigma} = \emptyset$.
- If $\tau \subsetneq \sigma$, then $\overline{B_\sigma} \subseteq B_\tau$.
- $diam(B_\sigma) \leq 2^{-|\sigma|}$.
- The center of the ball $B_\sigma$, denoted by $x_\sigma$, is a rational point of $M_e$.

If $M_e$ has such a Cantor scheme, then by the same argument as the standard proof of the fact that every Polish space contains an homeomorphic copy of $2^{\mathbb{N}}$ (see, e.g. [8]), $2^{\mathbb{N}}$ can be embedded homeomorphically into $M_e$ via the map $\psi : 2^{\mathbb{N}} \to M_e$ defined by

$$\psi(X) = \lim_{n \to \infty} x_{X \upharpoonright n} \quad \text{for all } X \in 2^{\mathbb{N}}.$$

**Construction of $S_e$**

For each $e \in \mathbb{N}$, we define a computable tree $S_e$ uniformly in $e$ as follows.

Let $\lambda \in S_e$ and for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $|\sigma| = n+1$, we let $\sigma$ be in $S_e$ if and only if there exist rational open balls $B_0, B_1, B_{00}, B_{01}, B_{10}, B_{11}, \ldots, B_{\underbrace{11\ldots1}_{(n+1)\text{-copies}}}$ in $M_e$, where the indices are finite binary strings ordered by the lexicographic order on $2^{<\mathbb{N}}$, such that the following conditions hold:

- $\sigma = (\langle B_0, B_1\rangle, \langle B_{00}, B_{01}, B_{10}, B_{11}\rangle, \ldots, \langle B_{\underbrace{00\ldots0}_{(n+1)\text{-copies}}}, B_{00\ldots01}, \ldots, B_{\underbrace{11\ldots1}_{(n+1)\text{-copies}}}\rangle)$
- $diam(B_\delta) \leq 2^{-|\delta|}$ for all $\delta \in 2^{<\mathbb{N}} \setminus \{\lambda\}$ with $|\delta| \leq n+1$.
- $(B_\delta, B_{\delta0})$ and $(B_\delta, B_{\delta1})$ satisfy (1) for all $\delta \in 2^{<\mathbb{N}}$ with $1 \leq |\delta| \leq n$.
- $(B_{\delta0}, B_{\delta1})$ satisfies (2) for all $\delta \in 2^{<\mathbb{N}}$ with $|\delta| \leq n$.

This ends the construction of $S_e$.

It is clear from the definition of $S_e$ that $S_e$ is a computable tree uniformly in $e$. Note that

$$S_e \text{ has an infinite path } \iff (\exists X \in \mathbb{N}^{\mathbb{N}})(\forall n \in \mathbb{N})(X \upharpoonright n \in S_e).$$

Therefore, since $S_e$ is computable uniformly in $e$, $\{e : S_e \text{ has an infinite path}\}$ is $\Sigma^1_1$.

It remains to show that for all $e \in PolSp$, $S_e$ has an infinite path $\iff 2^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically. Let $e \in PolSp$.

($\Longrightarrow$) Assume that $S_e$ has an infinite path, say $X \in \mathbb{N}^{\mathbb{N}}$. For each $n \in \mathbb{N}^+$, we can decode each finite string $X \upharpoonright n$ to get a finite collection of rational open balls $\{B_\alpha : \alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}, |\alpha| \leq n\}$. So the infinite path $X$ gives an infinite family $(B_\alpha)_{\alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$ of rational open balls in $M_e$. By the construction of $S_e$, it is easy to see that $(B_\alpha)_{\alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$ is a Cantor scheme. Therefore, $2^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically.

($\Longleftarrow$) Assume that $2^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically via an injective continuous map $f : (2^{\mathbb{N}}, d) \hookrightarrow (M_e, d_e)$. We construct a Cantor scheme $(B_\alpha)_{\alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$ that satisfies the following condition:

($*$) For all $\delta \in 2^{<\mathbb{N}}$, $(B_{\delta 0}, B_{\delta 1})$ satisfies (2), and if $|\sigma| > 0$, then $(B_\delta, B_{\delta 0})$ and $(B_\delta, B_{\delta 1})$ satisfy (1).

**Construction of** $(B_\alpha)_{\alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$

Stage 0: Do nothing.

Stage 1: Choose any $X_0, X_1 \in 2^{\mathbb{N}}$ with $X_0 \neq X_1$. Since $f$ is injective, $f(X_0) \neq f(X_1)$. Choose an $\varepsilon_1 \in \mathbb{Q}^+$ such that

$$\varepsilon_1 < \min\{2^{-2}, \frac{1}{4} d_e(f(X_0), f(X_1))\}.$$

Since $(\mathbb{N}, d_e)$ is dense in $M_e$, there exist rational points $p_0, p_1 \in \mathbb{N}$ such that

$$d_e(f(X_i), p_i) < \varepsilon_1 \quad \text{for all } i \in \{0, 1\}.$$

Choose an $r_1 \in \mathbb{Q}^+$ such that

$$\varepsilon_1 < r_1 < \min\{2^{-2}, \frac{1}{2} d_e(f(X_0), f(X_1)) - \varepsilon_1\}.$$

Let $B_0 := B(p_0, r_1)$ and $B_1 := B(p_1, r_1)$.

Note that for all $i \in \{0, 1\}$, $d_e(f(X_i), p_i) < \varepsilon_1 < r_1 < 2^{-2}$ and $diam(B_i) \leq 2r_1 < 2^{-1}$. Also note that $(B_0, B_1)$ satisfies (2) because

$$\begin{aligned} d_e(p_0, p_1) &\geq d_e(f(X_0), f(X_1)) - d_e(f(X_0), p_0) - d_e(f(X_1), p_1) \\ &> d_e(f(X_0), f(X_1)) - \varepsilon_1 - \varepsilon_1 \\ &> 2r_1. \end{aligned}$$

<u>Stage $n+1$ where $n > 0$:</u>

We have defined $\varepsilon_n, r_n$ and $X_\delta, p_\delta$ for all $\delta \in 2^{<\mathbb{N}}$ with $|\delta| = n$. We assume by induction that

$$d_e(f(X_\delta), p_\delta) < \varepsilon_n < r_n < 2^{-(n+1)} \quad \text{for all } \delta \in 2^{<\mathbb{N}} \text{ with } |\delta| = n.$$

For each $\delta \in 2^{<\mathbb{N}}$ with $|\delta| = n$, since $f$ is continuous at $X_\delta$, we can choose $X_{\delta 0}, X_{\delta 1} \in 2^{\mathbb{N}}$ such that

- $X_{\delta 0} \neq X_{\delta 1}$,
- $d_e(f(X_{\delta i}), f(X_\delta)) < \frac{1}{2}(r_n - \varepsilon_n)$ for all $i \in \{0, 1\}$.

Note that

$$d_e(f(X_{\delta 0}), f(X_{\delta 1})) \leq d_e(f(X_{\delta 0}), f(X_\delta)) + d_e(f(X_\delta), f(X_{\delta 1})) < r_n - \varepsilon_n < r_n < 2^{-(n+1)}.$$

Choose an $\varepsilon_{n+1} \in \mathbb{Q}^+$ such that

$$\varepsilon_{n+1} < \min_{\delta \in 2^n} \{\frac{1}{4} d_e(f(X_{\delta 0}), f(X_{\delta 1}))\}.$$

Note that

$$\varepsilon_{n+1} < \frac{1}{4} d_e(f(X_{\delta 0}), f(X_{\delta 1})) < \frac{1}{4}(r_n - \varepsilon_n) < \frac{1}{2}(r_n - \varepsilon_n),$$

and for all $i \in \{0, 1\}$,

$$\varepsilon_{n+1} < \frac{1}{4}(r_n - \varepsilon_n) = \frac{1}{2}[(r_n - \varepsilon_n) - \frac{1}{2}(r_n - \varepsilon_n)] < \frac{1}{2}[(r_n - \varepsilon_n) - d_e(f(X_{\delta i}), f(X_\delta))].$$

Since $(\mathbb{N}, d_e)$ is dense in $M_e$, for each $\delta \in 2^n$, there exist rational points $p_{\delta 0}, p_{\delta 1} \in \mathbb{N}$ such that

$$d_e(f(x_{\delta i}), p_{\delta i}) < \varepsilon_{n+1} \quad \text{for all } i \in \{0,1\}.$$

Note that $\varepsilon_{n+1} < \frac{1}{2}(r_n - \varepsilon_n) < \frac{1}{2}r_n < 2^{-(n+2)}$. For each $\delta \in 2^n$, since $\varepsilon_{n+1} < \frac{1}{4}d_e(f(X_{\delta 0}), f(X_{\delta 1}))$, we have $\varepsilon_{n+1} < \frac{1}{2}d_e(f(X_{\delta 0}), f(X_{\delta 1})) - \varepsilon_{n+1}$. Also, since $\varepsilon_{n+1} < \frac{1}{2}[(r_n - \varepsilon_n) - d_e(f(X_{\delta i}), f(X_\delta))]$, we have $\varepsilon_{n+1} < r_n - [d_e(f(X_{\delta i}), f(X_\delta)) + \varepsilon_n + \varepsilon_{n+1}]$. We conclude that

$$\varepsilon_{n+1} < \min_{\substack{\delta \in 2^n \\ i \in \{0,1\}}} \{2^{-(n+2)}, r_n - [d_e(f(X_{\delta i}), f(X_\delta)) + \varepsilon_n + \varepsilon_{n+1}], \frac{1}{2}d_e(f(X_{\delta 0}), f(X_{\delta 1})) - \varepsilon_{n+1}\}.$$

Choose an $r_{n+1} \in \mathbb{Q}^+$ such that

$$\varepsilon_{n+1} < r_{n+1} < \min_{\substack{\delta \in 2^n \\ i \in \{0,1\}}} \{2^{-(n+2)}, r_n - [d_e(f(X_{\delta i}), f(X_\delta)) + \varepsilon_n + \varepsilon_{n+1}], \frac{1}{2}d_e(f(X_{\delta 0}), f(X_{\delta 1})) - \varepsilon_{n+1}\}.$$

For each $\delta \in 2^n$ and $i \in \{0,1\}$, let $B_{\delta i} := B(p_{\delta i}, r_{n+1})$.

Note that for all $\delta \in 2^n$ and $i \in \{0,1\}$, $d_e(f(X_{\delta i}), p_{\delta i}) < \varepsilon_{n+1} < r_{n+1} < 2^{-(n+2)}$ and $diam(B_{\delta i}) \leq 2r_{n+1} < 2^{-(n+1)}$. Also note that $(B_\delta, B_{\delta i})$ satisfies (1) because

$$\begin{aligned} d_e(p_{\delta i}, p_\delta) &\leq d_e(p_{\delta i}, f(X_{\delta i})) + d_e(f(X_{\delta i}), f(X_\delta)) + d_e(f(X_\delta), p_\delta) \\ &< \varepsilon_{n+1} + d_e(f(X_{\delta i}), f(X_\delta)) + \varepsilon_n \\ &< r_n - r_{n+1}. \quad (\because r_{n+1} < r_n - [d_e(f(X_{\delta i}), f(X_\delta)) + \varepsilon_n + \varepsilon_{n+1}]) \end{aligned}$$

Also, $(B_{\delta 0}, B_{\delta 1})$ satisfies (2) because

$$\begin{aligned} d_e(p_{\delta 0}, p_{\delta 1}) &\geq d_e(f(X_{\delta 0}), f(X_{\delta 1})) - d_e(f(X_{\delta 0}), p_{\delta 0}) - d_e(f(X_{\delta 1}), p_{\delta 1}) \\ &> d_e(f(X_{\delta 0}), f(X_{\delta 1})) - \varepsilon_{n+1} - \varepsilon_{n+1} \\ &> 2r_{n+1}. \quad (\because r_{n+1} < \frac{1}{2} d_e(f(X_{\delta 0}), f(X_{\delta 1})) - \varepsilon_{n+1}) \end{aligned}$$

This ends the construction of $(B_\alpha)_{\alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$.

It is clear from the construction that $(B_\alpha)_{\alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$ is a Cantor scheme in $M_e$ that satisfies $(*)$. Recall that each infinite path $X \in [S_e]$ gives a Cantor scheme in $M_e$. Moreover, $[S_e]$ gives all possible Cantor schemes that satisfies $(*)$. So $S_e$ must have an infinite path corresponding to $(B_\alpha)_{\alpha \in 2^{<\mathbb{N}} \setminus \{\lambda\}}$. More specifically, for every $n \in \mathbb{N}$, the finite string $\sigma_n$ of length $n+1$ defined by

$$\sigma_n := (\langle B_0, B_1 \rangle, \langle B_{00}, B_{01}, B_{10}, B_{11} \rangle, \ldots, \langle B_{\underbrace{00\ldots0}_{(n+1)\text{-copies}}}, B_{00\ldots01}, \ldots, B_{\underbrace{11\ldots1}_{(n+1)\text{-copies}}} \rangle)$$

must be in the tree $S_e$, and so $\bigcup_{n \in \mathbb{N}} \sigma_n \in \mathbb{N}^{\mathbb{N}}$ is an infinite path in $S_e$.

We conclude that for all $e \in PolSp$, $S_e$ has an infinite path $\iff 2^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically. It follows that $\{e : 2^{\mathbb{N}} \hookrightarrow M_e \text{ homeomorphically}\}$ is $\Sigma^1_1$. $\square$

**Theorem 6.1.2.** *The set* $\{e \in \mathbb{N} : 2^{\mathbb{N}} \hookrightarrow M_e$ *homeomorphically*$\}$ *is* $\Sigma^1_1$*-hard.*

*Proof.* Recall that $(T_e)_{e \in \mathbb{N}}$ is a fixed effective enumeration of all primitive recursive trees $T_e \subseteq \mathbb{N}^{<\mathbb{N}}$. To show that $\{e : 2^{\mathbb{N}} \hookrightarrow M_e \text{ homeomorphically}\}$ is $\Sigma^1_1$-hard, it is enough to build a computable sequence $(X_e)_{e \in \mathbb{N}}$ of Polish metric spaces such that for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\iff 2^{\mathbb{N}} \hookrightarrow X_e$ homeomorphically.

For each $\sigma, \tau \in 2^{<\mathbb{N}}$ with $|\sigma| = |\tau|$, we let $\sigma \oplus \tau$ denote the string in $2^{<\mathbb{N}}$ defined by

$$(\sigma \oplus \tau)(n) = \begin{cases} \sigma(k) & \text{if } n = 2k \\ \tau(k) & \text{if } n = 2k+1 \end{cases}$$

For each $e \in \mathbb{N}$, let

$$S_e := \{\sigma \oplus \tau : \tau \in T_e \text{ and } \sigma \in 2^{<\mathbb{N}} \text{ with } |\sigma| = |\tau|\},$$
$$\widetilde{S}_e := \{\rho^\frown 0^{\mathbb{N}} : \rho \in S_e\} \subseteq 2^{\mathbb{N}}.$$

Define a metric $\widetilde{d}$ on $\widetilde{S}_e$ by

$$\widetilde{d}(X,Y) = 2^{-\frac{1}{2}\min\{n \in \mathbb{N} : X(n) \neq Y(n)\}} \quad \text{for all } X, Y \in \widetilde{S}_e,$$

where $\widetilde{d}(X,Y) = 0$ if $X = Y$. Let $X_e$ be the completion of $(\widetilde{S}_e, \widetilde{d})$.

The idea is that each $\tau \in T_e$ will correspond to an isometric copy of the full binary tree $2^{<\mathbb{N}}$ up to level $|\tau|$ in $S_e$. So if $T_e$ has an infinite path, then $S_e$ will contain an isometric copy of the full binary tree $2^{<\mathbb{N}}$, and so $2^{\mathbb{N}} \hookrightarrow X_e$ isometrically.

Since $(T_e)_{e \in \mathbb{N}}$ is a computable sequence of trees, $(S_e)_{e \in \mathbb{N}}$ is also a computable sequence of trees. It follows that $(X_e)_{e \in \mathbb{N}}$ is a computable sequence of Polish metric spaces.

We claim that $T_e$ has an infinite path $\iff 2^{\mathbb{N}} \hookrightarrow X_e$ homeomorphically.

($\Longrightarrow$) Assume $T_e$ has an infinite path, say $f \in \mathbb{N}^{\mathbb{N}}$. Let $A := \{\sigma^\frown 0^{\mathbb{N}} : \sigma \in 2^{<\mathbb{N}}\}$

and define a map $\psi : A \to \widetilde{S}_e$ by

$$\psi(\sigma^\frown 0^{\mathbb{N}}) = (\sigma \oplus (f \upharpoonright |\sigma|))^\frown 0^{\mathbb{N}} \quad \text{for all } \sigma \in 2^{<\mathbb{N}}.$$

Then for all $\sigma \in 2^{<\mathbb{N}}$ and $n \in \mathbb{N}$,

$$(\psi(\sigma^\frown 0^{\mathbb{N}}))(n) = \begin{cases} \sigma(k) & \text{if } n = 2k \text{ and } k < |\sigma| \\ f(k) & \text{if } n = 2k+1 \text{ and } k < |\sigma| \\ 0 & \text{otherwise} \end{cases}$$

It follows that for all $X, Y \in A$ with $X \neq Y$, we have

$$\min\{n : (\psi(X))(n) \neq (\psi(Y))(n)\} = 2\min\{n : X(n) \neq Y(n)\}, \text{ and so}$$
$$d(X,Y) = 2^{-\min\{n : X(n) \neq Y(n)\}} = 2^{-\frac{1}{2}\min\{n \in \mathbb{N} : (\psi(X))(n) \neq (\psi(Y))(n)\}} = \widetilde{d}(\psi(X), \psi(Y)).$$

So $\psi : (A, d) \to (\widetilde{S}_e, \widetilde{d})$ is a distance-preserving map. Thus, since $A$ is dense in $(2^{\mathbb{N}}, d)$, $\psi$ can be extended to an isometric embedding $\psi : (2^{\mathbb{N}}, d) \to (X_e, \widetilde{d})$. Therefore, $2^{\mathbb{N}} \hookrightarrow X_e$ isometrically. In particular, $2^{\mathbb{N}} \hookrightarrow X_e$ homeomorphically.

($\Longleftarrow$) Assume that $T_e$ has no infinite paths. By the definition of $S_e$, $S_e$ also has no infinite paths. It follows that $(\widetilde{S}_e, \widetilde{d})$ is countable and it has no limit points. So the completion $X_e$ is just $\widetilde{S}_e$, which is countable. Since $2^{\mathbb{N}}$ is uncountable, $2^{\mathbb{N}}$ does not embed into $X_e$ homeomorphically. In particular, $2^{\mathbb{N}}$ does not embed into $X_e$ isometrically.

We conclude that for all $e \in \mathbb{N}$, $T_e$ has an infinite path $\Longleftrightarrow$ $2^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically. Therefore, $\{e : 2^{\mathbb{N}} \hookrightarrow M_e \text{ homeomorphically}\}$ is $\Sigma^1_1$-hard. $\square$

By Theorem 6.1.1, Theorem 6.1.2 and the well-known fact that a Polish metric space is uncountable if and only if it contains a homeomorphic copy of the Canter space $2^{\mathbb{N}}$ (see, e.g. [8]), we have the following theorem.

**Theorem 6.1.3.** *The set*

$$\{e \in \mathbb{N} : 2^{\mathbb{N}} \hookrightarrow M_e \textit{ homeomorphically}\} = \{e \in \mathbb{N} : M_e \textit{ is uncountable}\}$$

*is* $\Sigma^1_1$*-complete.*

**Theorem 6.1.4.** *The set* $\{e \in \mathbb{N} : 2^{\mathbb{N}} \hookrightarrow M_e \textit{ isometrically}\}$ *is* $\Sigma^1_1$*-complete.*

*Proof.* By Proposition 3.1.6, $\{e : 2^{\mathbb{N}} \hookrightarrow M_e \text{ isometrically}\}$ is $\Sigma^1_1$. The proof of Theorem 6.1.2 also shows that $\{e : 2^{\mathbb{N}} \hookrightarrow M_e \text{ isometrically}\}$ is $\Sigma^1_1$-hard. □

## 6.2 Baire Space

We consider the Baire space $\mathbb{N}^{\mathbb{N}}$ equipped with the metric

$$d(X, Y) = 2^{-\min\{n \in \mathbb{N} : X(n) \neq Y(n)\}} \quad \text{for all } X, Y \in \mathbb{N}^{\mathbb{N}},$$

where $d(X, Y) = 0$ if $X = Y$.

The Baire space is a Polish metric space, and the infinite strings that are eventually 0 (i.e. the strings $\sigma^\frown 0^{\mathbb{N}}$ where $\sigma \in \mathbb{N}^{<\mathbb{N}}$) form a computable presentation.

First, we consider the embedding problem $\mathbb{N}^{\mathbb{N}} \hookrightarrow M_e$ homeomorphically. It is clear that $2^{\mathbb{N}} \hookrightarrow \mathbb{N}^{\mathbb{N}}$ isometrically and homeomorphically. It is also known that

$\mathbb{N}^\mathbb{N} \hookrightarrow 2^\mathbb{N}$ homeomorphically. For example, we can define a function $f : \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ by

$$f : (x_n)_{n\in\mathbb{N}} \mapsto 1^{x_0}01^{x_1}01^{x_2}0\ldots,$$

where $1^{x_n}$ is the finite string of length $x_n$ consisting only 1s. Then it is not hard to show that $f : \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ is a homeomorphic embedding. In fact, the range of $f$ is

$$\{(y_n)_{n\in\mathbb{N}} \in 2^\mathbb{N} : y_n = 0 \text{ for infinitely many } n\},$$

which is dense in $2^\mathbb{N}$. Therefore, we have that

$$\{e : \mathbb{N}^\mathbb{N} \hookrightarrow M_e \text{ homeomorphically}\} = \{e : 2^\mathbb{N} \hookrightarrow M_e \text{ homeomorphically}\},$$

which is $\Sigma^1_1$-complete by Theorem 6.1.3.

Also note that the proof of Theorem 6.1.2 for $2^\mathbb{N}$ also works for $\mathbb{N}^\mathbb{N}$ by using

$$S_e := \{\sigma \oplus \tau : \tau \in T_e \text{ and } \sigma \in \mathbb{N}^{<\mathbb{N}} \text{ with } |\sigma| = |\tau|\}.$$

Therefore, we have the following.

**Theorem 6.2.1.**

- *The set* $\{e \in \mathbb{N} : \mathbb{N}^\mathbb{N} \hookrightarrow M_e$ *homeomorphically*$\}$ *is* $\Sigma^1_1$*-complete.*
- *The set* $\{e \in \mathbb{N} : \mathbb{N}^\mathbb{N} \hookrightarrow M_e$ *isometrically*$\}$ *is* $\Sigma^1_1$*-complete.*

It was shown in [12] that the Cantor space $2^\mathbb{N}$ is computably categorical as a metric space. We show that the same is true for the Baire space. We start with proving the following lemmas.

**Lemma 6.2.2.** *For all* $n, l \in \mathbb{N}$*, if* $\beta_0, \ldots, \beta_n \in \mathbb{N}^{\mathbb{N}}$ *and* $\beta_i \restriction l = \beta_j \restriction l$ *for all* $i, j \in \{0, \ldots, n\}$*, then the set*

$$\{\rho \in \mathbb{N}^{\mathbb{N}} : (\forall i \leq n)(d(\beta_i, \rho) = 2^{-l})\}$$

*is an infinite open set of* $\mathbb{N}^{\mathbb{N}}$*.*

*Proof.* Assume $n, l \in \mathbb{N}$, $\beta_0, \ldots, \beta_n \in \mathbb{N}^{\mathbb{N}}$, and $\beta_i \restriction l = \beta_j \restriction l$ for all $i, j \in \{0, \ldots, n\}$. Let $V := \{\rho \in \mathbb{N}^{\mathbb{N}} : (\forall i \leq n)(d(\beta_i, \rho) = 2^{-l})\}$. Note that for each $i \leq n$,

$$[\![\beta_i \restriction l]\!] = \{\rho \in \mathbb{N}^{\mathbb{N}} : d(\beta_i, \rho) \leq 2^{-l}\}.$$

By the definition of the metric $d$, if $d(\beta_i, \rho) > 2^{-(l+1)}$, then $d(\beta_i, \rho) \geq 2^{-l}$. It follows that

$$V = \bigcap_{i \leq n} \big([\![\beta_i \restriction l]\!] \cap (\mathbb{N}^{\mathbb{N}} \setminus [\![\beta_i \restriction l+1]\!])\big).$$

Thus, since the basic open sets of $\mathbb{N}^{\mathbb{N}}$ are clopen, $V$ is open.

Let $M := \max\{\beta_i(l) : i \leq n\} + 1$ and $\sigma := (\beta_0 \restriction l)^\frown M$. Then $\beta_i(l) \neq M$ for all $i \leq n$. Thus, since $\beta_i \restriction l = \beta_0 \restriction l$ for all $i \leq n$, we have that $[\![\sigma]\!] \subseteq V$. So $V$ is infinite since $[\![\sigma]\!]$ is infinite. $\square$

**Lemma 6.2.3.** *For all* $\alpha, \beta, \gamma \in \mathbb{N}^{\mathbb{N}}$*, if* $d(\alpha, \beta) < d(\alpha, \gamma)$*, then* $d(\beta, \gamma) = d(\alpha, \gamma)$*.*

*Proof.* Let $\alpha, \beta, \gamma \in \mathbb{N}^{\mathbb{N}}$ be such that $d(\alpha, \beta) < d(\alpha, \gamma)$. If $\alpha = \beta$, then $d(\beta, \gamma) =$

$d(\alpha, \gamma)$ and we are done. Assume $\alpha \neq \beta$. Then $d(\alpha, \beta) = 2^{-l}$ for some $l \in \mathbb{N}$. Since $d(\alpha, \beta) < d(\alpha, \gamma)$, $d(\alpha, \gamma) = 2^{-m}$ for some $m < l$. By the definition of $d$, we have that $\alpha \upharpoonright l = \beta \upharpoonright l$, $\alpha(l) \neq \beta(l)$, $\alpha \upharpoonright m = \gamma \upharpoonright m$, and $\alpha(m) \neq \gamma(m)$. Since $\alpha \upharpoonright l = \beta \upharpoonright l$ and $m < l$, we have $\beta \upharpoonright m = \alpha \upharpoonright m = \gamma \upharpoonright m$ and $\beta(m) = \alpha(m) \neq \gamma(m)$. Therefore, $d(\beta, \gamma) = 2^{-m} = d(\alpha, \gamma)$. □

It was proved by Melnikov [12] that the Cantor space $2^{\mathbb{N}}$ is computably categorical. We show that the same is true for the Baire space $\mathbb{N}^{\mathbb{N}}$.

**Theorem 6.2.4.** *The Baire space $\mathbb{N}^{\mathbb{N}}$ is computably categorical.*

*Proof.* Let $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ be computable presentations of $\mathbb{N}^{\mathbb{N}}$. Without loss of generality, we can assume that $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ have no repetitions. We will build a computable bijection $f : \mathbb{N} \to \mathbb{N}$ such that

$$d(\alpha_i, \alpha_j) = d(\beta_{f(i)}, \beta_{f(j)}) \quad \text{for all } i, j \in \mathbb{N}.$$

Then the map $\psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by

$$\psi(\lim_{i\to\infty} \alpha_{g(i)}) = \lim_{i\to\infty} \beta_{f(g(i))} \quad \text{for all Cauchy names } (\alpha_{g(i)})_{i\in\mathbb{N}} \text{ in } (\alpha_i)_{i\in\mathbb{N}}$$

is a computable isometry w.r.t. $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$. So $\mathbb{N}^{\mathbb{N}}$ is computably categorical as a metric space.

**Construction of $f$**

We use a back-and-forth construction to build a computable sequence $(f_s)_{s\in\mathbb{N}}$ of isometries, that is,

$$d(\alpha_i, \alpha_j) = d(\beta_{f_s(i)}, \beta_{f_s(j)}) \quad \text{for all } i, j \in dom(f_s),$$

then we let $f := \bigcup_s f_s$.

Stage 0: Let $f_0 := \{(0,0)\}$.

Stage $s+1 = 2n+1$: We make sure that $n \in dom(f_{s+1})$. If $n \in dom(f_s)$, then let $f_{s+1} := f_s$. If $n \notin dom(f_s)$, then we do the following.

Let $l$ be the largest number such that $(\exists i \in dom(f_s))(d(\alpha_i, \alpha_n) = 2^{-l})$, and let

$$I := \{i \in dom(f_s) : d(\alpha_i, \alpha_n) = 2^{-l}\} \neq \emptyset.$$

Find the least $k \in \mathbb{N}$ such that

$$k \notin range(f_s) \text{ and } d(\beta_{f_s(i)}, \beta_k) = 2^{-l} \text{ for all } i \in I.$$

(We will show below that such a number $k$ exists.) Let $f_{s+1} := f_s \cup \{(n,k)\}$.

Stage $s+1 = 2n+2$: We make sure that $n \in dom(f_{s+1}^{-1}) = range(f_{s+1})$. If $n \in dom(f_s^{-1})$, then let $f_{s+1} := f_s$. If $n \notin dom(f_s^{-1})$, then we do the following.

Let $l$ be the largest number such that $(\exists i \in dom(f_s^{-1}))(d(\beta_i, \beta_n) = 2^{-l})$, and let

$$I := \{i \in dom(f_s^{-1}) : d(\beta_i, \beta_n) = 2^{-l}\} \neq \emptyset.$$

Find the least $k \in \mathbb{N}$ such that

$$k \notin range(f_s^{-1}) \text{ and } d(\alpha_{f_s^{-1}(i)}, \alpha_k) = 2^{-l} \text{ for all } i \in I.$$

Let $f_{s+1} := f_s \cup \{(k, n)\}$.

This ends the construction.

We will show by induction that for every $s \in \mathbb{N}$, $f_s$ is an isometry and we can find such a number $k$ in the construction at stage $s$ if $s > 0$.

Clearly, $f_0 = \{(0, 0)\}$ is an isometry. Now let $s \in \mathbb{N}$. By the induction hypothesis, $f_s$ is an isometry and we can such a number $k$ in the construction at stage $s$ if $s > 0$. We consider the construction at stage $s + 1$ as follows.

<u>Case $s + 1 = 2n + 1$:</u> We only need to consider when $n \notin dom(f_s)$. Note that for every $i \in I$, $d(\alpha_i, \alpha_n) = 2^{-l}$, and so $\alpha_i \restriction l = \alpha_n \restriction l$. So for all $i, j \in I$, $\alpha_i \restriction l = \alpha_j \restriction l$, and so $d(\alpha_i, \alpha_j) \leq 2^{-l}$. Thus, since $I \subseteq dom(f_s)$ and $f_s$ is an isometry, we have that for all $i, j \in I$, $d(\beta_{f_s(i)}, \beta_{f_s(j)}) = d(\alpha_i, \alpha_j) \leq 2^{-l}$, and so $\beta_{f_s(i)} \restriction l = \beta_{f_s(j)} \restriction l$. Also note that $I$ is finite because $dom(f_s)$ is finite. Thus, by Lemma 6.2.2, the set

$$V := \{\rho \in \mathbb{N}^{\mathbb{N}} : (\forall i \in I)(d(\beta_{f_s(i)}, \rho) = 2^{-l})\}$$

is an infinite open set. So, since $(\beta_i)_{i \in \mathbb{N}}$ is dense in $\mathbb{N}^{\mathbb{N}}$, $\{\beta_i\}_{i \in \mathbb{N}} \cap V$ is infinite. Hence, since $range(f_s)$ is finite, there must be the least $k \in \mathbb{N}$ such that

$$k \notin range(f_s) \text{ and } \beta_k \in V, \text{ and so } d(\beta_{f_s(i)}, \beta_k) = 2^{-l} \text{ for all } i \in I.$$

Then we will define $f_{s+1} := f_s \cup \{(n,k)\}$.

Next, we show that $f_{s+1}$ is an isometry. By the choice of $k$, we have that for all $i \in I$, $d(\beta_{f_{s+1}(i)}, \beta_{f_{s+1}(n)}) = d(\beta_{f_s(i)}, \beta_k) = 2^{-l} = d(\alpha_i, \alpha_n)$. Thus, since $dom(f_{s+1}) = dom(f_s) \sqcup \{n\}$ and $f_s$ is an isometry, it remains to show that for all $j \in dom(f_s) \setminus I$, $d(\beta_{f_{s+1}(j)}, \beta_{f_{s+1}(n)}) = d(\alpha_j, \alpha_n)$.

Let $j \in dom(f_s) \setminus I$.. Then $d(\alpha_j, \alpha_n) \neq 2^{-l}$. Since $n \notin dom(f_s)$, $j \neq n$, and so, since $(\alpha_i)_{i\in\mathbb{N}}$ has no repetitions, $\alpha_j \neq \alpha_n$. By the maximality of $l$, we must have $d(\alpha_j, \alpha_n) = 2^{-m}$ for some $m < l$. Since $I \neq \emptyset$, we can fix an $i \in I$. Then $d(\alpha_n, \alpha_i) = 2^{-l} < 2^{-m} = d(\alpha_n, \alpha_j)$. Thus, by Lemma 6.2.3, $d(\alpha_i, \alpha_j) = d(\alpha_n, \alpha_j) = 2^{-m}$. Since $i, j \in dom(f_s)$ and $f_s$ is an isometry, we have

$$d(\beta_{f_{s+1}(i)}, \beta_{f_{s+1}(j)}) = d(\beta_{f_s(i)}, \beta_{f_s(j)}) = d(\alpha_i, \alpha_j) = 2^{-m}.$$

Also, since $i \in I$, we have

$$d(\beta_{f_{s+1}(i)}, \beta_{f_{s+1}(n)}) = 2^{-l} < 2^{-m} = d(\beta_{f_{s+1}(i)}, \beta_{f_{s+1}(j)}).$$

Thus, by Lemma 6.2.3,

$$d(\beta_{f_{s+1}(n)}, \beta_{f_{s+1}(j)}) = d(\beta_{f_{s+1}(i)}, \beta_{f_{s+1}(j)}) = 2^{-m} = d(\alpha_n, \alpha_j).$$

We have shown that for all $j \in dom(f_s) \setminus I$, $d(\beta_{f_{s+1}(j)}, \beta_{f_{s+1}(n)}) = d(\alpha_j, \alpha_n)$. Therefore, $f_{s+1}$ is an isometry.

<u>Case $s+1 = 2n+2$:</u> We only need to consider when $n \notin dom(f_s^{-1})$. By a similar

argument as Case $s+1=2n+1$, there is the least $k \in \mathbb{N}$ such that

$$k \notin range(f_s^{-1}) \text{ and } d(\alpha_{f_s^{-1}(i)}, \alpha_k) = 2^{-l} \text{ for all } i \in I.$$

Since $f_s$ is an isometry, so is $f_s^{-1}$. Then we can use a similar argument as Case $s+1=2n+1$ to show that $f_{s+1}$ is an isometry.

We conclude that $f : \mathbb{N} \to \mathbb{N}$ is a computable bijection such that

$$d(\alpha_i, \alpha_j) = d(\beta_{f(i)}, \beta_{f(j)}) \quad \text{for all } i, j \in \mathbb{N}.$$

It follows that $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ are computably isometric. Therefore, $\mathbb{N}^{\mathbb{N}}$ is computably categorical as a metric space. $\square$

# Chapter 7

# Spaces of Continuous Functions

For a compact metric space $X$, we consider the Polish metric space $C(X)$ of continuous real-valued functions on $X$, equipped with the pointwise supremum metric:

$$d(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$

It is well-known that for any compact metric space $X$, the space $C(X)$ is a Banach space.

Recall that a *Banach space* is a complete normed vector space. We can write a Banach space $\mathbb{B}$ as the tuple $(B, d, 0, +, (r\cdot)_{r\in\mathbb{Q}})$, where $B$ is the underlying set, $d$ is the metric induced by the norm, 0 denotes the additive identity (or zero), $+$ denotes the vector addition, and for each $r \in \mathbb{Q}$, $r\cdot$ denotes the scalar multiplication by $r$. The signature of Banach spaces consists of $d, 0, +, (r\cdot)_{r\in\mathbb{Q}}$.

## 7.1 The Space $C[0,1]$

In this section, we provide several relevant results on Banach spaces and on $C[0,1]$ due to Melnikov (see [11] and [12]). We will refer to these results when we consider the space $C(2^{\mathbb{N}})$ in the next section.

**Fact 7.1.1** (Melnikov [12])**.** *Let $\mathbb{B} = (B, d, 0, +, (r\cdot)_{r\in\mathbb{Q}})$ be a Banach space. Suppose $(p_i)_{i\in\mathbb{N}}$ is a computable presentation of $(B, d)$ w.r.t. which $+$ and $(r\cdot)_{r\in\mathbb{Q}}$ are uniformly computable, and $(q_i)_{i\in\mathbb{N}}$ is a computable presentation of $(B, d)$ w.r.t. which $0$ is computable. If $(q_i)_{i\in\mathbb{N}}$ is computably isometric to $(p_i)_{i\in\mathbb{N}}$, then $+$ and $(r\cdot)_{r\in\mathbb{Q}}$ are uniformly computable w.r.t. $(q_i)_{i\in\mathbb{N}}$.*

**Fact 7.1.2** (Melnikov [11])**.** *In a computably separable Banach space, the operation $+$ and $d(\cdot,\cdot)$ effectively determine the operations $(r\cdot)_{r\in\mathbb{Q}}$, $-$ and the zero element $0$.*

**Definition 7.1.3** (Melnikov [11])**.** Two computable presentations $\mathcal{A}$ and $\mathcal{B}$ of a separable metric space $(M, d)$ are said to be *limit equivalent* if there is a total computable function $g : \mathcal{A}\times\mathbb{N} \to \mathcal{B}$ of two arguments such that $f(x) := \lim_{s\to\infty} g(x, s)$ is a surjective isometry from $\mathcal{A}$ onto $\mathcal{B}$, where the limit is taken with respect to the standard metric on $\mathbb{N}$ (i.e. the sequence $(g(x, s))_{s\in\mathbb{N}}$ is eventually stable on every $x$).

**Definition 7.1.4** (Melnikov [11])**.** A computable presentation $\mathcal{A}$ of a separable metric space $(M, d)$ is *rational-valued* if $d(x, y) \in \mathbb{Q}$ for every $x, y \in \mathcal{A}$, and the distance function $d$ is represented by a computable function of two arguments mapping each pair of rational points $(x, y)$ to the corresponding rational number $d(x, y)$.

**Theorem 7.1.5** (Melnikov [11])**.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two rational-valued computable presentations on a separable metric space $(M, d)$ which are not computably* *isometric. If $\mathcal{A}$ and $\mathcal{B}$ are limit equivalent, then $(M, d)$ has infinitely many computable presentations which are pairwise not computably isometric.*

**Theorem 7.1.6** (Melnikov [11]). *There exist infinitely many computable presentations of $(C[0,1], d)$ which are pairwise not computably isometric. In particular, the space $C[0,1]$ is not computably categorical as a metric space.*

**Theorem 7.1.7** (Melnikov [11]). *The space $C[0,1]$ is not computably categorical as a Banach space.*

## 7.2 The Space $C(2^{\mathbb{N}})$

Consider the Polish metric space $C(2^{\mathbb{N}})$ equipped with the pointwise supremum metric:

$$d(f, g) := \sup_{X \in 2^{\mathbb{N}}} |f(X) - g(X)|.$$

Since $2^{\mathbb{N}}$ is compact, $C(2^{\mathbb{N}})$ is a Banach space.

In this section, we modify Melnikov's idea for proving that the space $C[0,1]$ is computably categorical as a metric space and as a Banach space to prove the same results for $C(2^{\mathbb{N}})$.

For each $\sigma \in 2^{<\mathbb{N}}$, $[\![\sigma]\!] := \{X \in 2^{\mathbb{N}} : \sigma \subseteq X\}$ is the basic clopen set of $2^{\mathbb{N}}$ w.r.t. $\sigma$. We define $\chi_\sigma : 2^{\mathbb{N}} \to \mathbb{R}$ by

$$\chi_\sigma(X) = \chi_{[\![\sigma]\!]}(X) = \begin{cases} 1 & \text{if } \sigma \subseteq X \\ 0 & \text{if } \sigma \not\subseteq X \end{cases}$$

Let $D$ be the linear span of $\{\chi_\sigma\}_{\sigma\in 2^{<\mathbb{N}}}$ over $\mathbb{Q}$, that is,

$$D := \left\{ \sum_{i=0}^{n} q_i \chi_{\sigma_i} : n \in \mathbb{N}, \sigma_i \in 2^{<\mathbb{N}}, q_i \in \mathbb{Q} \right\}.$$

We will call the elements of $D$ the *rational simple functions.*

Note that each rational simple function can be written in the form $\sum\limits_{i=0}^{n} q_i \chi_{\sigma_i}$ where $\bigsqcup_{i=0}^{n} [\![\sigma_i]\!] = 2^{\mathbb{N}}$ and $q_i \in \mathbb{Q}$.

**Proposition 7.2.1.** *$D$ is a countable dense subset of $C(2^{\mathbb{N}})$.*

*Proof.* It is clear that $D$ is countable. It is easy to see that $D$ satisfies the following conditions:

(1) $f + g \in D$ for all $f, g \in D$.

(2) $f \cdot g \in D$ for all $f, g \in D$.

(3) $qf \in D$ for all $q \in \mathbb{Q}$ and $f \in D$.

(4) $D$ contains the constant function 1.

(5) $D$ separates points, i.e. for every $X, Y \in 2^{\mathbb{N}}$ with $X \neq Y$, there is an $f \in D$ such that $f(X) \neq f(Y)$.

Therefore, by the Stone-Weierstrass Theorem (see, e.g. [18]), $D$ is dense in $C(2^{\mathbb{N}})$. □

Let $\mathcal{L} := (l_i)_{i\in\mathbb{N}}$ be an effective list of all rational simple functions, without repetition.

**Proposition 7.2.2.** *$\mathcal{L}$ is a computable presentation of $C(2^{\mathbb{N}})$. Moreover, $\mathcal{L}$ is effectively closed under $+$ and $\times$. That is, there are computable functions $f$ and $g$ such* *that for every* $i, j \in \mathbb{N}$, *we have*

$$l_i + l_j = l_{f(i,j)} \quad \textit{and} \quad l_i \times l_j = l_{g(i,j)}.$$

*It follows that the operations* $+$ *and* $\times$ *are computable w.r.t.* $\mathcal{L}$.

*Proof.* To show that $\mathcal{L}$ is a computable presentation of $C(2^{\mathbb{N}})$, we need to show that $d(l_i, l_j) := \sup_{X \in 2^{\mathbb{N}}} |l_i(X) - l_j(X)|$ is a computable real uniformly in $i$ and $j$.

Let $i, j \in \mathbb{N}$. Then we can effectively find $n, m \in \mathbb{N}, q_k, r_l \in \mathbb{Q}$ and $\sigma_k, \tau_l \in 2^{<\mathbb{N}}$ such that

$$l_i = \sum_{k=0}^{n} q_k \chi_{\sigma_k} \quad \text{and} \quad l_j = \sum_{l=0}^{m} r_l \chi_{\tau_l}.$$

It is clear that

$$(\forall \sigma \in 2^{<\mathbb{N}})(\forall X, Y \in 2^{\mathbb{N}})(X \upharpoonright |\sigma| = Y \upharpoonright |\sigma| \implies \chi_\sigma(X) = \chi_\sigma(Y)).$$

Let $N := \max\{|\sigma_k|, |\tau_l| : k \in \{0, \ldots, n\}, l \in \{0, \ldots, m\}\}$. Then

$$(\forall X, Y \in 2^{\mathbb{N}})(X \upharpoonright N = Y \upharpoonright N \implies (l_i(X) = l_i(Y) \wedge l_j(X) = l_j(Y))).$$

It follows that

$$d(l_i, l_j) = \max_{\sigma \in 2^N} |l_i(\sigma^\frown 0^{\mathbb{N}}) - l_j(\sigma^\frown 0^{\mathbb{N}})|.$$

Thus, since $2^N = \{\sigma \in 2^{<\mathbb{N}} : |\sigma| = N\}$ is a finite set of strings, we can compute $d(l_i, l_j)$. Note that $d(l_i, l_j)$ can be computed in this way uniformly in $i$ and $j$. Therefore, $\mathcal{L}$ is a computable presentation of $C(2^{\mathbb{N}})$.

Next, we show that $\mathcal{L}$ is effectively closed under $+$ and $\times$. Let $i, j \in \mathbb{N}$. Then we can effectively write $l_i$ and $l_j$ as linear combinations of $\chi_\sigma$ as before. So we can effectively write $l_i$ and $l_j$ as linear combinations of $\chi_\sigma$ where $\sigma \in 2^N$. Since $[\![\sigma]\!] \cap [\![\tau]\!] = \emptyset$ for every distinct $\sigma, \tau \in 2^N$, $l_i$ and $l_j$ are constant on $[\![\sigma]\!]$ for all $\sigma \in 2^N$. So we can compute the value of $l_i + l_j$ on $[\![\sigma]\!]$ for each $\sigma \in 2^N$, and write $l_i + l_j$ as a linear combination of $\chi_\sigma$ where $\sigma \in 2^N$. Hence we can effectively find a number $n_{i,j}$ such that $l_i + l_j = l_{n_{i,j}}$, and then let $f(i,j) = n_{i,j}$. So $f$ is computable and $l_i + l_j = l_{f(i,j)}$ for all $i, j \in \mathbb{N}$. Therefore, $\mathcal{L}$ is effectively closed under $+$. By the same argument, $\mathcal{L}$ is also effectively closed under $\times$. □

Note that 1 is a computable point w.r.t. $\mathcal{L}$ since it is a rational point of $\mathcal{L}$. By Proposition 7.2.2 and Fact 7.1.2, we have the following corollary.

**Corollary 7.2.3.** *The constant functions* $0$ *and* $1$*, and the operations* $+, \times, (r\cdot)_{r\in\mathbb{Q}}$ *are uniformly computable w.r.t.* $\mathcal{L}$

**Theorem 7.2.4.** *There exists a rational-valued computable presentation* $\mathcal{A} = (f_i)_{i\in\mathbb{N}}$ *of* $(C(2^{\mathbb{N}}), d)$ *such that* $\mathcal{A}$ *is limit equivalent to* $\mathcal{L}$*, the constant zero function* $0$ *is computable w.r.t.* $\mathcal{A}$*, and the operation* $(\frac{1}{2}\cdot) : f \mapsto \frac{1}{2}f$ *is not computable w.r.t.* $\mathcal{A}$*.*

*Proof.* We use the idea of the proof of Theorem 7.1.6 (see Theorem 3.10 in [12]). Fix an effective list $(\Psi_e)_{e\in\mathbb{N}}$ of all Turing functionals of one argument. We build a computable presentation $\mathcal{A} = (f_i)_{i\in\mathbb{N}}$ of $(C(2^{\mathbb{N}}), d)$ by constructing a computable double sequence $(f_{i,s})_{i,s\in\mathbb{N}}$ of rational simple functions in stages and let $f_i := \lim_s f_{i,s}$. At the end of stage $s$, we will have a finite collection $f_{0,s}, \ldots, f_{n(s),s}$ of rational simple functions, where $n(s)$ is a nondecreasing function in $s$.

We need the following properties:

- $f_i := \lim_s f_{i,s}$ exists for every $i$.
- $(f_i)_{i\in\mathbb{N}}$ is dense in $(C(2^{\mathbb{N}}), d)$.
- $d(f_i, f_j)$ is computable uniformly in $i, j$.
- The zero function 0 is a computable point w.r.t $(f_i)_{i\in\mathbb{N}}$.
- The operation $(\frac{1}{2}\cdot) : f \mapsto \frac{1}{2}f$ is not computable w.r.t. $(f_i)_{i\in\mathbb{N}}$.
- $\mathcal{A}$ is limit equivalent to $\mathcal{L}$.

To satisfy the above properties, we will construct $(f_{i,s})_{i,s\in\mathbb{N}}$ that satisfies the following conditions:

(1) For every $i$, there is an $s_i$ such that $f_{i,t} = f_{i,s_i}$ for every $t \geq s_i$.

(2) For every $j$, there is a unique $k_j$ such that $f_{k_j} = l_j$.

(3) For every $s \in \mathbb{N}$ and $i, j \leq n(s)$, $d(f_{i,s}, f_{j,s}) = d(f_{i,s+1}, f_{j,s+1})$.

(4) For every $s$, $f_{0,s} = 0$.

(5) For every $e$, $\Psi_e$ does not represent $(\frac{1}{2}\cdot)$ in $(f_i)_{i\in\mathbb{N}}$.

After the construction, we can define a map using condition (1) to show that $\mathcal{A}$ is limit equivalent to $\mathcal{L}$.

To ensure that the operation $(\frac{1}{2}\cdot)$ is not computable w.r.t. $(f_i)_{i\in\mathbb{N}}$, we diagonalize against $\Psi_e$ potentially witnessing the computability of $(\frac{1}{2}\cdot)$ as follows. We choose a basic open set $U_e \subseteq 2^{\mathbb{N}}$ and a rational point $f_p$. Whenever the value of $\Psi_e$ on $f_p$ becomes close to $\frac{1}{2}f_p$ on $U_e$ in our current approximation (if ever), we change the approximation so that $\Psi_e$ on $f_p$ is far enough from $\frac{1}{2}f_p$ in the new approximation, and we will never change the approximation on $U_e$ again. This will make $\Psi_e$ on $f_p$ too far from $\frac{1}{2}f_p$, and so $\Psi_e$ does not represent $(\frac{1}{2}\cdot)$ in $(f_i)_{i\in\mathbb{N}}$.

To satisfy condition (2), we will satisfy the $P_j$-requirements:

$P_j$ : There is a unique $k_j$ such that $f_{k_j} = l_j$.

To satisfy the $P_j$-requirements, we use the following $P_j$-strategies.

**The $P_j$-strategy**

At stage $s+1 = 2\langle k, j\rangle$ where $k, j \in \mathbb{N}$: If $l_j$ is not among $f_{0,s}, \dots, f_{n(s),s}$, then let $f_{n(s)+1,v} := l_j$ for every $v \leq s+1$.

This ends the $P_j$-strategy.

Note that the $P_j$-strategies guarantee that for every $j \in \mathbb{N}$,

$$l_j \in \{f_{i,s} : i \leq n(s)\} \quad \text{for infinitely many } s \in \mathbb{N}.$$

After the construction, we verify that, in fact, $l_j \in \{f_{i,s} : i \leq n(s)\}$ for cofinitely many stages.

To make sure that $(\frac{1}{2}\cdot)$ is not computable w.r.t. $(f_i)_{i\in\mathbb{N}}$, we diagonalize against $\Psi_e$ so that $\Psi_e$ does not represent $(\frac{1}{2}\cdot)$ in $(f_i)_{i\in\mathbb{N}}$ for every $e \in \mathbb{N}$. If $\Psi_e$ represents $(\frac{1}{2}\cdot)$ in $(f_i)_{i\in\mathbb{N}}$, then for every $p \in \mathbb{N}$, since the constant sequence $(f_p, f_p, \dots)$ is a Cauchy name of $f_p$ in $(f_i)_{i\in\mathbb{N}}$, $\Psi_e^{(f_p,f_p,\dots)}$ enumerates a Cauchy name of $\frac{1}{2}f_p$ in $(f_i)_{i\in\mathbb{N}}$.

Fix an effective list $(\Phi_e)_{e\in\mathbb{N}}$ of all partial computable functions of two arguments.

By the *s*-*m*-*n* Theorem, there is a computable function $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$\Psi_e^{(f_p, f_p, \dots)}(n) = \Phi_{\alpha(e)}(p, n) \quad \text{for all } e, p, n \in \mathbb{N}.$$

So it is enough to make sure that for every $e \in \mathbb{N}$, $\Phi_e$ does not represent $(\frac{1}{2}\cdot)$ in $(f_i)_{i\in\mathbb{N}}$, i.e. there exists a $p \in \mathbb{N}$ such that $(f_{\Phi_e(p,n)})_{n\in\mathbb{N}}$ is not a Cauchy name for $\frac{1}{2}f_p$.

If $\Phi_e$ represents $(\frac{1}{2}\cdot)$ in $(f_i)_{i\in\mathbb{N}}$, then for every $p \in \mathbb{N}$, $(f_{\Phi_e(p,n)})_{n\in\mathbb{N}}$ is a Cauchy name for $\frac{1}{2}f_p$, and so

$$d(f_{\Phi_e(p,n)}, \frac{1}{2}f_p) \le 2^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, to ensure that $\Phi_e$ does not represent $(\frac{1}{2}\cdot)$ in $(f_i)_{i\in\mathbb{N}}$ for every $e \in \mathbb{N}$, it is enough to satisfy the $N_e$-requirements:

$$N_e : (\exists p)(\exists n)(\forall h)[\Phi_e(p, n) \downarrow = h \Longrightarrow d(f_h, \tfrac{1}{2}f_p) > 2^{-n}].$$

Fix an effective list $(U_e)_{e\in\mathbb{N}}$ of disjoint basic open sets of $2^{\mathbb{N}}$, say $U_e = [\![\tau_e]\!]$ where $\tau_e \in 2^{<\mathbb{N}}$. For example, we can let $U_e := [\![1^e 0]\!]$ for all $e \in \mathbb{N}$. The strategy for each $N_e$-requirement will act on its own basic open set $U_e$. This is to avoid any conflict with our attempt to satisfy the $P_j$-requirements and to preserve distances.

For each $e \in \mathbb{N}$, we let $\delta_e := 2^{-e-3}$ and we define a constant function $c_e \in C(2^{\mathbb{N}})$ by $c_e(X) = 2^{e+1}$ for all $X \in 2^{\mathbb{N}}$. The function $c_e$ will become our witness to satisfy the $N_e$-requirement.

**The $N_e$-strategy**

(a) At stage $t + 1 = 2\langle 0, e\rangle + 1$ where $e \in \mathbb{N}$:

If the constant function $c_e$ is not already among $f_{0,t}, \ldots, f_{n(t),t}$, then let $f_{n(t)+1,v} = c_e$ for every $v \leq t+1$.

So, in any case, there exists a $p \leq n(t)+1$ such that $f_{p,t+1} = c_e = 2^{e+1}$. We fix such $p$.

(b) At stage $s+1 = 2\langle k, e\rangle + 1$ where $k > 0$ and $e \in \mathbb{N}$:
We consider the computation for $\Phi_{e,s}(p, e+3)$:

If $\Phi_{e,s}(p, e+3)\uparrow$, then do nothing and go to the next stage.

If $\Phi_{e,s}(p, e+3)\downarrow = h$, then we have the following cases:

Case 1. $f_{h,s}$ has not been defined so far (i.e. $h > n(s)$):
Do nothing and go to the next stage.
(The $P_j$-strategies will ensure that $\lim_{s\to\infty} n(s) = \infty$. So we will wait until the first stage $s'+1 > s+1$ where $h \leq n(s')$.)

Case 2. $\sup_{X\in U_e} |f_{h,s}(X) - \frac{1}{2} f_{p,s}(X)| > 2^{-e-3} = \delta_e$:
Do nothing and stop the strategy.
(Note that we can compute $\sup_{X\in U_e} |f_{h,s}(X) - \frac{1}{2} f_{p,s}(X)|$) because $f_{h,s}$ and $f_{p,s}$ are rational simple functions and $U_e = [\![\tau_e]\!]$.)

Case 3. $\sup_{X\in U_e} |f_{h,s}(X) - \frac{1}{2} f_{p,s}(X)| \leq 2^{-e-3} = \delta_e$:
We effectively find a basic open set $V = [\![\rho_e]\!] \subseteq U_e$, a point $Y \in V$, and a sequence $(f_{0,s+1}, \ldots, f_{n(s),s+1})$ of rational simple functions such that

- $f_{0,s+1} = f_{0,s} = 0$ and $f_{i,s+1} = f_{i,s}$ on $2^{\mathbb{N}} \setminus V$ for all $i \leq n(s)$,
- for all $i \leq n(s)$, if $f_{i,s} \leq_V 2^e$, then $f_{i,s+1} = f_{i,s}$,
- $d(f_{i,s+1}, f_{j,s+1}) = d(f_{i,s}, f_{j,s})$ for all $i, j \leq n(s)$,

- $f_{p,s+1}(Y) = 2^e$.

(Here, $f \leq_V g$ means $f(x) \leq g(x)$ for all $x \in V$. Similarly for $<_V, >_V, =_V$, and $\geq_V$.) Stop the strategy.

The procedure for finding such $V, Y$ and $f_{i,s+1}$ is as follows. Since $f_{i,s}$'s are rational simple functions, we can effectively find a basic open set $V = [\![\rho_e]\!] \subseteq U_e$ such that $f_{i,s}$ is constant on $V$ for all $i \leq n(s)$. This implies that for all $i, j \leq n(s)$, we have $f_{i,s} <_V f_{j,s}$ or $f_{i,s} >_V f_{j,s}$ or $f_{i,s} =_V f_{j,s}$. We let $Y := \rho_e^\frown 0^{\mathbb{N}} \in V$.

Recall that we fixed $p \leq n(t) + 1$ such that $f_{p,t+1} = c_e = 2^{e+1}$. Since the value of $f_{p,t+1}$ on $U_e$ can be changed only by the $N_e$-strategy, and the $N_e$-strategy never changes the approximations before this stage $s+1$, we must have that $f_{p,s} =_{U_e} f_{p,t+1} =_{U_e} c_e = 2^{e+1}$. In particular, $f_{p,s}(Y) = 2^{e+1}$.

Let $V_0 := [\![\rho_e^\frown 0]\!]$ and $V_1 := [\![\rho_e^\frown 1]\!]$. Then $V = V_0 \sqcup V_1$ and $Y \in V_0$. For each $i \leq n(s)$, define a rational simple function $f_{i,s+1}$ as follows.

If $f_{i,s} \leq_V 2^e$, then let $f_{i,s+1} := f_{i,s}$.

If $f_{i,s} >_V 2^e$, then let

$$f_{i,s+1}(X) = \begin{cases} 2^e & \text{if } X \in V_0 \\ f_{i,s}(X) & \text{otherwise} \end{cases}$$

This ends the $N_e$-strategy.

We need to show that $V, Y$ and $f_{i,s+1}$ satisfy the desired properties. It is clear from the construction that $f_{0,s} = 0 <_V 2^e$. So, by the procedure, we have $f_{0,s+1} =$

$f_{0,s} = 0$. It is clear from the procedure that $f_{i,s+1} = f_{i,s}$ on $2^{\mathbb{N}} \setminus V$ for all $i \leq n(s)$. It is also clear that for all $i \leq n(s)$, if $f_{i,s} \leq_V 2^e$, then $f_{i,s+1} = f_{i,s}$, Recall that $f_{p,s} =_{U_e} 2^{e+1}$. So $f_{p,s} =_V 2^{e+1} >_V 2^e$. Thus, by the procedure, we have $f_{p,s+1} =_{V_0} 2^e$, in particular, $f_{p,s+1}(Y) = 2^e$.

It remains to show that $d(f_{i,s+1}, f_{j,s+1}) = d(f_{i,s}, f_{j,s})$ for all $i, j \leq n(s)$. Let $i, j \leq n(s)$. Let $M := \sup_{X \in V} |f_{i,s}(X) - f_{j,s}(X)|$. Since $f_{i,s}$ and $f_{j,s}$ are constant on $V$, we have that

$$\sup_{X \in V_0} |f_{i,s}(X) - f_{j,s}(X)| = M = \sup_{X \in V_1} |f_{i,s}(X) - f_{j,s}(X)|.$$

Note that $f_{i,s+1} = f_{i,s}$ and $f_{j,s+1} = f_{j,s}$ on $2^{\mathbb{N}} \setminus V_0$. So

$$\sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| = \sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s}(X) - f_{j,s}(X)| \geq M.$$

Hence $d(f_{i,s}, f_{j,s}) = \sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s}(X) - f_{j,s}(X)|$. Now it is enough to show that $\sup_{X \in V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| \leq M$.

Case 1. $f_{i,s} >_V 2^e$ and $f_{j,s} >_V 2^e$: Then $f_{i,s+1} =_{V_0} f_{j,s+1} =_{V_0} 2^e$, and so $\sup_{X \in V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| = 0 \leq M$.

Case 2. $f_{i,s} >_V 2^e$ and $f_{j,s} \leq_V 2^e$: Then $f_{i,s+1} =_{V_0} 2^e <_{V_0} f_{i,s}$ and $f_{j,s+1} = f_{j,s}$ Thus, for all $X \in V_0$, we have $f_{i,s+1}(X) - f_{j,s+1}(X) = 2^e - f_{j,s} \geq 0$, and so

$$|f_{i,s+1}(X) - f_{j,s+1}(X)| = 2^e - f_{j,s}(X) < f_{i,s}(X) - f_{j,s}(X) \leq M.$$

Hence $\sup_{X \in V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| \leq M$.

Case 3. $f_{i,s} \leq_V 2^e$ and $f_{j,s} >_V 2^e$: Similar to Case 2.

From all cases, we have that $\sup_{X \in V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| \leq M$. Therefore,

$$d(f_{i,s+1}, f_{j,s+1}) = \sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| = \sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s}(X) - f_{j,s}(X)| = d(f_{i,s}, f_{j,s}).$$

**Construction**

At stage 0: Let $f_{0,0} := 0$.

At even stage $s+1 = 2\langle k, j \rangle > 0$ where $k, j \in \mathbb{N}$: Use the $P_j$-strategy.

At odd stage $s+1 = 2\langle k, e \rangle + 1$ where $k, e \in \mathbb{N}$: Use the $N_e$-strategy.

At the end of stage $s+1$: For each $i \leq n(s)$, if $f_{i,s+1}$ has not been defined by any strategies, then we let $f_{i,s+1} := f_{i,s}$.

This ends the construction.

**Verification**

Our construction is effective because at each stage we have a finite collection of rational simple functions and all questions we ask about these collections are effectively decidable. Therefore, $(f_{i,s})_{i,s \in \mathbb{N}}$ is a computable double sequence.

It is clear from the construction that

$$f_{0,s} = f_{0,0} = 0 \text{ and } d(f_{i,s}, f_{j,s}) = d(f_{i,s+1}, f_{j,s+1}) \text{ for all } s \in \mathbb{N} \text{ and } i, j \leq n(s).$$

That is, conditions (3) and (4) are satisfied. So $f_0 = \lim_{s \to \infty} f_{0,s} = 0$ and $d(f_i, f_j) = d(f_{i,s}, f_{j,s})$ for all $s \in \mathbb{N}$ and $i, j \leq n(s)$. This implies that 0 is a computable point w.r.t. $(f_i)_{i\in\mathbb{N}}$ and $d(f_i, f_j)$ is computable uniformly in $i, j$. Note that, since $f_{i,s}$'s are rational simple functions, we have $d(f_i, f_j) \in \mathbb{Q}$ for all $i, j \in \mathbb{N}$, and so $\mathcal{A}$ is rational-valued.

We remark that only the $N_e$-strategies can change an approximations $(f_{i,s})_{s\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ in Case 3, and the change from $N_e$ occurs within its own basic open set $U_e$, which is disjoint from other basic open sets $U_j$ where $j \neq e$. Also, each $N_e$-strategy can change an approximations at most once.

Next, we show that condition (1) is satisfied. Let $i \in \mathbb{N}$ and let $t$ be the first stage at which $f_i$ gets its approximation, namely $f_{i,t}$. Fix a $C \in \mathbb{N}$ large enough so that $\|f_{i,t}\| = d(f_{i,t}, 0) < 2^C$. By conditions (3) and (4), we have that for all $i \in \mathbb{N}$ and $s \geq t$,

$$\|f_{i,s}\| = d(f_{i,s}, 0) = d(f_{i,s}, f_{0,s}) = d(f_{i,t}, f_{0,t}) = d(f_{i,t}, 0) = \|f_{i,t}\|.$$

At each stage $s+1$, the $N_e$-strategies can change an approximation of $f_i$ only if we are in Case 3 in the $N_e$-strategies where $f_{i,s} >_V 2^e$, in particular, $\|f_{i,s}\| > 2^e$. So only an $N_e$-strategy where $e < C$ can possibly change an approximation after stage $t$. Since each $N_e$-strategy acts at most once, there is a stage $s_0$ large enough so that $N_0, \ldots, N_{C-1}$ never act after stage $s_0$. So the approximation of $f_i$ will eventually reach its final value at or before stage $s_0$, and so condition (1) is satisfied. In particular, $f_i = \lim_s f_{i,s}$ exists for every $i$.

Next, we show that the $P_j$-requirements are satisfied. Let $j \in \mathbb{N}$. Fix a $C \in \mathbb{N}$ large enough so that $\|l_j\| < 2^C$. By the same argument as before, there is a stage $s_0$ after which $N_0, \cdots, N_{C-1}$ no longer act, and so every approximation $f_{i,s}$ with $\|f_{i,s}\| < 2^C$ and $s > s_0$ will become stable. Thus, since the $P_j$-strategy guarantees that $l_j \in \{f_{i,s} : i \leq n(s)\}$ for infinitely many $s \in \mathbb{N}$, there must be some $s > s_0$ and $k \leq n(s)$ such that $l_j = f_{k,s} = f_k$. The uniqueness of $k$ follows from the $P_j$-strategy and the observation that for every $s \in \mathbb{N}$ and $i, j \leq n(s)$, $f_{i,s+1} = f_{j,s+1} \iff f_{i,s} = f_{j,s}$. Hence the $P_j$-requirements are met, and so condition (2) is satisfied. Therefore, since $(l_j)_{j\in\mathbb{N}}$ is dense in $(C(2^{\mathbb{N}}), d)$, so is $(f_i)_{i\in\mathbb{N}}$.

Finally, to show that $(\frac{1}{2}\cdot)$ is not computable w.r.t $(f_i)_{i\in\mathbb{N}}$, it is enough to show that $(f_i)_{i\in\mathbb{N}}$ satisfies the $N_e$-requirements:

$$N_e : (\exists p)(\exists n)(\forall h)[\Phi_e(p, n) \downarrow = h \Longrightarrow d(f_h, \tfrac{1}{2} f_p) > 2^{-n}].$$

Let $e \in \mathbb{N}$. Choose $n := e+3$. So $\delta_e := 2^{-e-3} = 2^{-n}$. From (a) in the $N_e$-strategy, there exists a $p \leq n(t) + 1$ such that $f_{p,t+1} = 2^{e+1}$, where $t + 1 = 2\langle 0, e\rangle + 1$.

Assume that $\Phi_e(p, n) \downarrow = h$. Then $\Phi_{e,s}(p, n) \downarrow = h$ for some $s \in \mathbb{N}$. So we will eventually do Case 2 or Case 3 in the $N_e$-strategy at some stage $s$ large enough.

If the $N_e$-strategy stops in Case 2, then $\sup\limits_{X\in U_e} |f_{h,s+1}(X) - \frac{1}{2} f_{p,s+1}(X)| > \delta_e$, $f_h =_{U_e} f_{h,s+1}$, and $f_p =_{U_e} f_{p,s+1}$. So

$$d(f_h, \frac{1}{2} f_p) \geq \sup_{X\in U_e} |f_h(X) - \frac{1}{2} f_p(X)| = \sup_{X\in U_e} |f_{h,s+1}(X) - \frac{1}{2} f_{p,s+1}(X)| > \delta_e = 2^{-n}.$$

If the $N_e$-strategy stops in Case 3, then $\sup\limits_{X\in U_e} |f_{h,s}(X) - \frac{1}{2} f_{p,s}(X)| \leq \delta_e$, and there exist a basic open set $V \subseteq U_e$ and a point $Y \in V$ such that $f_{p,s+1}(Y) = 2^e$. Since the approximation on $U_e$ will never be changed again after the $N_e$-strategy acts, we have that $f_p(Y) = f_{p,s+1}(Y) = 2^e$ and $f_h(Y) = f_{h,s+1}(Y) = \min\{2^e, f_{h,s}(Y)\}$. Since

$\sup_{X \in U_e} |f_{h,s}(X) - \frac{1}{2}f_{p,s}(X)| \leq \delta_e$, $f_{p,s}(Y) = 2^{e+1}$ and $Y \in V \subseteq U_e$, we have

$$f_{h,s}(Y) \geq \frac{1}{2}f_{p,s}(Y) - \delta_e = 2^e - \delta_e.$$

So $f_h(Y) = f_{h,s+1}(Y) = \min\{2^e, f_{h,s}(Y)\} \geq 2^e - \delta_e$. Hence

$$\begin{aligned} f_h(Y) - \frac{1}{2}f_p(Y) &= f_h(Y) - \frac{1}{2} \cdot 2^e \\ &\geq 2^e - \delta_e - 2^{e-1} \\ &= 2^{e-1} - \delta_e \\ &> \delta_e. \quad (\because \delta_e = 2^{-e-3} < \frac{1}{2} \cdot 2^{e-1}) \end{aligned}$$

So $d(f_h, \frac{1}{2}f_p) > \delta_e = 2^{-n}$. Therefore, the $N_e$-requirements are satisfied, and so $(\frac{1}{2}\cdot)$ is not computable w.r.t. $(f_i)_{i\in\mathbb{N}}$.

Finally, we show that $\mathcal{A} := (f_i)_{i\in\mathbb{N}}$ is limit equivalent to $\mathcal{L} = (l_j)_{j\in\mathbb{N}}$. Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $g(x, s) =$ the unique number such that $f_{x,s} = l_{g(x,s)}$. Then $g$ is (total) computable. We think of $g$ as a function $g : \mathcal{A} \times \mathbb{N} \to \mathcal{L}$.

From condition (1), we have that for each $x \in \mathbb{N}$, for all $s \geq s_x$, $f_x = f_{x,s} = f_{x,s_x}$, and so $g(x, s) = g(x, s_x)$. Hence $\lim_{s\to\infty} g(x, s) = g(x, s_x)$.

Define $f : \mathcal{A} \to \mathcal{L}$ by $f(x) = \lim_{s\to\infty} g(x, s) = g(x, s_x)$, i.e. $f : f_x \mapsto l_{g(x,s_x)}$.

We claim that $f : \mathcal{A} \to \mathcal{L}$ is an isometry. To show that $f$ is distance-preserving, let $x, y \in \mathbb{N}$. Then $l_{g(x,s_x)} = f_{x,s_x} = f_x$ and $l_{g(y,s_y)} = f_{y,s_y} = f_y$. So $d(f_x, f_y) = d(l_{g(x,s_x)}, l_{g(y,s_y)}) = d(l_{f(x)}, l_{f(y)})$. Therefore, $f$ is distance-preserving. Thus, since $(l_j)_{j\in\mathbb{N}}$ and $(f_i)_{i\in\mathbb{N}}$ have no repetitions, $f$ is also injective.

It remains to show that $f$ is surjective. Let $j \in \mathbb{N}$. By condition (2), there is a unique $k_j \in \mathbb{N}$ such that $f_{k_j} = l_j$. Then $l_{g(k_j, s_{k_j})} = f_{k_j, s_{k_j}} = f_{k_j} = l_j$. So $f(k_j) = g(k_j, s_{k_j}) = j$. Therefore, $f$ is surjective.

We conclude that $f : \mathcal{A} \to \mathcal{L}$ is an isometry, and so $\mathcal{A}$ is limit equivalent to $\mathcal{L}$. □

**Theorem 7.2.5.** *There exist infinitely many computable presentations of* $(C(2^{\mathbb{N}}), d)$ *which are pairwise not computably isometric. In particular, the space* $C(2^{\mathbb{N}})$ *is not computably categorical as a metric space.*

*Proof.* By Fact 7.1.1 and Theorem 7.2.4, there exist two limit equivalent rational-valued computable presentations on $(C(2^{\mathbb{N}}), d)$ which are not computably isometric. The theorem then follows from Theorem 7.1.5. □

By Fact 7.1.2, we also have the following corollary.

**Corollary 7.2.6.** *There is a computable presentation of* $(C(2^{\mathbb{N}}), d)$ *in which* $+$ *is not computable.*

Next, we show that $C(2^{\mathbb{N}})$ is not computably categorical as a Banach space. Recall that the signature of Banach spaces consists of $d, 0, +, (r\cdot)_{r \in \mathbb{Q}}$. By Fact 7.1.2, we can assume that the signature of Banach spaces only contains $+$.

We will build a computable presentation $\mathcal{A}$ of $(C(2^{\mathbb{N}}), d, +)$ that is not computably isometric to the standard presentation $\mathcal{L}$ in the signature of Banach spaces. Suppose that $\mathcal{L}$ is computably isometric to $\mathcal{A}$ via a computable Banach space isomorphism $T : C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ w.r.t. $\mathcal{L}$ and $\mathcal{A}$. By the Banach-Stone Theorem (see, e.g.

[3], Theorem 2.1.1 on page 25), $T$ must have the form

$$Tf(x) = h(x)f(\varphi(x)) \quad \text{for all } x \in 2^{\mathbb{N}} \text{ and } f \in C(2^{\mathbb{N}}),$$

where $\varphi$ is a homeomorphism from $2^{\mathbb{N}}$ onto itself and $h$ is a (real-valued) continuous unimodular function on $2^{\mathbb{N}}$ (i.e. $h \in C(2^{\mathbb{N}})$ and $|h(x)| = 1$ for all $x \in 2^{\mathbb{N}}$). Since the constant function 1 is a computable point w.r.t. $\mathcal{L}$ and $T : C(2^{\mathbb{N}}) \to C(2^{\mathbb{N}})$ is a computable map w.r.t. $\mathcal{L}$ and $\mathcal{A}$, $T(1) = h$ is a computable point w.r.t. $\mathcal{A}$.

Let $|\cdot|$ denote the absolute value function $|\cdot| : f \mapsto |f|$ from $C(2^{\mathbb{N}})$ into $C(2^{\mathbb{N}})$. Then, since $h$ is unimodular, $|h| = 1$. So, if the operation $|\cdot|$ is computable w.r.t. $\mathcal{A}$, then the constant function 1 is also computable w.r.t. $\mathcal{A}$. Therefore, if we can build $\mathcal{A}$ so that the operation $|\cdot|$ is computable w.r.t. $\mathcal{A}$, but the constant function 1 is not computable w.r.t. $\mathcal{A}$, then we will have that $\mathcal{A}$ is not computably isometric to $\mathcal{L}$, and so $C(2^{\mathbb{N}})$ is not computably categorical as a Banach space.

**Theorem 7.2.7.** *There is a computable presentation $\mathcal{A} = (f_i)_{i\in\mathbb{N}}$ of $(C(2^{\mathbb{N}}), d, +)$ such that the operation $|\cdot|$ is computable w.r.t. $\mathcal{A}$, but the constant function 1 is not computable w.r.t. $\mathcal{A}$.*

*Proof.* We use the idea of the proof of Theorem 7.1.7 (see Theorem 4.2 in [12]). Fix an effective list $(\Phi_e)_{e\in\mathbb{N}}$ of all partial computable functions of one argument. We build a computable presentation $\mathcal{A} = (f_i)_{i\in\mathbb{N}}$ of $(C(2^{\mathbb{N}}), d)$ by constructing a computable double sequence $(f_{i,s})_{i,s\in\mathbb{N}}$ of rational simple functions in stages and then let $f_i := \lim_s f_{i,s}$. At the end of stage $s$, we will have a finite collection $f_{0,s}, \ldots, f_{n(s),s}$ of rational simple functions, where $n(s)$ is a nondecreasing function in $s$.

We need the following properties:

- $f_i := \lim_s f_{i,s}$ exists for every $i$.
- $(f_i)_{i\in\mathbb{N}}$ is dense in $(C(2^{\mathbb{N}}), d)$.
- $d(f_i, f_j)$ is computable uniformly in $i, j$.
- $+$ and $|\cdot|$ are computable w.r.t. $(f_i)_{i\in\mathbb{N}}$.
- The constant function 1 is not a computable point w.r.t $(f_i)_{i\in\mathbb{N}}$.

To satisfy the above properties, we will construct $(f_{i,s})_{i,s\in\mathbb{N}}$ that satisfies the following conditions:

(1) For every $i$, $\lim_s f_{i,s}$ exists.

(2) For every $j$ and $e$, there is some $k$ such that $d(f_k, l_j) \leq 2^{-e}$.

(3) For every $s \in \mathbb{N}$ and $i, j \leq n(s)$, $d(f_{i,s}, f_{j,s}) = d(f_{i,s+1}, f_{j,s+1})$.

(4) For every $s \in \mathbb{N}$ and $i, j, k \leq n(s)$, $f_{i,s} + f_{j,s} = f_{k,s} \Rightarrow f_{i,s+1} + f_{j,s+1} = f_{k,s+1}$.

(5) For every $s \in \mathbb{N}$ and $i, k \leq n(s)$, $|f_{i,s}| = f_{k,s} \Rightarrow |f_{i,s+1}| = f_{k,s+1}$.

(6) For every $e$, $\Phi_e$ is not a Cauchy name of the constant function 1 in $(f_i)_{i\in\mathbb{N}}$.

To ensure that condition (2) is satisfied and the operations $+$ and $|\cdot|$ are computable, we will implement at odd stages a strategy similar to the $P_j$-strategy in the proof of Theorem 7.2.4.

Recall that a point in a computable metric space is computable if it has a computable Cauchy name. To make sure that the constant function 1 is not computable w.r.t. $(f_i)_{i\in\mathbb{N}}$, we diagonalize against $\Phi_e$ so that $\Phi_e$ is not a Cauchy name of 1 in $(f_i)_{i\in\mathbb{N}}$ for every $e \in \mathbb{N}$. That is, we want to satisfy condition (6).

If $\Phi_e$ is a Cauchy name of 1, then $\Phi_e$ is total and $d(f_{\Phi_e(n)}, 1) \leq 2^{-n}$ for all $n \in \mathbb{N}$. Therefore, to ensure that 1 is not a computable point w.r.t. $(f_i)_{i\in\mathbb{N}}$, it is enough to satisfy the $N_e$-requirements:

$$N_e : (\exists n)(\forall h)[\Phi_e(n)\downarrow= h \implies d(f_h, 1) > 2^{-n}].$$

Fix an effective list $(U_e)_{e\in\mathbb{N}}$ of disjoint basic open sets of $2^{\mathbb{N}}$, say $U_e = [\![\tau_e]\!]$ where $\tau_e \in 2^{<\mathbb{N}}$. For each $e \in \mathbb{N}$, let $\delta_e := 2^{-e-2}$.

**The $N_e$-strategy**

Wait until a stage $s+1 = 2\langle k, e\rangle+2$ where $k, e \in \mathbb{N}$ such that $\Phi_{e,s}(e+2)$ converges to a natural number $h \leq n(s)$. Then we have the following cases:

Case 1. $\sup_{X\in U_e} |f_{h,s}(X) - 1| > 2^{-e-2} = \delta_e$:
Do nothing and stop the strategy.

Case 2. $\sup_{X\in U_e} |f_{h,s}(X) - 1| \leq 2^{-e-2} = \delta_e$:
Then for all $X \in U_e$, $|f_{h,s}(X)| \geq 1-|f_{h,s}(X)-1| \geq 1-\delta_e > 0$. Since $f_{i,s}$'s are rational simple functions, we can effectively find a basic open set $V = [\![\rho_e]\!] \subseteq U_e$ such that $f_{i,s}$ is constant on $V$ for all $i \leq n(s)$. Let $V_0 := [\![\rho_e{}^\frown 0]\!]$, $V_1 := [\![\rho_e{}^\frown 1]\!]$ and $Y := \rho_e{}^\frown 0^{\mathbb{N}} \in V_0$. Then $V = V_0 \sqcup V_1$ and $Y \in V_0$. For each $i \leq n(s)$, define a rational simple function $f_{i,s+1}$ by

$$f_{i,s+1}(X) = \begin{cases} (1-2^{-e-1})f_{i,s}(X) = (1-2^{-e-1})f_{i,s}(Y) & \text{if } X \in V_0 \\ f_{i,s}(X) & \text{if } X \in 2^{\mathbb{N}} \setminus V_0 \end{cases}$$

Stop the strategy.

This ends the $N_e$-strategy.

From the $N_e$-strategy, it is clear that for all $i \leq n(s)$, $f_{i,s+1} = f_{i,s}$ on $2^{\mathbb{N}} \setminus V_0$, $f_{h,s+1}(Y) = (1-2^{-e-1})f_{h,s}(Y)$, and $d(f_{i,s}, f_{i,s+1}) = 2^{-e-1}|f_{i,s}(Y)| \leq 2^{-e-1}\|f_{i,s}\|$. Note that for all $i \leq n(s)$, $|f_{i,s+1}(X)| = (1-2^{-e-1})|f_{i,s}(X)| \leq |f_{i,s}(X)|$ for all $X \in V_0$, and so $\|f_{i,s+1}\| \leq \|f_{i,s}\|$. It is also clear from the definition of $f_{i,s+1}$ that for every $i, j, k \leq n(s)$,

- $f_{i,s} + f_{j,s} = f_{k,s} \Rightarrow f_{i,s+1} + f_{j,s+1} = f_{k,s+1}$,
- $|f_{i,s}| = f_{k,s} \Rightarrow |f_{i,s+1}| = f_{k,s+1}$.

We claim that $d(f_{i,s+1}, f_{j,s+1}) = d(f_{i,s}, f_{j,s})$ for all $i, j \leq n(s)$. Let $i, j \leq n(s)$. Let $M := \sup\limits_{X \in V} |f_{i,s}(X) - f_{j,s}(X)|$. Since $f_{i,s}$ and $f_{j,s}$ are constant on $V$, we have that

$$\sup_{X \in V_0} |f_{i,s}(X) - f_{j,s}(X)| = M = \sup_{X \in V_1} |f_{i,s}(X) - f_{j,s}(X)|.$$

So $d(f_{i,s}, f_{j,s}) = \sup\limits_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s}(X) - f_{j,s}(X)|$. Recall that, on $2^{\mathbb{N}} \setminus V_0$, $f_{i,s+1} = f_{i,s}$ and $f_{j,s+1} = f_{j,s}$. Hence

$$\sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| = \sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s}(X) - f_{j,s}(X)| \geq M.$$

Note that for every $X \in V_0$,

$$\begin{aligned}
|f_{i,s+1}(X) - f_{j,s+1}(X)| &= |(1-2^{-e-1})f_{i,s}(X) - (1-2^{-e-1})f_{j,s}(X)| \\
&= (1-2^{-e-1})|f_{i,s}(X) - f_{j,s}(X)| \\
&\leq |f_{i,s}(X) - f_{j,s}(X)|
\end{aligned}$$

$$\leq M.$$

So $\sup_{X \in V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| \leq M$. Therefore,

$$d(f_{i,s+1}, f_{j,s+1}) = \sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s+1}(X) - f_{j,s+1}(X)| = \sup_{X \in 2^{\mathbb{N}} \setminus V_0} |f_{i,s}(X) - f_{j,s}(X)| = d(f_{i,s}, f_{j,s}).$$

We conclude that the operations $d(\cdot, \cdot)$, $+$, and $|\cdot|$ are preserved under the action of the $N_e$-strategies.

**Construction**

At stage 0: Let $f_{0,0} := 0$.

At odd stage $s + 1 = 2\langle p, q, r, e\rangle + 1$ where $p, q, r, e \in \mathbb{N}$ and $r = \langle r_0, r_1\rangle$:
We let

$$j = \begin{cases} p & \text{if } q \equiv 0 (\text{mod } 3) \\ \text{a number such that } l_j = f_{r_0,s} + f_{r_1,s} & \text{if } q \equiv 1 (\text{mod } 3) \\ \text{a number such that } l_j = |f_{r,s}| & \text{if } q \equiv 2 (\text{mod } 3) \end{cases}$$

If $l_j$ is not among $(f_{i,s})_{i \leq n(s)}$, we let $f_{n(s)+1,v} := l_j$ for all $v \leq s + 1$.

At even stage $s + 1 = 2\langle k, e\rangle + 2$ where $k, e \in \mathbb{N}$: Use the $N_e$-strategy.

At the end of stage $s + 1$: For each $i \leq n(s)$, if $f_{i,s+1}$ has not been defined, we let $f_{i,s+1} := f_{i,s}$.

This ends the construction.

**Verification**

Our construction is effective because at each stage we have a finite collection of rational simple functions, and all questions we ask about these collections are effectively decidable. Therefore, $(f_{i,s})_{i,s\in\mathbb{N}}$ is a computable double sequence.

We remark that only the $N_e$-strategies can change an approximation $(f_{i,s})_{s\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ in Case 2, and the change from $N_e$ occurs within its own basic open set $U_e$, which is disjoint from other basic open sets $U_j$ where $j \neq e$. Also, each $N_e$-strategy can change an approximations at most once, and if an $N_e$-strategy acts at some stage $s$, then $d(f_{i,s}, f_{i,s+1}) \leq 2^{-e-1}\|f_{i,s}\|$ and $\|f_{i,s+1}\| \leq \|f_{i,s}\|$ for all $i \leq n(s)$.

Next, we show that conditions (1) and (2) are satisfied. By the above remark, we have that for every $e$, there is a stage $s_e$ such that the strategies $N_0, \ldots, N_{e-1}$ never act at or after stage $s_e$. For each $i \in \mathbb{N}$, let $t_i$ be the first stage at which $f_i$ gets its approximation, namely $f_{i,t_i}$, and let $M_i := \|f_{i,t_i}\|$. It follows that for every $i, e, s \in \mathbb{N}$, if $s \geq s_e$ and $i \leq n(s)$, then for every $u, v \geq \max\{s, t_i\}$, $d(f_{i,u}, f_{i,v}) \leq 2^{-e}\|f_{i,v}\| \leq 2^{-e}M_i$. This implies that for every $i \in \mathbb{N}$, $(f_{i,s})_{s\in\mathbb{N}}$ is a Cauchy sequence, and so $\lim\limits_{s\to\infty} f_{i,s}$ exists, that is, condition (1) is satisfied. To show that condition (2) is satisfied, let $j, e \in \mathbb{N}$ and let $e' \in \mathbb{N}$ be large enough so that $2^{-e'}\|l_j\| < 2^{-e}$. Consider a stage of the form $s'+1 = 2\langle j, 3q, r, m\rangle + 1 \geq s_{e'}$ where $q, r, m \in \mathbb{N}$. The construction at this stage ensures that $l_j = f_{k,s'+1}$ for some $k \leq n(s'+1)$. Since $s'+1 \geq s_{e'}$, we have that $d(f_{k,s}, f_{k,s'+1}) \leq 2^{-e'}\|f_{k,s'+1}\| = 2^{-e'}\|l_j\|$ for every $s \geq s'+1$. So

$$d(f_k, l_j) = d(\lim_{s\to\infty} f_{k,s}, f_{k,s'+1}) \leq 2^{-e'}\|l_j\| < 2^{-e}.$$

Therefore, condition (2) is satisfied. Then condition (2) and the density of $(l_j)_{j\in\mathbb{N}}$ in $C(2^{\mathbb{N}})$ implies that $(f_i)_{i\in\mathbb{N}}$ is dense in $C(2^{\mathbb{N}})$.

Since the operations $d(\cdot,\cdot)$ is preserved under the action of the $N_e$-strategies, we have

$$d(f_{i,s}, f_{j,s}) = d(f_{i,s+1}, f_{j,s+1}) \text{ for all } s \in \mathbb{N} \text{ and } i, j \leq n(s).$$

That is, condition (3) is satisfied. Hence $d(f_i, f_j) = d(f_{i,s}, f_{j,s})$ for all $s \in \mathbb{N}$ and $i, j \leq n(s)$, and so $d(f_i, f_j)$ is computable uniformly in $i, j$.

Since the operations $+$ and $|\cdot|$ are preserved under the action of the $N_e$-strategies, we have that conditions (4) and (5) are satisfied. To see why this is true, let $i, j \in \mathbb{N}$ and let $s$ be large enough so that $f_{i,s}$ and $f_{j,s}$ are both defined (i.e. $i, j \leq n(s)$), and let $p$ be such that $f_{i,s} + f_{j,s} = l_p$. Then the construction at odd stages ensures that $f_i + f_j$ must receive a definition at or before stage $s' := 2\langle p, 3s+1, \langle i, j\rangle, e\rangle + 1 > s+1$ where $e \in \mathbb{N}$. That is, at the end of stage $s'$, we will have $f_{i,s'} + f_{j,s'} = f_{k,s'}$ for some $k \leq n(s')$, and so, by condition (4), we have that $f_i + f_j = f_k$. Therefore, $+$ is computable w.r.t. $(f_i)_{i\in\mathbb{N}}$. Similarly, condition (5) implies that $|\cdot|$ is computable w.r.t. $(f_i)_{i\in\mathbb{N}}$.

It remains to show that $(f_i)_{i\in\mathbb{N}}$ satisfies the $N_e$-requirements:

$$N_e : (\exists n)(\forall h)[\Phi_e(n) \downarrow= h \Longrightarrow d(f_h, 1) > 2^{-n}].$$

Let $e \in \mathbb{N}$ and choose $n := e + 2$. Then $\delta_e = 2^{-e-2} = 2^{-n}$. Assume that $\Phi_e(n) \downarrow= h$. Then $\Phi_{e,s}(n) \downarrow= h$ for some $s \in \mathbb{N}$. So we will eventually do Case 1 or Case 2 in the $N_e$-strategy at some stage $s$ large enough.

If the $N_e$-strategy stops in Case 1, then $\sup\limits_{X\in U_e} |f_{h,s+1}(X) - 1| > \delta_e$ and $f_h =_{U_e} f_{h,s+1}$. So

$$d(f_h, 1) \geq \sup_{X\in U_e} |f_h(X) - 1| = \sup_{X\in U_e} |f_{h,s+1}(X) - 1| > \delta_e = 2^{-n}.$$

If the $N_e$-strategy stops in Case 2, then $\sup_{X \in U_e} |f_{h,s}(X) - 1| \leq \delta_e$ and there exist a basic open set $V \subseteq U_e$ and a point $Y \in V$ such that $f_{h,s+1}(Y) = (1 - 2^{-e-1})f_{h,s}(Y)$.

Since the approximation on $U_e$ will never be changed again after the $N_e$-strategy acts, we have that $f_h(Y) = f_{h,s+1}(Y) = (1 - 2^{-e-1})f_{h,s}(Y)$. Since $Y \in V \subseteq U_e$ and $\sup_{X \in U_e} |f_{h,s}(X) - 1| \leq \delta_e$, we have $f_{h,s}(Y) \leq 1 + \delta_e$. Thus, since $1 - 2^{-e-1} > 0$, we have

$$\begin{aligned} f_h(Y) = f_{h,s+1}(Y) &= (1 - 2^{-e-1})f_{h,s}(Y) \\ &\leq (1 - 2^{-e-1})(1 + \delta_e) \\ &= (1 - 2^{-e-1})(1 + 2^{-e-2}) \\ &= 1 - 2^{-e-2} - 2^{-e-1} \cdot 2^{-e-2} \\ &< 1 - 2^{-e-2}. \end{aligned}$$

So $d(f_h, 1) \geq |f_h(Y) - 1| > 2^{-e-2} = 2^{-n}$. Therefore, the $N_e$-requirements are satisfied, and so the constant function 1 is not computable w.r.t. $(f_i)_{i \in \mathbb{N}}$.

This completes the proof of Theorem 7.2.7. □

By the discussion before Theorem 7.2.7, we have the following theorem.

**Theorem 7.2.8.** *The space $(C(2^{\mathbb{N}}), d, +)$ is not computably categorical. Equivalently, $C(2^{\mathbb{N}})$ is not computably categorical as a Banach space.*

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