EFFECTIVENESS FOR THE DUAL RAMSEY THEOREM

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Abstract. We analyze the Dual Ramsey Theorem for k partitions and ℓ colors (DRT^k_ℓ) in the context of reverse math, effective analysis, and strong reductions. Over RCA_0, the Dual Ramsey Theorem stated for Baire colorings $\mathsf{Baire\text{-}DRT}^k_\ell$ is equivalent to the statement for clopen colorings ODRT^k_ℓ and to a purely combinatorial theorem CDRT^k_ℓ.

When the theorem is stated for Borel colorings and $k \geq 3$, the resulting principles are essentially relativizations of CDRT^k_ℓ. For each α, there is a computable Borel code for a Δ^0_α coloring such that any partition homogeneous for it computes $\emptyset^{(\alpha)}$ or $\emptyset^{(\alpha-1)}$ depending on whether α is infinite or finite.

For $k = 2$, we present partial results giving bounds on the effective content of the principle. A weaker version for Δ^0_n reduced colorings is equivalent to D^n_2 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$ and in the sense of strong Weihrauch reductions.

1. Introduction

The Dual Ramsey Theorem states that for every sufficiently nice coloring of the k-block partitions of ω using ℓ colors, there is a partition of ω into infinitely many blocks such that every coarsening of it down to exactly k blocks has the same color. The theorem was proved for Borel colorings by Carlson and Simpson [3] (who also show it is not true for arbitrary colorings by a straightforward choice argument) and was extended to colorings with the Baire property by Prömel and Voigt [13].

Dual Ramsey Theorem ([3], [13])**.** *For all finite $k, \ell \geq 1$, if $(\omega)^k = C_0 \cup \cdots \cup C_{\ell-1}$, where each C_i is Borel (or more generally has the Baire property), then there exist $p \in (\omega)^\omega$ and $i < \ell$ such that $(p)^k \subseteq C_i$.*

In this statement, $(\omega)^k$ is the set of partitions of ω into k nonempty pieces, $(\omega)^\omega$ is the set of partitions of ω into infinitely many nonempty pieces and $(p)^k$ is the set of coarsenings of p down to exactly k many blocks. The partition p in the Dual Ramsey Theorem is said to be *homogeneous* for the coloring. Typically, we think of the colors C_i being disjoint although they do not have to be. Throughout this article, when talking about versions of the Dual Ramsey Theorem with parameters k and ℓ, we will assume $k, \ell \geq 2$.

To study the Dual Ramsey Theorem in computability theory or reverse mathematics, we must choose a method to code the coloring of k-partitions. Previous work in these areas focused on ODRT^k_ℓ (requiring each color to be open), or avoided coding by considering variants of the Carlson-Simpson Lemma $\mathsf{CSL}(k, \ell)$ (the main

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combinatorial lemma in [3], defined below in Section 3) such as the Variable and Ordered Variable Word principles $\mathsf{VW}(k,\ell)$ and $\mathsf{OVW}(k,\ell)$.

Miller and Solomon [11] defined the principle ODRT^k_ℓ to be the restriction of the Dual Ramsey Theorem for k partitions and ℓ colors in which each color is represented by a code for an open set (and hence each color is clopen). Formalizing arguments from [3], they showed that ODRT^{k+1}_ℓ implies RT^k_ℓ over RCA_0, and hence when $k \geq 4$, ODRT^k_ℓ implies ACA_0.

For the variants of Carlson-Simpson Lemma, Erhard [6] proved that COH does not imply $\mathsf{VW}(2,2)$ and that SRT^2_2 does not imply $\mathsf{OVW}(2,2)$, while Solomon and Miller [11] showed that WKL_0 does not imply $\mathsf{VW}(2,2)$. However, the methods used in these proofs do not extend to the more general Carlson-Simpson Lemma and it remains an open question whether $\mathsf{CSL}(2,2)$ is computably true.

In the present paper, we consider a broader range of representations for the colorings in the Dual Ramsey Theorem. After fixing notation in Section 2, we specify four version of the Dual Ramsey Theorem in Section 3. Three of the versions are directly related to the formal method of coding the coloring: $\mathsf{Borel\text{-}DRT}^k_\ell$ uses Borel codes, ODRT^k_ℓ uses codes for open sets and $\mathsf{Baire\text{-}DRT}^k_\ell$ uses a Baire approximation to the coloring. The fourth variant, CDRT^k_ℓ is a combinatorial statement similar to the Carlson-Simpson Lemma (with a slight shift in parameters) but is more closely tied to the topological versions. In particular, CDRT^k_ℓ implies the Carlson-Simpson Lemma (with appropriate parameters) and it follows from transfinitely many nested applications of the Carlson-Simpson Lemma (again, with appropriate parameters).

In Section 3.1, we show that $\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ and CDRT^k_ℓ are equivalent over RCA_0, and that when $k = 2$, these principles are provable in RCA_0, and hence are computably true. The coding issues for $\mathsf{Borel\text{-}DRT}^k_\ell$ are more subtle. We discuss the connections between $\mathsf{Borel\text{-}DRT}^k_\ell$, $\mathsf{Baire\text{-}DRT}^k_\ell$ and ATR_0 in Section 3.2 but delay the formal reverse mathematics proofs until Section 6.

Simpson noted a connection between $\mathsf{CSL}(2,\ell)$ and Hindman's Theorem (see [3, page 268]), and we thank Ludovic Patey for showing us a proof of $\mathsf{CSL}(2,\ell)$ from Hindman's Theorem. With minor modifications, we adapt this proof in Section 3.4 to show that Hindman's Theorem for ℓ colorings implies CDRT^3_ℓ and hence ACA^+_0 implies CDRT^3_ℓ by Blass, Hirst and Simpson [2]. We also show that the method does not generalize for $k > 3$.

The earliest claim we are aware of for a proof of CDRT^k_ℓ is in [13] where a generalization of CDRT^k_ℓ labeled Theorem A is attributed to a preprint of Voigt titled "Parameter words, trees and vector spaces". However, as far as we can tell, this paper never appeared in print. Another proof of CDRT^k_ℓ can be found in [17] where it comes as a corollary of a larger theory. Therefore, in Section 3.5, we present a self contained classical proof of CDRT^k_ℓ for $k \geq 3$ (since CDRT^2_ℓ is provable in RCA_0) in which the only non-constructive steps are $\omega\cdot(k-2)$ nested applications of the Carlson-Simpson Lemma.

In Section 4, we consider $\mathsf{Borel\text{-}DRT}^k_2$ for $k \geq 3$ from the perspective of effective combinatorics rather than reverse mathematics. For each ordinal $0 < \alpha < \omega_1^{CK}$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring $(\omega)^k = R \cup \overline{R}$ for which any homogeneous infinite partition p computes $\emptyset^{(\alpha)}$. We pull the $\emptyset^{(\alpha)}$ description of the open set R down to a computable code for R at the expense of describing R as a topologically Δ^0_α or $\Delta^0_{\alpha+1}$ set in the Borel hierarchy depending on whether α is infinite or finite.

Figure 1. Implications over RCA_0 between variants of the Dual Ramsey Theorem considered in this paper and some related principles. The parameter $k \geq 4$ is arbitrary.

$$\begin{tikzcd}
 & & \Pi^1_1\text{-}\mathsf{CA}_0 \arrow[dll] \arrow[drr] & & \\
\mathsf{ATR}_0 \arrow[d] & & & & \mathsf{Borel\text{-}DRT}^k_2 \arrow[d] \\
\mathsf{ACA}_0^+ \arrow[d] & & & & \mathsf{CDRT}^k_2 \leftrightarrow \mathsf{Baire\text{-}DRT}^k_2 \leftrightarrow \mathsf{ODRT}^k_2 \arrow[ddllll] \arrow[ddll] \arrow[dd] \\
\mathsf{HT} \arrow[d] \arrow[drr] & & & & \\
\mathsf{ACA}_0 \arrow[d] & & \mathsf{CDRT}^3_2 \leftrightarrow \mathsf{Baire\text{-}DRT}^3_2 \leftrightarrow \mathsf{ODRT}^3_2 \arrow[dll] \arrow[drr] & & \mathsf{CSL}(k-1,2) \arrow[d] \\
\mathsf{RT}^2_2 \arrow[d] & & & & \mathsf{CSL}(2,2) \arrow[ddll] \\
\mathsf{SRT}^2_2 \leftrightarrow \Delta^0_2\text{-}\mathsf{rDRT}^2_2 \arrow[drr] & & & & \\
 & & \mathsf{RCA}_0 \leftrightarrow \mathsf{CDRT}^2_2 \leftrightarrow \mathsf{Baire\text{-}DRT}^2_2 \leftrightarrow \mathsf{ODRT}^2_2 & &
\end{tikzcd}$$

Therefore, for each $0 < \alpha < \omega_1^{CK}$, there is a computable Borel code for a set R as a topologically Δ^0_α set (although R is classically open) such that every infinite homogeneous partition for the coloring $(\omega)^k = R \cup \overline{R}$ computes $\emptyset^{(\alpha-1)}$ or $\emptyset^{(\alpha)}$ depending on whether α is finite or infinite.

We interpret the results in Sections 3 and 4 as indicating that Baire codes are a more natural representation than Borel codes for studying computational properties of the Dual Ramsey Theorem and that the Borel version of the Dual Ramsey Theorem can be thought of as a relativization of the Baire version.

In Section 5, we study $\mathsf{Borel\text{-}DRT}^2_2$ and give upper bounds on the complexity of finding an infinite homogeneous partition for colorings $(\omega)^2 = R \cup \overline{R}$ where R is coded as a set at a finite level of the Borel hierarchy. If R is a computable open set, then there is a computable infinite homogeneous partition, although the construction of this partition is necessarily non-uniform. If R has a computable code as a Σ^0_{n+2} set in the Borel hierarchy, then there is either a $\emptyset^{(n)}$-computable homogeneous partition for R or a $\emptyset^{(n+1)}$-computable homogeneous partition for $\overline{R}$.

Because of the non-uniformity in these results, we end Section 5 by characterizing a restriction of $\mathsf{Borel\text{-}DRT}^2_2$ under strong Weihrauch reducibility. For this reducibility, we think of $\mathsf{Borel\text{-}DRT}^2_2$ as an *instance-solution problem*. Such a problem consists of a collection of subsets of ω called the *instances* of this problem, and for each instance, a collection of subsets of ω called the *solutions* to this instance (for this problem). A problem P is *strongly Weihrauch reducible* to a problem Q if there are fixed Turing functionals Φ and Ψ such that given any instance A of P, Φ^A is an instance of Q, and given any solution B to Φ^A in Q, Ψ^B is a solution to A in P. There are a number of variations on this reducibility and we refer to the reader to [5] and [8] for background on these reductions and for connections to reverse mathematics. In this paper, we will only be interested in problems arising out of Π^1_2 statements of second order arithmetic. Any such statement can be put in the form $\forall X(\varphi(X) \to \exists Y \psi(X,Y))$, where ϕ and ψ are arithmetical. We can then regard this as a problem, with instances being all X such that $\varphi(X)$, and the solutions to X being all Y such that $\psi(X,Y)$. Note that while the choice of φ and ψ is not unique, we always have a fixed such choice in mind for a given Π^1_2 statement, and so also a fixed assignment of instances and solutions.

We formulate a version of $\mathsf{Borel\text{-}DRT}^2_2$, denoted $\Delta^0_n\text{-}\mathsf{rDRT}^2_2$, for which the instances are reduced colorings $(\omega)^2 = R \cup \overline{R}$ where R and $\overline{R}$ are given by Borel codes for Σ^0_n sets in the Borel hierarchy and the solutions are homogeneous infinite partitions. (We define a reduced coloring in Section 3.) We show this problem is strong Wiehrauch equivalent (and equivalent over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$) to D^n_2 which states that every stable coloring $c : [\omega]^n \to 2$ has an infinite limit-homogeneous set. In particular, since D^2_2 is equivalent to SRT^2_2 over RCA_0 by Chong, Lempp and Yang [4], it follows that $\Delta^0_2\text{-}\mathsf{rDRT}^2_2$ is equivalent to SRT^2_2 over RCA_0.

In Section 6, we present a number of technical results in reverse mathematics connected to Borel and Baire codes. In particular, we show that ATR_0 is equivalent to the statement that every Borel code has a Baire approximation and to the statement that for every Borel code B, there is some point x such that $x \in B$ or $x \notin B$. The proofs use a method of effective transfinite recursion, ETR, which is available in ACA_0 (and possibly in weaker systems). Variations of these results are known in some branches of effective mathematics. For example, Ash and Knight [1] prove similar results in the context of computable fragments of $\mathcal{L}_{\omega_1,\omega}$ rather than Borel codes. Greenberg and Montalbán [7] use ETR to establish equivalences of ATR_0 and claim that ETR is provable in RCA_0. However, their proof of ETR overlooks an application of Σ^0_1 transfinite induction, and in general, transfinite induction for Σ^0_1 formulas does not hold in RCA_0. While the main results in [7] continue to hold because Greenberg and Montalbán show the classified theorems imply ACA_0 without reference to ETR (and hence can use ETR in ACA_0 to complete the equivalence with ATR_0), we have included a proof of ETR to make explicit the use of transfinite induction.

We end this section with two comments on notation. First, we use ω to denote the natural numbers, which in subsystems of Z_2 is the set $\{x : x = x\}$, often denoted by $\mathbb{N}$ in the literature. Despite this notation, we do not restrict ourselves to ω-models. Second, when we refer to the parameters k and ℓ in versions of the Dual Ramsey Theorem, we assume k and ℓ are arbitrary standard numbers with $k, \ell \geq 2$. By a statement such as "RCA_0 proves $\mathsf{Borel\text{-}DRT}^k_\ell$ implies $\mathsf{Baire\text{-}DRT}^k_\ell$", we mean, for all $k, \ell \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{Borel\text{-}DRT}^k_\ell \to \mathsf{Baire\text{-}DRT}^k_\ell$. For many results, the quantification over k and ℓ can be pulled inside the formal system. However, in some cases, issues of induction arise and we wish to set those aside in this work.

2. Notation

For $k \leq \omega$, let $k^{<\omega}$ denote the set of finite strings over k and let k^ω denote the set of functions $f : \omega \to k$. As noted above, unless explicitly stated otherwise, we will always assume that $k \geq 2$. For $\sigma \in k^{<\omega}$, $|\sigma|$ denotes the length of σ, and if $|\sigma| > 0$, $\sigma(0), \dots, \sigma(|\sigma| - 1)$ denote the entries of σ in order. For $p \in k^\omega$ and $\sigma \in k^{<\omega}$, we write $\sigma \prec p$ if σ is an initial segment of p. Similarly, if $\sigma, \tau \in k^{<\omega}$, we write $\sigma \preceq \tau$ if σ is an initial segment of τ and $\sigma \prec \tau$ if σ is a proper initial segment of τ. We write $p \upharpoonright n$ to denote the string obtained by restricting the domain of p to n. The standard (product) topology on k^ω is generated by basic clopen sets of the form

$$[\sigma] = \{p \in k^\omega : \sigma \prec p\}$$

for $\sigma \in k^{<\omega}$.

Informally, for $k \leq \omega$, a *k-partition* p of ω is a collection of k many pairwise disjoint nonempty sets $B^p_i \subseteq \omega$ (called *blocks* or *p-blocks*) such that $\cup_{i<k} B^p_i = \omega$. When the partition is clear from context, we may drop the superscript p. We denote the least element of B^p_i by $\mu^p(i)$ or simply $\mu(i)$. To fix a unique representation for each k-partition, we assume the blocks are indexed such that $\mu^p(i) < \mu^p(i+1)$. With this convention, each k-partition is represented by a unique surjective function $p : \omega \to k$ with $p^{-1}(i) = B^p_i$.

More formally, we say $f \in k^\omega$ is *ordered* if

$$\forall n\, \forall i < k\, \big(f(n) = i \to \forall j < i\, \exists m < n\, (f(m) = j)\big)$$

and we say that $\sigma \in k^{<\omega}$ is *ordered* if it satisfies the analogous condition for all $n < |\sigma|$. We let $(\omega)^k \subseteq k^\omega$ denote the set of ordered surjective functions $f \in k^\omega$. In second order arithmetic, we view the notation $p \in (\omega)^k$ as shorthand for the formal statement that $p : \omega \to k$ is an ordered surjective function. Similar comments apply to many of the sets defined below. In RCA_0, we define a k-partition as follows.

Definition 2.1. Let $k \leq \omega$. A *k-partition of* ω is a function $p \in (\omega)^k$. If $k < \omega$, we say that p is a *finite partition* and if $k = \omega$, we say that p is an *infinite partition*.

For finite k, $(\omega)^k_{\text{fin}}$ denotes the set of ordered $\sigma \in k^{<\omega}$ in which all $i < k$ appear. That is, $\sigma \in (\omega)^k_{\text{fin}}$ represents a k-partition of an initial segment of ω. The set $(\omega)^k$ inherits the subspace topology from k^ω with basic open sets of the form $[\sigma] \cap (\omega)^k$ for $\sigma \in k^{<\omega}$. The space $(\omega)^k$ is not compact since, for example, the collection of open sets $[0^n 1]$ for $n \geq 1$ cover $(\omega)^2$ but this collection has no finite subcover. However, if $\sigma \in (\omega)^k_{\text{fin}}$, then $[\sigma] \subseteq (\omega)^k$ and $[\sigma]$ is a compact clopen subset of $(\omega)^k$. To generate the topology on $(\omega)^k$, it suffices to restrict to the basic clopen sets of the form $[\sigma]$ with $\sigma \in (\omega)^k_{\text{fin}}$. Although the notation $[\sigma]$ is ambiguous about whether the ambient space is k^ω or $(\omega)^k$ (or ℓ^ω or $(\omega)^\ell$ for some $\ell > k$), the meaning will be clear from context.

Definition 2.2. Let $p \in (\omega)^\omega$ and $k \leq \omega$. We say q is a *k-coarsening of* p if $q \in (\omega)^k$ and for all $n, m \in \omega$, if $p(n) = p(m)$, then $q(n) = q(m)$. In other words, q is a k-partition and each p-block is contained in a q-block. We let $(p)^k$ denote the set of all k-coarsenings of p.

Similarly, for $\tau \in (\omega)^h_{\text{fin}}$ with $h \in \omega$, $\sigma \in (\omega)^k_{\text{fin}}$ is a *k-coarsening of* τ if $k \leq h$, $|\sigma| = |\tau|$ and for all $n, m < |\tau|$ such that $\tau(n) = \tau(m)$, we have $\sigma(n) = \sigma(m)$.

As with $(\omega)^k$, $(p)^k$ inherits the subspace topology from k^ω. For $k < \omega$, we let $(p)^k_{\text{fin}}$ denote the set of all $\sigma \in (\omega)^k_{\text{fin}}$ which are coarsenings of $p \upharpoonright \mu^p(n)$ for some $n \geq k$. The topology on $(p)^k$ is generated by $[\sigma]$ for $\sigma \in (p)^k_{\text{fin}}$.

Coarsenings have a natural composition operation. Let $p \in (\omega)^\omega$, $k \in \omega$ and $r \in (\omega)^k$. Viewing p and r as functions, the composition $r \circ p : \omega \to k$ is an ordered surjective map with $r \circ p \in (p)^k$. Intuitively, the partition coded by $r \circ p$ uses r to describe how to combine the p-blocks. If $r(m) = r(n) = i$, then the p-blocks B^p_n

and B^p_m will be contained in the $(r \circ p)$-block $B^{r \circ p}_i$. The map from $(\omega)^k$ to $(p)^k$ defined by $r \mapsto r \circ p$ gives a canonical homeomorphism between these sets.

We can also compose elements of $(\omega)^k_{\text{fin}}$ and $(\omega)^h_{\text{fin}}$ under the right conditions. Let $k < h$, $\sigma \in (\omega)^k_{\text{fin}}$, and $\tau \in (\omega)^h_{\text{fin}}$. If $|\sigma| \geq h$, then $\sigma \circ \tau : |\tau| \to k$ is ordered. Moreover, if $|\sigma| \geq h > \mu^\sigma(k-1)$, then $\sigma \circ \tau \in (\omega)^k_{\text{fin}}$ because the least element of the last σ-block appears in the range of τ and thus $\sigma \circ \tau$ is onto k.

We will use this compositional structure in two ways. First, we will use effective versions of the following lemma which states that an open coloring $\cup_{i<\ell} O_i = (p)^k$ of the k-coarsenings of a fixed partition $p \in (\omega)^\omega$ can be turned into a coloring $\cup_{i<\ell} \widehat{O}_i = (\omega)^k$ of the k-coarsenings of ω such that the set of all k-coarsenings of $q \in (\omega)^\omega$ are contained in $\widehat{O}_i$ if and only if the set of all k-coarsenings of $q \circ p$ are contained in O_i.

Lemma 2.3. *Fix $p \in (\omega)^\omega$ and $k, \ell \in \omega$. Let O_i, $i < \ell$, be open sets in $(p)^k$ such that $(p)^k = \cup_{i<\ell} O_i$. There are open sets $\widehat{O}_i$, $i < \ell$, in $(\omega)^k$ such that $\cup_{i<\ell} \widehat{O}_i = (\omega)^k$ and for any $q \in (\omega)^\omega$ and $i < \ell$, $(q)^k \subseteq \widehat{O}_i$ if and only if $(q \circ p)^k \subseteq O_i$.*

Proof. This follows from the continuity of the canonical homeomorphism $\phi : (\omega)^k \to (p)^k$, where $\phi(r) = r \circ p$. Letting $\widehat{O}_i = \phi^{-1}(O_i)$, it is straightforward to check that these are as required. $\square$

Later we will need to use the fact that this lemma holds in RCA_0. For that purpose it is useful to express O_i as a union of basic open sets and describe the inverse image of each. Let $S_i \subseteq (p)^k_{\text{fin}}$ be such that $O_i = \bigcup_{\sigma \in S_i} [\sigma]$. For each $\sigma \in (p)^k_{\text{fin}}$, let n be such that σ is a coarsening of $p \upharpoonright \mu^p(n)$ and define $\tau_\sigma \in (\omega)^k_{\text{fin}}$ to be the string such that $|\tau_\sigma| = n$ and $\tau_\sigma(i) = \sigma(\mu^p(i))$ for all $i < n$. This definition ensures that $\tau_\sigma \circ p = \sigma$. We have for any $r \in (\omega)^k$, $r \in [\tau_\sigma]$ if and only if $r \circ p \in [\sigma]$. Therefore, $\phi^{-1}([\sigma]) = [\tau_\sigma]$ and $\widehat{O}_i = \bigcup_{\sigma \in S_i} [\tau_\sigma]$.

Second, we will use the compositional structure to describe the k-coarsenings of a given $\tau \in (\omega)^{s+1}_{\text{fin}}$.

Lemma 2.4. *Let $s \in \omega$ and $\tau \in (\omega)^{s+1}_{fin}$. For $k \leq s+1$, the k-coarsenings of τ are exactly the strings $\sigma \circ \tau$ where $\sigma \in (\omega)^k_{fin}$ with $|\sigma| = s+1$.*

Proof. Fix s, τ and $k \leq s+1$. Let $\sigma \in (\omega)^k_{\text{fin}}$ be such that $|\sigma| = s+1$. Because $\tau \in (\omega)^{s+1}_{\text{fin}}$ and $|\sigma| = s+1 > \mu^\sigma(k-1)$, we have $\sigma \circ \tau \in (\omega)^k_{\text{fin}}$ by the comments preceding Lemma 2.3. Therefore, $\sigma \circ \tau$ is a k-coarsening of τ.

Conversely, let τ' be a k-coarsening of τ. Define $\sigma \in (\omega)^k_{\text{fin}}$ with $|\sigma| = s+1$ by $\sigma(i) = \tau'(\mu^\tau(i))$ for all $i < s+1$. By calculations similar to those in the proof of Lemma 2.3, we have that for all $n < |\tau| = |\tau'|$, $\sigma(\tau(n)) = \tau'(n)$ as required. $\square$

3. The Dual Ramsey Theorem

3.1. **Four versions of the Dual Ramsey Theorem.**

We formulate four versions of the Dual Ramsey Theorem in second order arithmetic and examine how they are related in reverse mathematics.

Definition 3.1 (RCA_0)**.** A *code for an open set in* $(\omega)^k$ is a set $O \subseteq \omega \times (\omega)^k_{\text{fin}}$. We say that a partition $p \in (\omega)^k$ is *in the open set coded by* O (or just *in* O and write $p \in O$) if there is a pair $\langle n, \sigma \rangle \in O$ such that $p \in [\sigma]$.

A *code for an closed set in* $(\omega)^k$ is also a set $V \subseteq \omega \times (\omega)^k_{\text{fin}}$. In this case, we say $p \in (\omega)^k$ *is in* V (and write $p \in V$) if for all pairs $\langle n, \sigma \rangle \in V$, $p \notin [\sigma]$.

We code a sequence of open sets $\{O_i\}_{i<\omega}$ by $O \subseteq \omega \times \omega \times (\omega)^k_{\text{fin}}$ with $p \in O_i$ if there is a triple $\langle i, n, \sigma \rangle \in O$ such that $p \in [\sigma]$. The codes in Definition 3.1 can be generalized to Borel sets which we describe in Section 3.2, although we delay the formal definition until Section 6 when we prove the results about Borel codes.

It will be useful to consider not only open colorings of $(\omega)^k$ but also open colorings of $(p)^k$ for $p \in (\omega)^\omega$. Modifying Definition 3.1, a code for an open set in $(p)^k$ is a set $O \subseteq \omega \times (p)^k_{\text{fin}}$ and we write $x \in O$ if there is a pair $\langle n, \sigma \rangle \in O$ such that $\sigma \prec x$. With these definitions, the proof of Lemma 2.3 goes through in RCA_0 with the modification that $\widehat{O} = \{\langle n, \tau_\sigma \rangle : \langle n, \sigma \rangle \in O\}$.

Coding colorings or sets with the Baire property in second order arithmetic is complicated by the fact that there are $2^{\mathfrak{c}}$ (where $\mathfrak{c} = 2^{\aleph_0}$) many subsets of $(\omega)^k$ or k^ω with the Baire property. Our definition for Baire codes (given below) is motivated by considering how facts about sets with the Baire property are typically proved.

Definition 3.2 (RCA_0)**.** An open set $O \subseteq (\omega)^k$ is *dense* if for all $\tau \in (\omega)^k_{\text{fin}}$, $[\tau] \cap O \neq \emptyset$. That is, for all τ, there is a pair $\langle n, \sigma \rangle \in O$ such that σ and τ are comparable as strings.

RCA_0 suffices to prove the Baire Category Theorem: if $\{D_n\}_{n<\omega}$ is a sequence of dense open sets, then $\cap_{n<\omega} D_n$ is dense. Classically, if a coloring $\cup_{i<\ell} C_i = (\omega)^k$ has the Baire property, then it has a comeager approximation given by sequences of open sets $\{O_i\}_{i<\ell}$ and $\{D_n\}_{n<\omega}$ such that each D_n is dense and for each $p \in \cap_{n<\omega} D_n$, $p \in C_i$ if and only if $p \in O_i$. The fact that $\cup_{i<\ell} C_i = (\omega)^k$ implies that $\cup_{i<\ell} O_i$ is dense. Often, a classical proof about colorings or sets with the Baire property will start by fixing a comeager approximation and will proceed by working exclusively with this approximation. This classical observation motivates our definition of a code for a Baire coloring.

Definition 3.3 (RCA_0)**.** A *code for a Baire ℓ-coloring of* $(\omega)^k$ is a sequence of dense open sets $\{D_n\}_{n<\omega}$ together with a sequence of open sets $\{O_i\}_{i<\ell}$ such that $\bigcup_{i<\ell} O_i$ is dense in $(\omega)^k$.

In Definition 3.3, the code consists of a comeager approximation to the intended coloring and thus avoids the difficulties of explicitly describing the coloring in second order arithmetic. Note that if we define (classically) $C_i = O_i \cap \bigcap_{n<\omega} D_n$, then $\cup_{i<\ell} C_i$ will differ from $(\omega)^k$ on a meager set. Thus, a single code for a Baire coloring will represent many different classical colorings, each of which admits the same comeager approximation.

We abuse terminology and refer to the Baire code as a *Baire ℓ-coloring* of $(\omega)^k$. Similarly, an *open* (or *Borel*) *ℓ-coloring* is a coloring $(\omega)^k = \cup_{i<\ell} C_i$ in which each C_i is given by an open (or Borel, respectively) code.

Definition 3.4. For each (standard) $k, \ell \geq 2$, we define $\mathsf{Borel\text{-}DRT}^k_\ell$, $\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ and CDRT^k_ℓ in RCA_0.

(1) $\mathsf{Borel\text{-}DRT}^k_\ell$: For every Borel ℓ-coloring $(\omega)^k = \cup_{i<\ell} C_i$, there is a partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for all $x \in (p)^k$, $x \in C_i$.
(2) $\mathsf{Baire\text{-}DRT}^k_\ell$: For every Baire ℓ-coloring $\{O_i\}_{i<\ell}$ and $\{D_n\}_{n<\omega}$ of $(\omega)^k$, there is a partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for all $x \in (p)^k$, $x \in O_i \cap \bigcap_n D_n$.
(3) ODRT^k_ℓ: For every open ℓ-coloring $(\omega)^k = \cup_{i<\ell} O_i$, there is a partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for all $x \in (p)^k$, $x \in O_i$.

(4) CDRT^k_ℓ: For every coloring $(\omega)^{k-1}_{\text{fin}} = \cup_{i<\ell} C_i$, there is a partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for all $x \in (p)^k$, $x \restriction \mu^x(k-1) \in C_i$.

In the statement of CDRT^k_ℓ, each color $C_i \subseteq (\omega)^{k-1}_{\text{fin}}$ is a set of strings in second order arithmetic and there is no assumption that the colors are disjoint. However, in RCA_0, given a coloring $(\omega)^{k-1}_{\text{fin}} = \cup_{i<\ell} C_i$, we can define a partition $(\omega)^{k-1}_{\text{fin}} = \cup_{i<\ell} \widehat{C}_i$ where $\sigma \in \widehat{C}_i$ if and only if i is the least index such that $\sigma \in C_i$. Thus, it is equivalent over RCA_0 to state CDRT^k_ℓ in terms of colorings given as functions $c : (\omega)^k_{\text{fin}} \to \ell$. Similarly, in RCA_0, we can replace open sets O_i, $i < \ell$, by pairwise disjoint open sets $\widehat{O}_i$ such that $\widehat{O}_i \subseteq O_i$ and $\cup_{i<\ell} O_i = \cup_{i<\ell} \widehat{O}_i$. Therefore, in RCA_0, we can assume without loss of generality that the individual colors in $\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ and CDRT^k_ℓ are pairwise disjoint.

Our first goal is to show that the instances of CDRT^k_ℓ are in one-to-one canonical correspondence with those instances of ODRT^k_l for which the coloring of $(\omega)^k$ is *reduced*. We define a reduced coloring without considering the coding method and note that any reduced coloring is classically open. In RCA_0, we will use the notion of a reduced coloring only in the context of an open coloring.

Definition 3.5. Let $y \in (\omega)^\omega$ and $m < k$. A coloring of $(y)^k$ is *m-reduced* if whenever $p, q \in (y)^k$ and $p \restriction \mu^p(m) = q \restriction \mu^q(m)$, p and q have the same color. A coloring of $(y)^k$ is *reduced* if it is $(k-1)$-reduced.

Proposition 3.6 (RCA_0)**.** *The following are equivalent.*

(1) CDRT^k_ℓ.
(2) *For every $y \in (\omega)^\omega$ and open reduced coloring $(y)^k = \cup_{i<\ell} O_i$, there are $p \in (y)^\omega$ and $i < \ell$ such that for all $x \in (p)^k$, $x \in O_i$.*

Proof. Assume (2) and fix $c : (\omega)^{k-1}_{\text{fin}} \to \ell$. Let $y \in (\omega)^\omega$ be the trivial partition with blocks $\{0\}, \{1\}, \dots$ and note that $(y)^k = (\omega)^k$. For each $i < \ell$, let

$$O_i = \{\langle 0, \sigma^\frown(k-1)\rangle : \sigma \in (\omega)^{k-1}_{\text{fin}} \text{ and } c(\sigma) = i\}$$

be an open code for the union of clopen sets $[\sigma^\frown(k-1)]$ such that $c(\sigma) = i$. $(\omega)^k = \cup_{i<\ell} O_i$ is an open reduced coloring of $(\omega)^k$ such that any $p \in (\omega)^\omega$ which is homogeneous for this coloring is also homogeneous for c.

For the other direction, assume CDRT^k_ℓ. Fix $y \in (\omega)^\omega$ and a reduced open coloring $(y)^k = \cup_{i<\ell} O_i$. By Lemma 2.3, let $\cup_{i<\ell} \widehat{O}_i = (\omega)^k$ be an open coloring such that for any $q \in (\omega)^\omega$, $(q)^k \subseteq \widehat{O}_i$ if and only if $(q \circ y)^k \subseteq O_i$. It is straightforward to check that the coloring $(\omega)^k = \cup_{i<\ell} \widehat{O}_i$ is also reduced.

We claim that for each $\sigma \in (\omega)^{k-1}_{\text{fin}}$, there is a triple $\langle n, \tau, i\rangle \in \omega \times (\omega)^k_{\text{fin}} \times \ell$ such that $\langle n, \tau\rangle \in \widehat{O}_i$ and $\sigma^\frown(k-1) \preceq \tau$. To see why, let $p \in (\omega)^k$ be any partition extending $\sigma^\frown(k-1)$. Because $(\omega)^k = \cup_{i<\ell} \widehat{O}_i$, there is a color $i < \ell$ and a pair $\langle n, \tau\rangle \in \widehat{O}_i$ such that $\tau \prec p$. Since $\tau \in (\omega)^k_{\text{fin}}$, we have $\sigma^\frown(k-1) \preceq \tau$, proving the existence of the triple $\langle n, \tau, i\rangle$. Because the coloring $\{\widehat{O}_i\}_{i<\ell}$ is reduced, for any $x \in (\omega)^k$, if $x \restriction \mu^x(k-1) = \sigma = p \restriction \mu^p(k-1)$, then $x \in \widehat{O}_i$ as well.

For each $\sigma \in (\omega)^{k-1}_{\text{fin}}$, let $\langle n_\sigma, \tau_\sigma, i_\sigma\rangle$ be the least triple satisfying the conditions in the previous paragraph. Define $c : (\omega)^{k-1}_{\text{fin}} \to \ell$ by $c(\sigma) = i_\sigma$. It follows that for each $x \in (\omega)^k$ and each $i < \ell$, if $c(x \restriction \mu(k-1)) = i$, then $x \in \widehat{O}_i$. Applying CDRT^k_ℓ to c, there are $i < \ell$ and $q \in (\omega)^\omega$ such that for all $x \in (q)^k$, $c(x \restriction \mu(k-1)) = i$. Therefore, $(q)^k \subseteq \widehat{O}_i$. Setting $p = q \circ y \in (y)^\omega$, we have $(p)^k = (q \circ y)^k \subseteq O_i$. $\square$

It is now routine to show that the number of colors does not matter.

Proposition 3.7 (RCA_0). *CDRT^k_ℓ and CDRT^k_2 are equivalent.*

Proof. Collapse colors and iterate CDRT^k_2 finitely many times using the canonical correspondence in Lemma 2.3. $\square$

Lemma 3.8 (RCA_0). *For any Baire ℓ-coloring of $(\omega)^k$ given by $\{O\}_{i<\ell}$ and $\{D_n\}_{n\in\omega}$, there is a function $f : (\omega)^k_{\text{fin}} \times \omega \to (\omega)^k_{\text{fin}} \times \ell$ such that for all $\tau \in (\omega)^k_{\text{fin}}$ and $s \in \omega$, $f(\tau, s) = \langle \delta, i\rangle$ where $\tau \prec \delta$ and $[\delta] \subseteq O_i \cap \bigcap_{n<s} D_n$.*

Proof. The function f is defined in a straightforward way by primitive recursion and minimization using the density of each D_n and $\cup_{i<\ell} O_i$. $\square$

The next proof is essentially an effective version of an argument in [13].

Theorem 3.9 (RCA_0). *$\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ and CDRT^k_ℓ are equivalent.*

Proof. By setting $D_n = (\omega)^k$ in $\mathsf{Baire\text{-}DRT}^k_\ell$, ODRT^k_ℓ is a special case of $\mathsf{Baire\text{-}DRT}^k_\ell$, and by Proposition 3.6, CDRT^k_ℓ is a special case of ODRT^k_ℓ. It remains to prove in RCA_0 that CDRT^k_ℓ implies $\mathsf{Baire\text{-}DRT}^k_\ell$.

Let $\{O_i\}_{i<\ell}$, $\{D_n\}_{n<\omega}$ be a Baire ℓ-coloring of $(\omega)^k$ for which the open sets O_i are pairwise disjoint. We construct a partition $y \in (\omega)^\omega$ such that $(y)^k \subseteq \cap_n D_n$ and $\cup_i O_i$ restricted to $(y)^k$ is reduced. By Proposition 3.6 and CDRT^k_ℓ, there is a homogeneous $z \in (y)^\omega$ for this open reduced coloring. Since $(z)^k \subseteq (y)^k \subseteq \cap_n D_n$, this partition z is homogeneous for the original Baire coloring.

To build $y \in (\omega)^\omega$, we construct a sequence $\tau_{k-1} \prec \tau_k \prec \cdots$ of strings $\tau_s \in (\omega)^s_{\text{fin}}$ in stages starting with τ_{k-1} for notational convenience and set $y = \cup_{s\geq k-1}\tau_s$. We define τ_{k-1} by $|\tau_{k-1}| = k-1$ and $\tau_{k-1}(n) = n$ for $n < k-1$, so τ_{k-1} corresponds to the trivial partition $\{0\}, \{1\}, \ldots, \{k-1\}$ of k.

At stage $s+1$, assume we have defined $\tau_s \in (\omega)^s_{\text{fin}}$ with $|\tau_s| = m_s$. We extend τ_s finitely many times to obtain τ_{s+1}. For the initial extension, let $\tau^0_s = \tau_s{}^\frown s \in (\omega)^{s+1}_{\text{fin}}$ which ensures that $\mu^x(s) = m_s$.

For the remaining extensions, consider the ways to coarsen a finite partition with $(s+1)$ many blocks down to a partition with k many blocks. By Lemma 2.4, these coarsenings correspond to composing with strings $\sigma \in (\omega)^k_{\text{fin}}$ of length $s+1$. Let M_s denote the number of $\sigma \in (\omega)^k_{\text{fin}}$ with $|\sigma| = s+1$ and let $\sigma^0_s, \ldots, \sigma^{M_s-1}_s$ list these strings. We define extensions τ^j_s for $1 \leq j \leq M_s$ (with τ^0_s defined above) and set $\tau_{s+1} = \tau^{M_s}_s$.

Assume τ^j_s has been defined. By Lemma 3.8, let δ^j_s and i^j_s be such that $\sigma^j_s \circ \tau^j_s \prec \delta^j_s$ and $[\delta^j_s] \subseteq O_{i^j_s} \cap \bigcap_{n<s+1} D_n$. That is, we coarsen τ^j_s by the j-th canonical way to coarsen the $(s+1)$ many blocks to k blocks and then we take an extension of this coarsening that lies in $O_{i^j_s} \cap \bigcap_{n<s+1} D_n$ for some $i^j_s < \ell$.

To define τ^{j+1}_s, we want to "uncollapse" δ^j_s by reversing the coarsening done to τ^j_s by σ^j_s. Define τ^{j+1}_s with $|\tau^{j+1}_s| = |\delta^j_s|$ by considering each $n < |\delta^j_s|$. If $n < |\tau^j_s|$ set $\tau^{j+1}_s(n) = \tau^j_s(n)$, guaranteeing that $\tau^j_s \prec \tau^{j+1}_s$. For $n \geq |\tau^j_s|$, let $\tau^{j+1}_s(n) = \tau^j_s(m)$ where m is least such that $\delta^j_s(m) = \delta^j_s(n)$. That is, $m = \mu^{\delta^j_s}(\delta^j_s(n))$. (Below we show that $m \leq m_s$ and $\sigma^j_s \circ \tau^{j+1}_s = \delta^j_s$.) This completes the definition of τ_{s+1} and hence the construction of y in RCA_0 since the initial segments of y are defined by primitive recursion using the function f from Lemma 3.8.

Claim. In the definition of $\tau^{j+1}_s(n)$ when $n \geq |\tau^j_s|$, the number m satisfies $m \leq m_s$.

Since $\tau_s^0 \preceq \tau_s^j$, if m is the least element of a τ_s^j-block, then $m \leq m_s$. Because $\sigma_j \circ \tau_s^j$ is a coarsening of τ_s^j, it follows that if m is the least element of a $(\sigma_j \circ \tau_s^j)$-block, then $m \leq m_s$. Therefore, since $\sigma_s^j \circ \tau_s^j \prec \delta_s^j$ and m is the least element of a δ_s^j-block, we have $m \leq m_s$.

Claim. For all $s \geq k$ and $j < M_s$, $\sigma_s^j \circ \tau_s^{j+1} = \delta_s^j$.

We show that for all $n < |\delta_s^j|$, $\sigma_s^j(\tau_s^{j+1}(n)) = \delta_s^j(n)$. For $n < |\tau_s^j|$,

$$\sigma_s^j(\tau_s^{j+1}(n)) = \sigma_s^j(\tau_s^j(n)) = \delta_s^j(n)$$

where the last equality holds because $\sigma_s^j \circ \tau_s^j \prec \delta_s^j$. For $n \geq |\tau_s^j|$, $\tau_s^{j+1}(n) = \tau_s^j(m)$ for an $m \leq m_s$ such that $\delta_s^j(m) = \delta_s^j(n)$. Therefore,

$$\sigma_s^j(\tau_s^{j+1}(n)) = \sigma_s^j(\tau_s^j(m)) = \delta_s^j(m) = \delta_s^j(n)$$

where the second equality holds because $m \leq m_s < |\tau_s^j|$ and $\sigma_s^j \circ \tau_s^j \prec \delta_s^j$.

Claim. $(y)^k \subseteq \cup_{i<\ell} O_i$ and the coloring $\cup_{i<\ell} O_i$ restricted to $(y)^k$ is reduced.

Let $x \in (y)^k$. We need to show that there is an i such that $x \in O_i$. (Since the open sets O_i are pairwise disjoint, there is at most one such index i.) Furthermore, for any $x' \in (y)^k$ with $x' \upharpoonright \mu^{x'}(k-1) = x \upharpoonright \mu^x(k-1)$, we need to show that $x' \in O_i$.

Let $m = \mu^x(k-1)$. Since m is the least element of an x-block, it must be the least element of a y-block. Fix s such that $m = \mu^y(s)$ and note that $s \geq k-1$. In the notation of the construction, $m = m_s$ and $y(m) = s$ was first decided at stage $s+1$. In particular, $\tau_s^0 = \tau_s^\frown s$ is the initial segment of y ending with the least element of B_s^y. Similarly, $x \upharpoonright (\mu^x(k-1)+1)$ is the initial segment of x ending with the least element of B_{k-1}^x.

Since B_s^y is the least y-block collapsed into B_{k-1}^x, we can fix the index $j < M_s$ from stage $s+1$ such that $\sigma_s^j \in (\omega)^k_{\text{fin}}$ with $|\sigma_s^j| = s+1$ satisfies $\sigma_s^j \circ \tau_s^0 = x \upharpoonright (\mu^x(k-1)+1)$. That is, σ_s^j described how x collapses the first $(s+1)$-many blocks of y into the k-many blocks of x. For any initial segment γ of y with $\tau_s^0 \preceq \gamma$ (so γ contains elements from each of the first $(s+1)$-many y-blocks) and with $\gamma \preceq \tau_{s+1}$ (so γ only containing elements from the first $(s+1)$-many y-blocks), $\sigma_s^j \circ \gamma$ will collapse the $(s+1)$-many γ-blocks as x specifies and so will satisfy $\sigma_s^j \circ \gamma \prec x$. In particular, since $\tau_s^0 \preceq \tau_s^{j+1} \prec y$ and $|\tau_s^{j+1}| \leq |\tau_{s+1}| = \mu^y(s+1)$, we have $\sigma_s^j \circ \tau_s^{j+1} = \delta_s^j \prec x$. However, $[\delta_s^j] \subseteq O_{i_s^j}$ by construction, so $x \in O_{i_s^j}$. If $x' \in (y)^k$ satisfies $x' \upharpoonright \mu^{x'}(k-1) = x \upharpoonright \mu^x(k-1)$, then x' determines the same string $\sigma_s^j \in (\omega)^k_{\text{fin}}$ and hence $x' \in O_{i_s^j}$ as well. This completes the proof of the claim.

Finally, we show that if $x \in (y)^k$, then $x \in \cap_{n<\omega} D_n$. Fix x and let m and s be as in the proof of the previous claim. Since $\delta_s^j \prec x$ and $[\delta_s^j] \subseteq \cap_{n<s+1} D_n$, we have $x \in \cap_{n<s+1} D_n$. The following claim will complete the proof that $x \in \cap_{n<\omega} D_n$.

Claim. For all $t > s$, by the end of stage $t+1$, it is forced that $x \in D_t$.

The proof of this claim is almost identical to the proof that the coloring $\cup_{i<\ell} O_i$ restricted to $(y)^k$ is reduced. Fix a stage $t+1$ for $t > s$. The string $\tau_t^0 = \tau_t^\frown t$ is the initial segment of y ending with the least element of B_t^y. Because $t > s$, x coarsens τ_t^0 down to k-many blocks. Fix the index j for the string $\sigma_t^j \in (\omega)^k_{\text{fin}}$ with $|\sigma_t^j| = t+1$ that describes this coarsening, so $\sigma_t^j \circ \tau_t^0 = x \upharpoonright (m_t + 1)$. As before, $\sigma_t^j \circ \tau_t^{j+1} = \delta_t^j \prec x$. Since $[\delta_t^j] \subseteq \cap_{n<t+1} D_n$, we have $x \in D_t$ completing the proof of the claim and the fact that $x \in \cap_{n<\omega} D_n$. Because $(y)^k \subseteq \cap_n D_n$ and $\cup_{i<\ell} O_i$

restricted to $(y)^k$ is an open reduced coloring, we have also completed the proof that CDRT^k_ℓ implies $\mathsf{Baire\text{-}DRT}^k_\ell$. $\square$

Since ODRT^{k+1}_ℓ implies RT^k_ℓ over RCA_0, we have the following corollary.

Corollary 3.10 (RCA_0)**.** CDRT^{k+1}_ℓ *implies* RT^k_ℓ.

Proposition 3.11. *For any $\ell \geq 2$, RCA_0 proves CDRT^2_ℓ and hence also ODRT^2_ℓ.*

Proof. Let $c : (\omega)^1_{\text{fin}} \to \ell$. Since $(\omega)^1_{\text{fin}} = \{0^n : n \geq 1\}$, c can be viewed as an ℓ-coloring of ω. By RT^1_ℓ, there is a color i and an infinite set X such that for every $n \in X$, $c(0^n) = i$. Let z be the partition which has a block of the form $\{n\}$ for each $n \in X$ and puts all the other numbers in B^z_0. Then z is homogeneous for c. $\square$

3.2. **Relationships with $\mathsf{Borel\text{-}DRT}^k_\ell$.** We give the formal definition of a Borel code in Section 6. Informally, a Borel code B for a subset of $(\omega)^k$ is a well founded tree in $\omega^{<\omega}$ in which each leaf codes a clopen set and the interior nodes code either an intersection or a union. Given a point $x \in (\omega)^k$, an evaluation map for B at x is a function $f : B \to \{0,1\}$ such that $f(\sigma) = 1$ for a leaf σ if x is in the clopen set coded by σ and f correctly propagates down the tree respecting unions and intersections. We say $x \in B$ if there is an evaluation function with value 1 at the root, and we say $x \notin B$ if there is an evaluation function with value 0 at the root. Therefore, both $x \in B$ and $x \notin B$ are Σ^1_1 statements, and in general, ATR_0 is required to show that evaluation maps exists. Similarly, $(\omega)^k = B_0 \cup \ldots \cup B_{\ell-1}$ is the Π^1_2 statement that for every $x \in (\omega)^k$ and $i < \ell$, there is an evaluation map for B_i at x and for some $i < \ell$, $x \in B_i$.

Proposition 3.12 (RCA_0)**.** $\mathsf{Borel\text{-}DRT}^k_\ell$ *implies* $\mathsf{Baire\text{-}DRT}^k_\ell$.

Proof. In Section 6, we show in RCA_0 that if $(\omega)^k = O_0 \cup \cdots \cup O_{\ell-1}$ is an open coloring, then each O_i has an equivalent Borel code B_i such that $(\omega)^k = B_0 \cup \cdots \cup B_{\ell-1}$. It follows that $\mathsf{Borel\text{-}DRT}^k_\ell$ implies ODRT^k_ℓ and hence implies $\mathsf{Baire\text{-}DRT}^k_\ell$. $\square$

Definition 3.13 (RCA_0)**.** Let B be a Borel (or open or closed) code for subset of $(\omega)^k$. A *Baire code for B* consists of open sets U and V and a sequence $\langle D_n : n \in \omega \rangle$ of dense open sets such that $U \cup V$ is dense and for every $x \in \cap_{n\in\omega} D_n$, if $p \in U$ then $p \in B$ and if $p \in V$ then $p \notin B$.

In Section 6, we will prove the following theorem.

Theorem 3.14 (RCA_0)**.** *The following are equivalent.*

1. ATR_0.
2. *For every Borel code B for a subset of $(\omega)^k$, there is an $x \in (\omega)^k$ such that $x \in B$ or $x \notin B$.*
3. *Every Borel code B for a subset of $(\omega)^k$ has a Baire code.*

Definition 3.15 (RCA_0)**.** A *Baire code* for a Borel coloring $(\omega)^k = C_0 \cup \cdots \cup C_{\ell-1}$ consists of open sets O_i, $i < \ell$, and a sequence $\langle D_n : n \in \omega \rangle$ of dense open sets such that $\cup_{i<\ell} O_i$ is dense and for every $p \in \cap_{n\in\omega} D_n$ and $i < \ell$, if $p \in O_i$ then $p \in C_i$.

Proposition 3.16 (ATR_0)**.** *Every Borel coloring $(\omega)^k = C_0 \cup \cdots \cup C_{\ell-1}$ has a Baire code.*

Proof. By Theorem 3.14, fix Baire codes U_i, V_i and $D_{n,i}$ for each C_i. We claim that the open sets U_i for $i < \ell$ and the sequence of dense open sets $D_{n,i}$ for $i < \ell$ and $n < \omega$ form a Baire code for this coloring. Note that if $i < \ell$ and $x \in \cap_{n,i} D_{n,i}$, then $x \in U_i$ implies $x \in C_i$. Therefore, it suffices to show that $\cup_{i<\ell} U_i$ is dense.

Suppose not. Then there is τ such that $[\tau] \cap U_i = \emptyset$ for all i. Because each set $U_i \cup V_i$ is open and dense, by the Baire Category Theorem there is $x \in [\tau]$ such that $x \in \cap_{n\in\omega, i<\ell} D_{n,i}$ and $x \in \cap_{i<\ell}(U_i \cup V_i)$. Since x is not in any U_i, we have $x \in V_i$ for each i. Therefore, for each i, $x \notin C_i$, contradicting that $(\omega)^k = C_0 \cup \cdots \cup C_{\ell-1}$. □

Proposition 3.17 (ATR_0)**.** $\mathsf{Baire\text{-}DRT}^k_\ell$ *implies* $\mathsf{Borel\text{-}DRT}^k_\ell$*.*

Proof. By Proposition 3.16, each Borel coloring has a Baire code. $\mathsf{Baire\text{-}DRT}^k_\ell$ guarantees a homogeneous partition for the coloring given by this Baire code and this partition is homogeneous for the Borel coloring. □

3.3. **Alternate coding methods for two complementary colors.** While we believe our formal statement of the Borel Dual Ramsey Theorem is the most natural, there is an alternate formal version of this theorem for colorings (classically) of the form $(\omega)^k = B \cup \overline{B}$ that avoids explicitly stating that every partition $p \in (\omega)^k$ has an evaluation map. The material in this subsection is somewhat tangential to our main story and nothing from it is used later in the paper.

Theorem 3.18 (RCA_0)**.** *The statement "for every Borel code B for a subset of $(\omega)^k$, there is a partition $p \in (\omega)^\omega$ such that either $\forall x \in (p)^k (x \in B)$ or $\forall x \in (p)^k (x \notin B)$" implies* ATR_0.

Proof. Fix a Borel code B. The given statement implies there is an $x \in (\omega)^k$ such that $x \in B$ or $x \notin B$. (Let x be any coarsening of p down to k-blocks.) By Theorem 3.14, this suffices to prove ATR_0. □

Note that this argument does not suffice to prove an implication from $\mathsf{Borel\text{-}DRT}^k_2$ to ATR_0 because the hypotheses of $\mathsf{Borel\text{-}DRT}^k_2$ include that $(\omega)^k = C_0 \cup C_1$ which requires the existence of an evaluation map for every $x \in (\omega)^k$ witnessing $x \in C_0$ or $x \in C_1$.

There is an analogous variant of the Dual Ramsey Theorem for colorings $(\omega)^k = O \cup \overline{O}$ where O is a code for an open set. Here, $\overline{O} = O$ as sets but $\overline{O}$ is viewed as a code for a closed set. That is, for $p \in (\omega)^k$, $p \in \overline{O}$ if for every $\langle s, \sigma\rangle \in O$, $\sigma \not\prec p$. Therefore $\overline{O}$ is a code for the complement of O. (Note that this version differs significantly from ODRT^k_2 because one of the colors is closed.)

For $0 < a < b$, let $O_{a,b} = \{x \in (\omega)^3 : \mu^x(1) = a \wedge \mu^x(2) = b\}$. $O_{a,b}$ is a finite union of basic open sets $[\sigma]$ with $\sigma \in (\omega)^3_{\text{fin}}$ and $|\sigma| = b+1$. For notational convenience, we write $\sigma \in O_{a,b}$ if $|\sigma| = b+1$, $\mu^\sigma(1) = a$ and $\mu^\sigma(2) = b$.

The use of exponent (at least) 3 in the following theorem is important. In Theorem 5.1, we will see that RCA_0 suffices to prove that for every open set O in $(\omega)^2$, there is an infinite homogeneous partition for $(\omega)^2 = O \cup \overline{O}$.

Theorem 3.19 (RCA_0)**.** *The statement "for every open set O in $(\omega)^3$, there is an infinite homogeneous partition for the coloring $(\omega)^3 = O \cup \overline{O}$" implies* ACA_0.

Proof. Fix a 1-to-1 function g and we show the range of g exists. Define O by

$$\langle s, \sigma\rangle \in O \Leftrightarrow \exists 0 < a < b < s \left(\sigma \in O_{a,b} \wedge \exists u \le a \, \exists b < t \le s \, (g(t) = u)\right).$$

Note that for all $0 < a < b$, either $O_{a,b} \subseteq O$ (when $\exists u \leq a\, \exists t > b(g(t) = u)$) or $O_{a,b} \subseteq \overline{O}$ (when $\forall u \leq a \neg\exists t > b(g(t) = u)$). Let $p \in (\omega)^\omega$ be a homogeneous partition for $(\omega)^3 = O \cup \overline{O}$. We claim that p is homogeneous for $\overline{O}$.

We prove the claim by constructing $x \in (p)^3 \cap \overline{O}$. Let $n = g(0)$, let m_1 be such that $n < \mu^p(m_1)$ and let $a = \mu^p(m_1)$. Since g is 1-to-1, the set $C = \{t : g(t) \leq a\}$ is finite. Let $m_2 > m_1$ be such that $\mu^p(m_2) > \max(C)$ and let $b = \mu^p(m_2)$. Define $x \in (p)^3$ by $B^x_0 = \cup_{i<m_1} B^p_i$, $B^x_1 = \cup_{m_1 \leq i < m_2} B^p_i$ and $B^x_2 = \cup_{i \geq m_2} B^p_i$. By definition, $\mu^x(1) = a$ and $\mu^x(2) = b$, so $x \in O_{a,b}$. By our choice of a and b, $\forall u \leq a\, \neg\exists t \geq b\, (g(t) = u)$ and therefore $x \in O_{a,b} \subseteq \overline{O}$ as required.

Since the function μ^p is strictly increasing, we can define the function $f(n) =$ the least m such that $\mu^p(m) > n$. We claim that $n \in \text{range}(g)$ if and only if $\exists t \leq \mu^p(f(n) + 1)\, (g(t) = n)$. Suppose $n = g(t)$ but $t > \mu^p(f(n) + 1)$. In this case, $\langle t, O_{\mu^p(f(n)),\mu^p(f(n)+1)}\rangle \in O$. But, collapsing p as above, there is an $x \in (p)^3$ such that $\mu^x(1) = \mu^p(f(n))$ and $\mu^x(2) = \mu^p(f(n) + 1)$ and hence $x \in O$ contradicting the fact that p is homogeneous for $\overline{O}$. □

We have seen that obtaining Baire codes for Borel colorings codes a significant amount of information. The next theorem shows that even obtaining Baire codes for open colorings codes a non-trivial amount of information.

Theorem 3.20 (RCA_0)**.** *The following are equivalent.*

(1) ACA_0.
(2) *Every closed subset of* $(\omega)^k$ *has a Baire code.*
(3) *Every open subset of* $(\omega)^k$ *has a Baire code.*

Proof. (2) and (3) are equivalent by trading the roles of U and V in their respective Baire codes. To see (1) implies (2), fix a closed set C, so C is a set of pairs $\langle s, \sigma\rangle$ and $x \in C$ if for all $\langle s, \sigma\rangle \in C$, $\sigma \not\prec x$. To define a Baire code for C, set $V = C$ as sets, but view V as a code for the open set $\overline{C}$. That is, $x \in V$ if there is $\langle s, \sigma\rangle \in V = C$ such that $\sigma \prec x$. Let $U = \{\langle 0, \tau\rangle : \neg\exists\langle s, \sigma\rangle \in V\, (\sigma \preceq \tau \vee \tau \prec \sigma)\}$ and set $D_n = (\omega)^k$ for all $n \in \omega$. It is straightforward to check that $U \cup V$ is dense.

Suppose $x \in \cap_n D_n = (\omega)^k$. If $x \in V$, then by definition, $x \in \overline{C}$. On the other hand, suppose $x \in U$ and fix $\langle 0, \tau\rangle \in U$ with $\tau \prec x$. For every $\langle s, \sigma\rangle \in V = C$, τ is incomparable with σ and hence $\sigma \not\prec x$. Therefore, $x \in C$ as required.

We show (3) implies (1) for the case when $k = 2$. The proof is similar for other values of k. Fix a 1-to-1 function g. Let $O = \{\langle n, 0^{m+1}1\rangle : g(n) = m\}$. Let $U, V, \{D_n\}_{n\in\omega}$ be a Baire code for O. Then the range of g has a Δ^0_1 definition:

$$\exists n[g(n) = m] \iff \exists\sigma[0^{m+1}1 \prec \sigma \text{ and } \sigma \in U] \iff \neg(\exists\sigma[0^{m+1}1 \prec \sigma \text{ and } \sigma \in V]).$$

The complementarity of the above Σ^0_1 formulas does not require any induction. For each m, either $[0^{m+1}1] \subseteq O$ or $[0^{m+1}1] \cap O = \emptyset$. The density of $U \cup V$ implies there is a $\sigma \in U \cup V$ with $0^{m+1} \prec \sigma$. If $[0^{m+1}1] \subseteq O$ then $\sigma \in U$, and if $[0^{m+1}1] \cap O = \emptyset$, then $\sigma \in V$. □

3.4. **Connections to Hindman's theorem.** In this section, we show that Hindman's Theorem for ℓ-colorings implies CDRT^3_ℓ.

Definition 3.21 (RCA_0)**.** Let $\mathcal{P}_{\text{fin}}(\omega)$ denote the set of (codes for) all non-empty finite subsets of ω. $X \subseteq \mathcal{P}_{\text{fin}}(\omega)$ is an *IP set* if X is closed under finite unions and contains an infinite sequence of pairwise disjoint sets.

Theorem 3.22 (Hindman's theorem for ℓ-colorings)**.** *For every $c : \mathcal{P}_{fin}(\omega) \to \ell$ there is an IP set X and a color $i < \ell$ such that $c(F) = i$ for all $F \in X$.*

Proposition 3.23 (RCA_0)**.** *Hindman's theorem for ℓ-colorings implies CDRT^3_ℓ. In particular, CDRT^3_ℓ is provable in ACA^+_0.*

Proof. Fix $\ell \geq 2$ and assume Hindman's Theorem for ℓ-colorings. Since Hindman's Theorem for 2-colorings implies ACA_0, we reason in ACA_0. By Proposition 3.6 and Lemma 2.3, it suffices to fix an open reduced coloring $(\omega)^3 = \cup_{i<\ell} O_i$ and produce $p \in (\omega)^\omega$ and $i < \ell$ such that for all $x \in (p)^3$, $x \in O_i$. We write the coloring as $c : (\omega)^3 \to \ell$ with the understanding that $c(x) = i$ is shorthand for $x \in O_i$.

For a nonempty finite set $F \subseteq \omega$ with $0 \notin F$ and a number $n > \max F$, we let $x_{F,n} \in (\omega)^3$ be the following partition.

$$x_{F,n}(k) = \begin{cases} 0 & \text{if } k \notin F \text{ and } k \neq n \\ 1 & \text{if } k \in F \\ 2 & \text{if } k = n \end{cases}$$

Thus, $B_0^{x_{F,n}} = \omega - (F \cup \{n\})$, $B_1^{x_{F,n}} = F$ and $B_2^{x_{F,n}} = \{n\}$. Note that we can determine the color $c(x_{F,n})$ as a function of F and n and that since c is reduced, if $x \in (\omega)^3$ and $x \restriction \mu^x(2) = x_{F,n} \restriction n$, then $c(x) = c(x_{F,n})$.

The remainder of the proof is most naturally presented as a forcing construction. After giving a classical description of this construction, we indicate how to carry out the construction in ACA_0. The forcing conditions are pairs (F, I) such that

- F is a non-empty finite set such that $0 \notin F$,
- I is an infinite set such that $\max F < \min I$, and
- for every nonempty subset U of F there is an $i < \ell$ such that $c(x_{U,n}) = i$ for all $n \in F \cup I$ with $\max U < n$.

Extension of conditions is defined as for Mathias forcing: $(\widehat{F}, \widehat{I}) \leq (F, I)$ if $F \subseteq \widehat{F} \subseteq F \cup I$ and $\widehat{I} \subseteq I$.

By the pigeonhole principle, there is an $i < l$ such that $c(x_{\{1\},n}) = i$ for infinitely many $n > 1$. For any such i, the pair $(\{1\}, \{n \in \omega : n > 1 \text{ and } c(x_{\{1\},n}) = i\})$ is a condition. More generally, given a condition (F, I) there is an infinite set $\widehat{I} \subseteq I$ such that $(F \cup \{\min I\}, \widehat{I})$ is also a condition. To see this, let $U_0, \ldots, U_{s-1}$ be the nonempty subsets of $F \cup \{\min I\}$ containing $\min I$. By arithmetic induction, for each positive $k \leq s$, there exist colors $i_0, \ldots, i_{k-1} < \ell$ such that there are infinitely many $n \in I$ with $c(x_{U_j,n}) = i_j$ for all $j < k$. (If not, fix the least k for which the fact fails, and apply the pigeonhole principle to obtain a contradiction.) Let $i_0, \ldots, i_{s-1}$ be the colors corresponding to $k = s$ and let $\widehat{I}$ be the infinite set $\{n \in I : \forall j < s \; (c(x_{U_j,n}) = i_j)\}$.

Fix a sequence of conditions $(F_1, I_1) > (F_2, I_2) > \cdots$ with $|F_k| = k$ and let $G = \bigcup_k F_k$. To complete the proof, we use G to define a coloring $d : \mathcal{P}_{\text{fin}}(\omega) \to \ell$ to which we can apply Hindman's Theorem. However, first we indicate why we can form G in ACA_0.

The conditions (F, I) used to form G can be specified by the finite set F, the number $m = \min I$ and the finite sequence $\delta \in \ell^M$ where $M = 2^{|F|} - 1$ such that if $F_0, \ldots, F_{M-1}$ is a canonical listing of the nonempty subsets of F, then $I = \{n \geq m : \forall j < M \, (c(x_{F_j,n}) = \delta(j))\}$. The extension procedure above can be captured by an arithmetically definable function $f(F, m, \delta) = \langle F \cup \{m\}, m', \delta' \rangle$ where $F \cup \{m\}$,

m' and δ' describe the extension $(F \cup \{m\}, \widehat{I})$. Because the properties of this extension where verified using arithmetic induction and the pigeonhole principle, both of which are available in ACA_0, we can define f in ACA_0 and form a sequence of conditions $(F_1, m_1, \delta_1) > (F_2, m_2, \delta_2) > \cdots$ giving $G = \bigcup_k F_k$.

It remains to use $G = \{g_0 < g_1 < \cdots\}$ to complete the proof. By construction, for each non-empty finite subset U of G, there is color $i_U < \ell$ such that $c(x_{U,n}) = i_U$ for all $n \in G$ with $n > \max U$. Define $d : \mathcal{P}_{\text{fin}}(\omega) \to \ell$ by $d(F) = i_{\{g_m : m \in F\}}$. We apply Hindman's theorem to d to obtain an IP set X and a color $i < \ell$. Since X contains an infinite sequence of pairwise disjoint members, we can find a sequence $E_1, E_2, \ldots$ of members of X such that $\max E_k < \min E_{k+1}$. Define $p \in (\omega)^\omega$ to be the partition whose blocks are $B_0^p = \omega - \bigcup_k \{g_m : m \in E_k\}$ and, for each $k \geq 1$, $B_k^p = \{g_m : m \in E_k\}$. Note that for all $k \geq 1$,

$$\max B_k^p = \max\{g_m : m \in E_k\} < \min B_{k+1}^p = \min\{g_m : m \in E_{k+1}\}.$$

It remains to verify that p and i have the desired properties. Consider any $x \in (p)^3$; we must show that $c(x) = i$. Let $U = B_1^x \upharpoonright \mu^x(2)$ and let $n = \mu^x(2) = \min B_2^x$. Then $n = \mu^{x_{U,n}}(2)$ and $x \upharpoonright n = x_{U,n} \upharpoonright n$, so since c is reduced, $c(x) = c(x_{U,n})$. Therefore, it suffices to show $c(x_{U,n}) = i$.

We claim U is a finite union of p-blocks. Because x is a coarsening of p, B_1^x is a (possibly infinite) union of p-blocks $B_{j_1}^p \cup B_{j_2}^p \cup \cdots$ with $0 < j_1 < j_2 < \cdots$ and $n = \mu^x(2) = \min B_2^x = \min B_b^p$ for some $b \geq 2$. Let j_a be the largest index such that $j_a < b$. Since the p-blocks are finite and increasing, $U = B_1^x \upharpoonright \mu^x(2) = B_{j_1}^p \cup \cdots \cup B_{j_a}^p$. Note that $n \in G$ (because $B_b^p \neq B_0^p$) and $\max U < n$.

It follows that $U = \{g_m \mid m \in F\}$ where $F = E_{j_1} \cup \cdots \cup E_{j_a}$. Since our fixed IP set X is closed under finite unions, $F \in X$ and therefore $d(F) = i$. By the definition of d, $d(F) = i_{\{g_m | m \in F\}} = i_U$, so $i = i_U$. Finally, U is a finite subset of G, $n \in G$ and $\max U < n$, so $c(x_{U,n}) = i_U = i$ as required. $\square$

Observe that this proof of CDRT^3_ℓ from HT produces a homogeneous p with a special property: $\max B_i^p < \min B_{i+1}^p$ for all $i > 0$. We show that this strengthened "ordered finite block" version of CDRT^3_ℓ is equivalent to HT. However, there is no finite block version of CDRT^k_ℓ for $k > 3$.

Proposition 3.24 (RCA_0)**.** *If for every ℓ-coloring of $(\omega)^2_{fin}$ there is an infinite homogeneous partition p with $\max B_i < \min B_{i+1}$ for all $i > 0$, then Hindman's Theorem for ℓ-colorings holds.*

Proof. Given $c : \mathcal{P}_{\text{fin}}(\omega) \to \ell$, define $\widehat{c} : (\omega)^2_{\text{fin}} \to \ell$ by $\widehat{c}(\sigma) = c(\{i < |\sigma| : \sigma(i) = 1\})$. Let p be a homogeneous partition for $\widehat{c}$ with $\max B_i^p < \min B_{i+1}^p$ for all $i > 0$. The set of all finite unions of the blocks B_i^p for $i > 0$ satisfies the conclusion of Hindman's Theorem. $\square$

Proposition 3.25. *There is a 2-coloring of $(\omega)^3_{fin}$ such that any infinite homogeneous partition p has B_i^p infinite for all $i > 0$.*

Proof. For $\sigma \in (\omega)^3_{\text{fin}}$, set $c(\sigma) = 1$ if σ contains more 1's than 2's and set $c(\sigma) = 0$ otherwise. Let p be homogeneous for this coloring. Suppose for contradiction that $i > 0$ is such that B_i^p is finite. Let $N = i + 2 + |B_i^p|$ and let x be the coarsening of p with nonzero blocks

$$B_1^x = B_i^p, B_2^x = \cup_{j=i+1}^N B_j^p \text{ and } B_3^x = B_{N+1}^p.$$

Since $|B^x_2| > |B^x_1|$, $c(x \restriction \mu^x(3)) = 0$. Now coarsen in a different way: let $h \in [i+1, N]$ be chosen so that the size of $B^p_h \cap [0, \mu^x(3)]$ is minimized. Let y be the coarsening of p whose nonzero blocks are

$$B^y_1 = \cup_{j=i}^N B^p_i \setminus B^p_h, B^y_2 = B^p_h \text{ and } B^y_3 = B^p_{N+1}.$$

Since at least one p-block has moved from B^x_2 to B^y_1 and since B^y_2 contains only the smallest p-block from B^x_2, $c(y \restriction \mu^y(3)) = 1$. So p was not homogeneous. □

3.5. CDRT and the Carlson-Simpson Lemma.

The Carlson-Simpson Lemma is the main technical tool in the original proof of the Borel version of the Dual Ramsey Theorem. The principle is usually stated in the framework of variable words, but it can also be understood as a special case of the Combinatorial Dual Ramsey Theorem.

Carlson-Simpson Lemma ($\mathsf{CSL}(m,\ell)$)**.** *For every coloring $(\omega)^m_{\text{fin}} = \cup_{i<\ell} C_i$, there is a partition $p \in (\omega)^\omega$ and a color i such that for all $x \in (p)^{m+1}$, if $B^p_j \subseteq B^x_j$ for each $j < m$, then $x \restriction \mu^x(m) \in C_i$.*

The condition $B^p_j \subseteq B^x_j$ for $j < m$ captures those $x \in (p)^k$ which keep the first m many blocks of p distinct in x. Therefore, $\mathsf{CSL}(m,\ell)$ is a special case of CDRT^{m+1}_ℓ. Two related principles, $\mathsf{OVW}(m,\ell)$ and $\mathsf{VW}(m,\ell)$ have also been studied (see [11, 6, 10]). We do not deal with these principles, but it may be useful to note that $\mathsf{VW}(m,\ell)$ is the strengthening of $\mathsf{CSL}(m,\ell)$ which requires each nonzero block B^p_j to be finite, and $\mathsf{OVW}(m,\ell)$ is the further strengthening which requires $\max B^p_j < \min B^p_{j+1}$ for all $j > 0$.

As with the Combinatorial Dual Ramsey Theorem, we can assume the coloring in $\mathsf{CSL}(m,\ell)$ is given in the form $c : (\omega)^m_{\text{fin}} \to \ell$. In Proposition 3.26, we give three equivalent versions of the Carlson-Simpson Lemma. The version in Proposition 3.26(2) is (up to minor notational changes which are easily translated in RCA_0) the statement from Carlson and Simpson [3].

Proposition 3.26 (RCA_0)**.** *The following are equivalent.*

(1) $\mathsf{CSL}(m,\ell)$.
(2) *For each coloring $(\omega)^m_{\text{fin}} = \cup_{i<\ell} C_i$, there is a partition $p \in (\omega)^\omega$ and a color i such that for all $a < m$, $a \in B^p_a$ and for all $x \in (p)^{m+1}$, if $B^p_j \subseteq B^x_j$ for each $j < m$, then $x \restriction \mu^x(m) \in C_i$.*
(3) *For each $y \in (\omega)^\omega$ and open reduced coloring $(y)^{m+1} = \cup_{i<\ell} O_i$, there is a partition $p \in (y)^\omega$ and a color i such that for all $a < m$, $B^y_a \subseteq B^p_a$ and for all $x \in (p)^{m+1}$, if $B^p_j \subseteq B^x_j$ for each $j < m$, then $x \in O_i$.*

Proof. (2) implies (1) because $\mathsf{CSL}(m,\ell)$ is a special case of (2). The extra condition in (2) that $a \in B^p_a$ for $a < m$ says that the partition p does not collapse any of the first m-many blocks of the trivial partition defined by $B_n = \{n\}$. The equivalence between (2) and (3) is proved in a similar way to Proposition 3.6 using the transformation in Lemma 2.3.

It remains to prove (1) implies (2). Fix an ℓ-coloring $c : (\omega)^m_{\text{fin}} \to \ell$. Define $\tilde{c} : (\omega)^m_{\text{fin}} \to \ell$ by $\tilde{c}(\sigma) = c(0^\frown 1^\frown \cdots ^\frown (m-1)^\frown \sigma)$. Apply $\mathsf{CSL}(m,\ell)$ to $\tilde{c}$ to get $\tilde{p} \in (\omega)^\omega$ and $i < \ell$ such that for all $\tilde{x} \in (\tilde{p})^{m+1}$, if $B^{\tilde{p}}_j \subseteq B^{\tilde{x}}_j$ for all $j < m$, then $\tilde{c}(\tilde{x} \restriction \mu^{\tilde{x}}(m)) = i$. We treat $\tilde{p}$ as an infinite string $\langle \tilde{p}(0), \tilde{p}(1), \cdots \rangle$ with entries in ω.

Let $p \in (\omega)^\omega$ be the partition corresponding to the infinite string $p = 0^\frown 1^\frown \cdots^\frown (m-1)^\frown \tilde{p}$. We claim that p satisfies the conditions in (2) for the coloring c with the fixed color i. By the definition of p, $a \in B^p_a$ for all $a < m$.

Fix $x \in (p)^{m+1}$ such that $B^p_j \subseteq B^x_j$ for all $j < m$. We need to show that $c(x \upharpoonright \mu^x(m)) = i$. Since x does not collapse any of the first m-many p-blocks, $a \in B^x_a$ for all $a < m$ and x (as an infinite string) has the form $x = 0^\frown 1^\frown \cdots^\frown (m-1)^\frown \widehat{x}$ such that the infinite string $\widehat{x}$ is an ordered function from $[m, \infty)$ onto $m+1$. Letting $\tilde{x}(n) = \widehat{x}(n+m)$, we obtain a partition $\tilde{x} \in (\tilde{p})^{m+1}$ such that $B^{\tilde{p}}_j \subseteq B^{\tilde{x}}_j$ for all $j < m$. Therefore, $\tilde{c}(\tilde{x} \upharpoonright \mu^{\tilde{x}}(m)) = i$. Translating back through $\widehat{x}$ to x, we have $\mu^x(m) = \mu^{\tilde{x}}(m) + m$ and $x \upharpoonright \mu^x(m) = 0^\frown 1^\frown \cdots^\frown (m-1)^\frown \tilde{x} \upharpoonright \mu^{\tilde{x}}(m)$. Translating from $\tilde{c}$ to c, we have

$$i = \tilde{c}(\tilde{x} \upharpoonright \mu^{\tilde{x}}(m)) = c(0^\frown 1^\frown \cdots^\frown (m-1)^\frown \tilde{x} \upharpoonright \mu^{\tilde{x}}(m)) = c(x \upharpoonright \mu^x(m))$$

as required to complete the proof that (1) implies (2). $\square$

We will also use a variant of Lemma 2.4. Let $p \in (\omega)^\omega$ and $s \geq m$. Consider the ways to collapse the first s-many blocks $B^p_0, \ldots, B^p_{s-1}$ of p to exactly m-many blocks while leaving the remaining p-blocks unchanged. Collapsing s-many blocks to m-many blocks is described by a string $\sigma \in (\omega)^m_{\text{fin}}$ with $|\sigma| = s$. To leave the remaining p-blocks unchanged, we extend σ to $\sigma^* \in (\omega)^\omega$ to renumber the blocks B^p_a for $a \geq s$ starting with index m. Formally, $\sigma^*(n) = \sigma(n)$ for $n < s$ and $\sigma^*(n) = n - (s - m)$ for $n \geq s$. An argument similar to the proof of Lemma 2.4 gives the next lemma.

Lemma 3.27. *Fix $s \geq m \geq 2$. Let $\sigma_{s,0}, \ldots, \sigma_{s,M_s-1}$ list the strings $\sigma_{s,j} \in (\omega)^m_{fin}$ with $|\sigma_{s,j}| = s$. For any $p \in (\omega)^\omega$, the coarsenings of p which collapse the first s-many blocks of p to m-many blocks and leave the remaining p-blocks unchanged are $\sigma^*_{s,0} \circ p, \ldots, \sigma^*_{s,M_s-1} \circ p$.*

Let $y \in (\omega)^\omega$ and $(y)^k = \cup_{i<\ell} C_i$ be an m-reduced coloring for some $1 < m < k$. We define the *induced coloring* $(y)^{m+1} = \cup_{i<\ell} \widehat{C}_i$ as follows. For $\widehat{q} \in (y)^{m+1}$, $\widehat{q} \in \widehat{C}_i$ if and only if $q \in C_i$ for some (or equivalently all) $q \in (y)^k$ such that $\widehat{q} \upharpoonright \mu^{\widehat{q}}(m) = q \upharpoonright \mu^q(m)$. This induced coloring is a reduced coloring of $(y)^{m+1}$ and therefore we can apply $\mathsf{CSL}(m, \ell)$ to it.

Lemma 3.28. *Let $1 < m < k$, $y \in (\omega)^\omega$ and $(y)^k = \cup_{i<\ell} C_i$ be an m-reduced coloring. Let $(y)^{m+1} = \cup_{i<\ell} \widehat{C}_i$ be the induced coloring and let $z \in (y)^\omega$ and $i < \ell$ be obtained by applying $\mathsf{CSL}(m, \ell)$ as in Proposition 3.26(3) to the induced coloring. If $x \in (z)^k$ with $B^z_a \subseteq B^x_a$ for $a < m$, then $x \in C_i$.*

Proof. Given $x \in (z)^k$ as in the lemma, let $\widehat{x} \in (x)^{m+1}$ be the coarsening of x with blocks $B^{\widehat{x}}_a = B^x_a$ for $a < m$ and $B^{\widehat{x}}_m = \cup_{m \leq a < k} B^x_a$. By definition, $\widehat{x} \in (z)^{m+1}$ with $B^z_a \subseteq B^{\widehat{x}}_a$ for $a < m$, and therefore $\widehat{x} \in \widehat{C}_i$. Since $\widehat{x} \upharpoonright \mu^{\widehat{x}}(m) = x \upharpoonright \mu^x(m)$, $x \in C_i$ by the definition of the induced coloring. $\square$

Our proof of CDRT^k_ℓ from the Carlson-Simpson Lemma will use repeated applications of the following lemma which is proved using ω many nested applications of $\mathsf{CSL}(m, \ell)$.

Lemma 3.29. *Fix $1 < m < k$ and $y \in (\omega)^\omega$. Let $(y)^k = \cup_{i<\ell} C_i$ be an m-reduced coloring. There is an $x \in (y)^\omega$ such that the coloring restricted to $(x)^k$ is $(m-1)$-reduced.*

Proof. Fix an m-reduced coloring $(y)^k = \cup_{i<\ell} C_i$. We define a sequence of infinite partitions $x_m, x_{m+1}, \cdots$ starting with index m such that $x_m = y$ and x_{s+1} is a coarsening of x_s for which $B_a^{x_s} \subseteq B_a^{x_{s+1}}$ for all $a < s$. That is, we do not collapse any of the first s-many blocks of the partition x_s when we coarsen it to x_{s+1}. This property guarantees that the sequence has a well-defined limit $x \in (\omega)^\omega$. We show this limiting partition x satisfies the conclusion of the lemma. The process of passing from x_s to x_{s+1} will use finitely many nested applications of $\mathsf{CSL}(m, \ell)$.

Assume x_s has been defined for a fixed $s \geq m$ and we construct x_{s+1}. Let $\sigma^*_{s,0}, \ldots, \sigma^*_{s,M_s-1}$ be the infinite partitions from Lemma 3.27. Set $x_s^0 = x_s$. We define a sequence of coarsenings $x_s^1, \ldots, x_s^{M_s}$ and set $x_{s+1} = x_s^{M_s}$. The definition of x_s^{j+1} from x_s^j will use one application of $\mathsf{CSL}(m, \ell)$.

Assume that x_s^j has been defined. Let $w_s^j = \sigma^*_{s,j} \circ x_s^j$ be the result of collapsing the first s-many blocks of x_s^j into m-many blocks in the j-th possible way and leaving the remaining blocks of x_s^j unchanged. Since w_s^j is a coarsening of y, the coloring $(y)^k = \cup_{i<\ell} C_i$ restricts to an m-reduced coloring of $(w_s^j)^k$ which induces a reduced coloring of $(w_s^j)^{m+1}$ as described above. Let z_s^j be the result of applying $\mathsf{CSL}(m, l)$ as stated in Proposition 3.26(3) to this reduced coloring of $(w_s^j)^{m+1}$.

To define x_s^{j+1}, we want to "uncollapse" the first m-many blocks of z_s^j to reverse the action of $\sigma^*_{s,j}$ in defining w_s^j. Since w_s^j collapsed the first s-blocks of x_s^j to m-many blocks and since z_s^j is a coarsening of w_s^j, if $x_s^j(u) < s$, then $z_s^j(u) < m$. We define x_s^{j+1} by cases as follows.

(1) If $x_s^j(u) < s$, then $x_s^{j+1}(u) = x_s^j(u)$.
(2) If $x_s^j(u) \geq s$ and $z_s^j(u) = a < m$, then $x_s^{j+1}(u) = x_s^j(\mu^{z_s^j}(a))$.
(3) If $z_s^j(u) \geq m$, then $x_s^{j+1}(u) = z_s^j(u) + (s - m)$.

Below we verify that x_s^{j+1} is an infinite partition coarsening x_s^j which does not collapse any of the first s-many blocks of x_s^j. This completes the construction of x_s^{j+1} and hence of x_{s+1} and x.

We verify the required properties of x_s^{j+1}. By (1), $B_a^{x_s^j} \subseteq B_a^{x_s^{j+1}}$ for all $a < s$, so we do not collapse any of the first s-many blocks of x_s^j in x_s^{j+1}. There is no conflict between (1) and (3) because $x_s^j(u) < s$ implies $z_s^j(u) < m$. Furthermore, (3) renumbers the z_s^j-blocks starting with index m to x_s^{j+1}-blocks starting with index s without changing any of these blocks. Therefore, x_s^{j+1} is an infinite partition.

In (2), we handle the case when the x_s^j-block containing u is not changed by w_s^j (except to renumber its index) but is collapsed by z_s^j into one of the first m-many z_s^j-blocks. In this case, $\mu^{z_s^j}(a) = \mu^{x_s^j}(b)$ for some $b < s$ and we have set $x_s^{j+1}(u) = b$. It is straightforward to check (as in the proof of Theorem 3.9) that x_s^{j+1} is a coarsening of x_s^j and that $\sigma^*_{s,j} \circ x_s^{j+1} = z_s^j$.

To complete the proof, we verify that the restriction of $\cup_{i<\ell} C_i$ to $(x)^k$ is $(m-1)$-reduced. Fix $p \in (x)^k$ and we show the color of p depends only on $p \upharpoonright \mu^p(m-1)$.

Fix $s \in \omega$ such that $\mu^x(s-1) = \mu^p(m-1)$. The partition p collapses the first s-many x-blocks into the first m-many p-blocks. The string $\sigma \in (\omega)^m_{\text{fin}}$ with $|\sigma| = s$ defined by $\sigma(a) = b$ if $B_a^x \subseteq B_b^p$ describes this collapse. Fix an index j such that $\sigma = \sigma_{s,j}$ in our fixed enumeration of such strings in Lemma 3.27. Note that $\sigma_{s,j}$ is determined by $p \upharpoonright \mu^p(m-1)$ and that p is a coarsening of $\sigma^*_{s,j} \circ x$.

Consider how x_s^{j+1} was defined from x_s^j in the construction. We set $w_s^j = \sigma^*_{s,j} \circ x_s^j$ and applied $\mathsf{CSL}(m, \ell)$ to the induced coloring of $(w_s^j)^{m+1}$ to get z_s^j. Although we did not use it in the construction, this application of $\mathsf{CSL}(m,\ell)$ also determined a homogeneous color $i^j_s < \ell$ which we will use below.

Since x is a coarsening of x^{j+1}_s for which $B^{x^{j+1}_s}_a \subseteq B^x_a$ for all $a < s$, $\sigma^*_{s,j} \circ x$ is a coarsening of $\sigma^*_{s,j} \circ x^{j+1}_s = z^j_s$. Because p is a coarsening of $\sigma^*_{s,j} \circ x$, it follows that p is a coarsening of z^j_s.

We claim that the first m-many blocks of z^j_s remain distinct in p. That is, if $z^j_s(u) < m$, then $p(u) = z^j_s(u)$. To see why, assume $z^j_s(u) < m$. It follows that $x^{j+1}_s(u) < s$ and hence that $x(u) = x^{j+1}_s(u) < s$ because x does not collapse any of the first s-many blocks of x^{j+1}_s. Therefore, $\sigma_{s,j}(x(u))$ is defined and we have

$$p(u) = \sigma_{s,j}(x(u)) = \sigma_{s,j}(x^{j+1}_s(u)) = z^j_s(u)$$

as required.

We obtained z^j_s and the color $i^j_s < \ell$ by applying $\mathsf{CSL}(m,\ell)$ to the induced coloring of $(w^j_s)^{m+1}$. Since $p \in (z^j_s)^k$ satisfies $B^{z^j_s}_a \subseteq B^p_a$ for $a < m$, we can apply Lemma 3.28 to conclude that $p \in C_{i^j_s}$. This completes the proof that the restriction of $\cup_{i<\ell} C_i$ to $(x)^k$ is $(m-1)$-reduced because the indices s and j in z^j_s are determined by $p \upharpoonright \mu^p(m-1)$ and the color of p is equal to the homogeneous color i^j_s obtained when we applied $\mathsf{CSL}(m,\ell)$ to obtain z^j_s. □

We end this section with the proof of CDRT^k_ℓ.

Theorem 3.30. *For all for $k \geq 2$ and all ℓ, CDRT^k_ℓ holds.*

Proof. For $k = 2$, CDRT^k_ℓ follows from the pigeonhole principle as in Proposition 3.11. Now assume $k \geq 3$. Consider CDRT^k_ℓ is the form given in Proposition 3.6. Let $y \in (\omega)^\omega$ and $(y)^k = \cup_{i<\ell} O_i$ be an open reduced coloring. These satisfy the assumptions of Lemma 3.29 with $m = k-1$. After $k-2$ applications of Lemma 3.29, we obtain $x \in (y)^\omega$ such that the restriction of $\cup_{i<\ell} O_i$ to $(x)^k$ is 1-reduced and hence the color of $p \in (x)^k$ depends only on $p \upharpoonright \mu^p(1)$. Since the numbers $n < \mu^p(1)$ must lie in B^p_0, the color of p is determined by the value of $\mu^p(1)$. By the pigeonhole principle, there is an infinite set $X \subseteq \{\mu^x(a) : a \geq 1\}$ and a color i such that for all $p \in (x)^k$, if $\mu^p(1) \in X$, then $p \in C_i$. It follows that for any $z \in (x)^\omega$ such that $\mu^z(a) \in X$ for all $a \geq 1$, $(z)^k \subseteq C_i$ as required. □

It is interesting to note that the only non-constructive steps in this proof are the $\omega \cdot (k-2)$ nested applications of the Carlson-Simpson Lemma.

4. The Borel Dual Ramsey Theorem for $k \geq 3$

In the next two sections we consider the Borel Dual Ramsey Theorem from the perspective of effective mathematics. For continuity with Section 3, we define a code for an open set in $(\omega)^k$ to be a set $O \subseteq \omega \times (\omega)^k_{\text{fin}}$. We say O is a Δ^0_n code if O is Δ^0_n as a set of natural numbers. Equivalently, a Δ^0_n code for an open set is a subset of $(\omega)^k_{\text{fin}}$ which is c.e. relative to $\emptyset^{(n-1)}$, or by replacing elements of O as they are enumerated with sets of sufficiently long strings, is a subset of $(\omega)^k_{\text{fin}}$ which is computable in $\emptyset^{(n-1)}$. We will shift between these coding methods in Sections 4 and 5.

We define Borel codes for topologically Σ^0_α subsets of $(\omega)^k$ by induction on the ordinals below ω_1. This definition gives another method of coding an open (topologically Σ^0_1) set which is easily translated into the codes described above. Let B_n, $n \in \omega$, be an effective listing of the clopen sets $\emptyset$, $(\omega)^k$ and $[\sigma]$ and $\overline{[\sigma]}$ for $\sigma \in (\omega)^k_{\text{fin}}$.

Definition 4.1. We define a *Borel code for a (topologically)* Σ^0_α *or* Π^0_α *set.*

- A *Borel code for a* Σ^0_0 *or a* Π^0_0 *set* is a labelled tree T consisting of just a root λ in which the root is labeled by a clopen set B_{n_λ}. The code represents the set B_{n_λ}.
- For $\alpha \geq 1$, a *Borel code for a* Σ^0_α *set* is a labelled tree with a root labelled by $\cup$ and attached subtrees at level 1, each of which is a Borel code for a $\Sigma^0_{\beta_n}$ or $\Pi^0_{\beta_n}$ set A_n for some $\beta_n < \alpha$. The code represents the set $\cup_n A_n$.
- For $\alpha \geq 1$, a *Borel code for a* Π^0_α *set* is the same, except the root is labelled $\cap$. The code represents the set $\cap_n A_n$.

For $\alpha \geq 1$, a *Borel code for a* Δ^0_α *set* is a pair of labelled trees which encode the same set, where one encodes it as a Σ^0_α set and the other encodes it as a Π^0_α set.

The codes are faithful to the Borel hierarchy in the sense that every code for a Σ^0_α set represents a Σ^0_α set and every Σ^0_α set is represented by a Borel code for a Σ^0_α set. There is a uniform procedure to transform a Borel code B for a Σ^0_α set A into a Borel code $\overline{B}$ for a Π^0_α set $\overline{A}$: leave the underlying tree structure the same, swap the $\cup$ and $\cap$ labels and replace the leaf labels by their complements.

If a Borel set A has a computable code (i.e. the labeled subtree of $\omega^{<\omega}$ is computable), then the Turing machine Φ_e giving the computable labelled tree is a *computable Borel code for A*.

We recall some notation from hyperarithmetic theory. Let $\mathcal{O}$ denote Kleene's set of computable ordinal notations. The ordinal represented by $a \in \mathcal{O}$ is denoted $|a|_{\mathcal{O}}$, with $|1|_{\mathcal{O}} = 0$, $|2^a|_{\mathcal{O}} = |a|_{\mathcal{O}} + 1$, and $|3 \cdot 5^e|_{\mathcal{O}} = \sup_j |\varphi_e(j)|_{\mathcal{O}}$. The H-sets are defined by effective transfinite recursion on $\mathcal{O}$ as follows: $H_1 = \emptyset$, $H_{2^a} = H'_a$ and $H_{3\cdot 5^a} = \{\langle i,j\rangle \mid i \in H_{\varphi_a(j)}\}$. The reader referred to Sacks [15] for more details. As usual, ω_1^{CK} denotes the least noncomputable ordinal.

It is well-known that an open set of high hyperarithmetic complexity can be represented by a computable Borel code for a Σ^0_α set, where α is an appropriate computable ordinal. In the following proposition, we use a standard technique to make this correspondence explicit. Let

$$\text{height}(a) = \begin{cases} |a|_{\mathcal{O}} & \text{if } |a|_{\mathcal{O}} < \omega \\ |a|_{\mathcal{O}} \dot{-} 1 & \text{if } |a|_{\mathcal{O}} \geq \omega. \end{cases}$$

where $\alpha \dot{-} 1 = \alpha$ if α is a limit and $\alpha - 1$ otherwise. Note that for $|a|_{\mathcal{O}} < \omega$, $\text{height}(2^a) = |a|_{\mathcal{O}} + 1$ and for $|a|_{\mathcal{O}} \geq \omega$, $\text{height}(2^a) = |a|_{\mathcal{O}}$. Fix an effective 1-to-1 enumeration τ_n for the strings $\tau \in (\omega)^k_{\text{fin}}$.

Proposition 4.2. *There is a partial computable function $p(x,y)$ such that $p(a,e)$ is defined for all $a \in \mathcal{O}$ and $e \in \omega$ and such that if $a \in \mathcal{O}$ and $R = \bigcup\{[\tau_n] : n \in W_e^{H_a}\}$, then $\Phi_{p(a,e)}$ is a computable Borel code for R as a $\Sigma^0_{\text{height}(2^a)}$ set.*

Proof. We define $p(a,e)$ for all e by effective transfinite recursion on $a \in \mathcal{O}$. Since $H_1 = \emptyset$, let $\Phi_{p(1,e)}$ be a Borel code for the open set $R = \bigcup\{[\tau_n] : n \in W_e\}$.

For the successor step, consider $R = \bigcup\{[\tau_n] : n \in W_e^{H_{2^a}}\}$. Each set which is Σ^0_1 in H_{2^a} is Σ^0_2 in H_a and for such sets, we can effectively pass from a $\Sigma^{0,H_{2^a}}_1$ index to a Σ_2^{0,H_a} description. Specifically, uniformly in e, we compute an index e' such that for all oracles X, $\Phi_{e'}^X(x,y)$ is a total $\{0,1\}$-valued function and

$$n \in W_e^{X'} \text{ if and only if } \exists t \, \forall s \geq t \, (\Phi_{e'}^X(n,s) = 1).$$

Let $R_t = \bigcup\{[\tau_n] : \exists s \geq t \, (\Phi_{e'}^{H_a}(n,s) = 0)\}$. $R_0 \supseteq R_1 \supseteq \cdots$ is a decreasing sequence of sets such that $x \notin R$ if and only if $\forall t \, (x \in R_t)$. Therefore, $R = \cup_t \overline{R_t}$. Each set R_t can be represented as $R_t = \bigcup\{[\tau_n] : n \in W_{e_t}^{H_a}\}$, where e_t is uniformly computable from e and t. Applying the induction hypothesis, we define $p(2^a, e)$ to encode a tree whose root is labelled by a union and whose t-th subtree at level 1 is the Borel code representing the complement of $\Phi_{p(a,e_t)}$.

For the limit step, consider $R = \bigcup\{[\tau_n] : n \in W_e^{H_{3\cdot 5^a}}\}$. Uniformly in e, we construct a sequence of indices e_t for $t \in \omega$ such that for all oracles X, $\Phi_{e_t}^X(x)$ converges if and only if $\Phi_e^X(x)$ converges and only asks oracle questions about numbers in the first t many columns of X. Let $R_t = \bigcup\{[\tau_n] : n \in W_{e_t}^{\oplus_{i \leq t} H_{\varphi_a(i)}}\}$ and note that $R = \cup_t R_t$. We can effectively pass to a sequence of indices e_t' such that $R_t = \bigcup\{[\tau_n] : n \in W_{e_t'}^{H_{\varphi_a(t)}}\}$. By induction, each $p(\varphi_a(t), e_t')$ is the index for a computable Borel code for R_t as a $\Sigma^0_{\text{height}(2^{\varphi_a(t)})}$ set, so we may define $p(3 \cdot 5^a, e)$ to be the index of a tree which has $\Phi_{p(\varphi_a(t), e_t')}$ as its subtrees. Since $\text{height}(2^{\varphi_a(t)}) < |3 \cdot 5^a|_{\mathcal{O}} = \text{height}(2^{3 \cdot 5^a})$ for all t, the resulting Borel code has the required height. □

To force the Dual Ramsey Theorem to output computationally powerful homogeneous sets, we use the following definition and a result of Jockusch [9].

Definition 4.3. For functions $f, g : \omega \to \omega$, we say g *dominates* f, and write $g \succeq f$, if $f(n) \leq g(n)$ for all but finitely many n.

Theorem 4.4 (Jockusch [9], see also [14, Exercise 16-98])**.** *For each $a \in \mathcal{O}$, there is a function f_a such that $f_a \equiv_T H_a$ and for every $g \succeq f_a$, we have $H_a \leq_T g$.*

In Theorem 4.7, we use these functions f_a to show that for every $a \in \mathcal{O}$, there is a computable Borel code for a set $R \subseteq (\omega)^3$ such that any homogeneous partition $p \in (\omega)^\omega$ for the coloring $(\omega)^3 = R \cup \overline{R}$ computes H_a.

Theorem 4.5. *Let A be a set and f_A be a function such that $A \equiv_T f_A$ and for every $g \succeq f_A$, we have $A \leq_T g$. There is an A-computable clopen coloring $(\omega)^3 = R \cup \overline{R}$ for which every homogeneous partition p satisfies $p \geq_T A$.*

Proof. Fix A and f_A as in the statement of the theorem. Without loss of generality, we assume that if $n < m$, then $f_A(n) < f_A(m)$. For $x \in (\omega)^3$, let $a_x = \mu^x(1)$ and $b_x = \mu^x(2)$. As in the proof of Theorem 3.19, let $O_{a,b} = \{x \in (\omega)^3 : a_x = a \wedge b_x = b\}$. Set $R = \{x \in (\omega)^3 : f_A(a_x) \leq b_x\}$. Since $R = \bigcup\{O_{n,m} \mid f_A(n) \leq m\}$ and $\overline{R} = \bigcup\{O_{n,m} \mid f_A(n) > m\}$ both R and $\overline{R}$ are A-computable open sets.

Claim. If $p \in (\omega)^\omega$ is homogeneous, then $(p)^3 \subseteq R$.

It suffices to show that there is an $x \in (p)^3$ with $x \in R$. Let $u = \mu^p(1)$. Because p has infinitely many blocks, there must be a p-block $V \neq B_1^p$ with least element $v \geq f(u)$. Consider the partition $x \in (p)^3$ with $B_0^x = \omega \setminus (B_1^p \cup V)$, $B_1^x = B_1^p$ and $B_2^x = V$. Since $a_x = u$ and $b_x = v$, we have $x \in (p)^3$ with $f(a_x) \leq b_x$, so $x \in R$.

Claim. If $p \in (\omega)^\omega$ is homogeneous, then $A \leq_T p$.

Fix p and let $g(n) = \mu^p(n+2)$. Since g is p-computable, it suffices to show $g \succeq f_A$. Because $n < \mu^p(n+1)$ and f_A is increasing, we have $f_A(n) < f_A(\mu^p(n+1))$. Therefore, to show $g \succeq f_A$, it suffices to show $f_A(\mu^p(n+1)) \leq \mu^p(n+2) = g(n)$.

Let $x_n \in (p)^3$ be the coarsening with blocks $\omega \setminus (B^p_{n+1} \cup B^p_{n+2})$, B^p_{n+1} and B^p_{n+2}. Note that $a_{x_n} = \mu^p(n+1)$ and $b_{x_n} = \mu^p(n+2)$. By the previous claim, $x_n \in R$, so $f_A(a_{x_n}) \leq b_{x_n}$. In other words, $f_A(\mu^p(n+1)) \leq \mu^p(n+2)$ as required. □

Corollary 4.6. *For each $k \geq 3$ and each $a \in \mathcal{O}$, there is an H_a-computable clopen set $R \subseteq (\omega)^k$ such that if $p \in (\omega)^\omega$ is homogeneous for $(\omega)^k = R \cup \overline{R}$, then $H_a \leq_T p$.*

Proof. For $k = 3$, this corollary follows from Theorems 4.4 and 4.5. For $k > 3$, use similiar definitions for R and $\overline{R}$ ignoring what happens after the first three blocks of the partition. □

Theorem 4.7. *For every recursive $\alpha > 0$, and every $k \geq 3$, there is a computable Borel code for a Δ^0_α set $R \subseteq (\omega)^k$ such that every $p \in (\omega)^\omega$ homogeneous for the coloring $(\omega)^k = R \cup \overline{R}$ computes $\emptyset^{(\alpha-1)}$ if $\alpha < \omega$ and computes $\emptyset^{(\alpha)}$ if $\alpha \geq \omega$.*

Proof. Given $a \in \mathcal{O}$ with $\mathrm{height}(2^a) = \alpha$, let $R, \overline{R}$ be H_a-computable clopen sets from the previous corollary. By Proposition 4.2, both R and $\overline{R}$ have computable Borel codes as $\Sigma^0_{\mathrm{height}(2^a)}$ subsets of $(\omega)^k$. Therefore, R has a computable Borel code as Δ^0_α set. By the previous corollary, if p is homogeneous for $(\omega)^k = R \cup \overline{R}$, then $p \geq_T H_a$, which corresponds to the indicated number of Turing jumps. □

For $\alpha = 2$, Theorem 4.7 says there is a classically clopen set $R \subseteq (\omega)^3$ such that R and $\overline{R}$ have computable Borel codes as Σ^0_2 sets (and hence as Δ^0_2 sets) and any homogeneous partition for $(\omega)^3 = R \cup \overline{R}$ computes $\emptyset'$. Theorem 3.19 gives an analogous result at a slightly better coding level in the sense that the coloring $(\omega)^3 = O \cup \overline{O}$ is given in terms of an open code and a closed code.

5. The Borel Dual Ramsey Theorem for $k = 2$

5.1. **Effective Analysis.** We consider the complexity of finding infinite homogeneous partitions for colorings $(\omega)^2 = R \cup \overline{R}$ when R is a computable code for a set at a finite level of the Borel hierarchy. We begin by showing that if R is a computable open set, there is a computable homogeneous partition.

Theorem 5.1. *Let R be a computable code for an open set in $(\omega)^2$. There is a computable $p \in (\omega)^\omega$ such that $(p)^2 \subseteq R$ or $(p)^2 \subseteq \overline{R}$.*

Proof. If there is an $n \geq 1$ such that $[0^n] \cap R = \emptyset$, then the partition $x \in (\omega)^\omega$ with blocks $\{0, 1, \ldots, n\}, \{n+1\}, \{n+2\}, \ldots$ satisfies $(x)^2 \subseteq \overline{R}$. Otherwise, for arbitrarily large n there are $\tau \succ 0^n 1$ with $[\tau] \subseteq R$, and hence there is a computable sequence $\tau_1, \tau_2, \ldots$ of such τ with $0^i \prec \tau_i$. Computably thin this sequence so that for each i, $0^{|\tau_i|} \prec \tau_{i+1}$. The partition x with blocks $B^x_i = \{j : \tau_i(j) = 1\}$ for $i > 0$ satisfies $(x)^2 \subseteq R$. □

To extend to sets coded at higher finite levels of the Borel hierarchy, we will need the following generalization of the previous result.

Theorem 5.2. *Let R be a computable code for an open set in $(\omega)^2$ such that $R \cap [0^n] \neq \emptyset$ for all n. Let $\{D_i\}_{i<\omega}$ be a uniform sequence of computable codes for open sets such that each D_i is dense in R. There is a computable $x \in (\omega)^\omega$ such that $(x)^2 \subseteq R \cap (\cap_i D_i)$.*

Proof. We build x as the limit of an effective sequence $\tau_0 \prec \tau_1 \prec \cdots$ with $\tau_s \in (\omega)^{s+1}_{\text{fin}}$. We define the strings τ_s in stages starting with $\tau_0 = \langle 0 \rangle$ which puts $0 \in B^x_0$. For $s \geq 1$, we ensure that at the start of stage $s+1$, we have $[\sigma \circ \tau_s] \subseteq R$ for all $\sigma \in (\omega)^2_{\text{fin}}$ with $|\sigma| = s+1$. That is, the open sets in $(\omega)^2$ determined by each way of coarsening the $s+1$ many blocks of τ_s to two blocks is contained in R.

At stage $s+1$, assume we have defined $\tau_s \in (\omega)^{s+1}_{\text{fin}}$. If $s \geq 1$, assume that for all $\sigma \in (\omega)^2_{\text{fin}}$ with $|\sigma| = s+1$, $[\sigma \circ \tau_s] \subseteq R$. Let $\sigma_0, \ldots, \sigma_{M_s - 1}$ list the strings $\sigma \in (\omega)^2_{\text{fin}}$ such that $|\sigma| = s+2$. We define a sequence of strings $\tau^0_s, \ldots, \tau^{M_s}_s$ and set $\tau_{s+1} = \tau^{M_s}_s$.

We define τ^0_s to start the block B^x_{s+1}. Since $[0^{|\tau_s|}] \cap R \neq \emptyset$, we effectively search for $\gamma_s \in (\omega)^2_{\text{fin}}$ such that $0^{|\tau_s|} \prec \gamma_s$ and $[\gamma_s] \subseteq R$. Since $\gamma_s \in (\omega)^2_{\text{fin}}$, there is at least one $m < |\gamma_s|$ such that $\gamma_s(m) = 1$. Define $\tau^0_s \in (\omega)^{s+2}_{\text{fin}}$ with $|\tau^0_s| = |\gamma_s|$ by

$$\tau^0_s(m) = \begin{cases} \tau_s(m) & \text{if } m < |\tau_s| \\ s+1 & \text{if } \gamma_s(m) = 1 \text{ (and hence } m \geq |\tau_s|) \\ 0 & \text{if } m \geq |\tau_s| \text{ and } \gamma_s(m) = 0. \end{cases}$$

Note that $\tau_s \prec \tau^0_s$. Intuitively, τ^0_s partitions $\{0, \ldots, |\tau^0_s| - 1\}$ into $(s+2)$ many blocks as follows. It leaves the blocks $B^{\tau_s}_1, \ldots, B^{\tau_s}_s$ unchanged, starts a new block $B^{\tau^0_s}_{s+1} = B^{\gamma_s}_1$ and puts the remaining elements in $B^{\tau^0_s}_0$.

Before proceeding, we claim that if $\sigma \in (\omega)^2_{\text{fin}}$ with $|\sigma| = s+2$, then $[\sigma \circ \tau^0_s] \subseteq R$. First, suppose $s = 0$. In this case, $\tau^0_s \in (\omega)^2_{\text{fin}}$ and $\tau^0_s = \gamma_s$ because $s+1 = 1$. Therefore, $[\tau^0_s] = [\gamma_s] \subseteq R$. Furthermore, the only string $\sigma \in (\omega)^2_{\text{fin}}$ with $|\sigma| = 2$ is $\sigma = \langle 0, 1 \rangle$. Therefore, $\sigma \circ \tau^0_s = \tau^0_s = \gamma_s$ and the claim follows. Second, suppose $s \geq 1$ and let j be the least number such that $\sigma(j) = 1$. If $j = s+1$, then $\sigma \circ \tau^0_s = \gamma_s$ and the claim follows. If $j < s+1$, then let $\sigma' \prec \sigma$ with $|\sigma'| = s+1$. Since $\sigma' \in (\omega)^2_{\text{fin}}$, we have by induction that $[\sigma' \circ \tau_s] \subseteq R$ and since $\sigma' \circ \tau_s \prec \sigma \circ \tau^0_s$, we have $[\sigma \circ \tau^0_s] \subseteq R$.

We continue to define the τ^j_s strings by induction. Assume that τ^j_s has been defined and consider the j-th string σ_j enumerated above describing how to collapse $(s+1)$ many blocks into 2 blocks. Since $\tau^0_s \preceq \tau^j_s$, we have $\sigma_j \circ \tau^0_s \preceq \sigma_j \circ \tau^j_s$ and hence $[\sigma_j \circ \tau^j_s] \subseteq R$. Because $\cap_{n<s+1} D_n$ is dense in R, we can effectively search for a string $\delta^j_s \in (\omega)^2_{\text{fin}}$ such that $\sigma_j \circ \tau^j_s \preceq \delta^j_s$ and $[\delta^j_s] \subseteq \cap_{n<s+1} D_n$. To define τ^{j+1}_s, we uncollapse δ^j_s. Let j^* be the least number such that $\sigma_j(j^*) = 1$. Define

$$\tau^{j+1}_s(m) = \begin{cases} \tau^j_s(m) & \text{if } m < |\tau^j_s| \\ j^* & \text{if } m \geq |\tau^j_s| \text{ and } \delta^j_s(m) = 1 \\ 0 & \text{if } m \geq |\tau^j_s| \text{ and } \delta^j_s(m) = 0 \end{cases}$$

It is straightforward to check that $\tau^j_s \preceq \tau^{j+1}_s$ and that $\sigma_j \circ \tau^{j+1}_s = \delta^j_s$. This completes the construction of the sequence $\tau^0_s \preceq \cdots \preceq \tau^{M_s}_s$ and of the computable partition x. It remains to show that if $p \in (x)^2$, then $p \in R$ and $p \in \cap_{n \in \omega} D_n$. Fix $p \in (x)^2$ and let s_0 be the least number such that $B^x_{s_0+1}$ is not collapsed into B^p_0.

Claim. $p \in R$.

Let $\sigma \in (\omega)^2_{\text{fin}}$ with $|\sigma| = s_0 + 2$ be the sequence defined by $\sigma(m) = 0$ for all $m < s_0 + 1$ and $\sigma(s_0+1) = 1$. Thus σ describes how the blocks $B^x_0, \ldots, B^x_{s_0+1}$ are collapsed in p. At stage $s_0 + 1$, we defined $\tau^0_{s_0} \prec x$ with the property that $[\sigma \circ \tau^0_{s_0}] \subseteq R$. Since $\sigma \circ \tau^0_s \prec p$, we have $p \in R$.

Claim. $p \in \cap_{n<\omega} D_n$.

Fix $k \in \omega$ and we show $p \in D_k$. Let $s = \max\{k, s_0\}$. Consider the action during stage $s+1$ of the construction. Let $\sigma_j \in (\omega)^2_{\mathrm{fin}}$ with $|\sigma_j| = s+2$ describe how p collapses $B^x_0, \dots, B^x_{s+1}$ into B^p_0 and B^p_1. We defined δ^j_s and τ^{j+1}_s such that $\sigma_j \circ \tau^{j+1}_s = \delta^j_s$ and $[\delta^j_s] \subseteq \cap_{n<s+1} D_n$, so in particular, $[\delta^s_j] \subseteq D_k$. Since $\tau^{j+1}_s \prec x$ contains the least elements of the first $(s+2)$ many x-blocks, we have $\delta^j_s = \sigma_j \circ \tau^{j+1}_s \prec p$, so $p \in D_k$ as required. $\square$

The next proposition is standard, but we present the proof because some details will be relevant to Theorem 5.4. In the proof, we use codes for open sets as in Definition 3.1 and we equate a partition $p \in (\omega)^2$ (namely, a surjection $p : \omega \to 2$) with the set for which p is the characteristic function.

Proposition 5.3. *Let $n \in \omega$ and let $A \subseteq 2^\omega$ be defined by a Σ^0_{n+1} predicate. There is a Δ^0_{n+1} code U for an open set in $(\omega)^2$, a Δ^0_{n+2} code V for an open set in $(\omega)^2$ and a uniformly Δ^0_{n+1} sequence $\langle D_i : i \in \omega\rangle$ of codes for dense open sets such that $U \cup V$ is dense and for all $p \in \cap_{i\in\omega} D_i$, if $p \in U$, then $p \in A$ and if $p \in V$ then $p \notin A$. Furthermore, the Δ^0_{n+1} and Δ^0_{n+2} indices for U, V and $\langle D_i : i \in \omega\rangle$ can be obtained uniformly from a Σ^0_{n+1} index for A.*

Proof. We proceed by induction on n. Throughout this proof, σ, τ, ρ and δ denote elements of $(\omega)^2_{\mathrm{fin}}$. In addition to the properties stated in the proposition, we ensure that if $\langle m, \sigma\rangle \in U$ (or V) and $\tau \succeq \sigma$, then there is a k such that $\langle k, \tau\rangle \in U$ (or V respectively). Thus, if $U \cap [\sigma] \neq \emptyset$, then there is $\langle k, \tau\rangle \in U$ with $\sigma \preceq \tau$.

For $n = 0$, we have $X \in A \Leftrightarrow \exists k\, \exists m\, P(m, X \upharpoonright k)$ where $P(x, y)$ is a Π^0_0 predicate. Without loss of generality, we assume that if $P(m, X \upharpoonright k)$ holds, then $P(m', Y \upharpoonright k')$ holds for all $k' \geq k$, $m' \geq m$ and $Y \in 2^\omega$ such that $Y \upharpoonright k = X \upharpoonright k$. Let $U = \{\langle n, \sigma\rangle : P(\sigma, n)\}$, $V = \{\langle 0, \sigma\rangle : \forall x\, \forall \tau \succeq \sigma\, (\neg P(\tau, x))\}$ and $D_i = (\omega)^2_{\mathrm{fin}}$ for $i \in \omega$. It is straightforward to check these codes have the required properties.

For the induction case, let $A \subseteq 2^\omega$ be defined by a Σ^0_{n+2} predicate, so $X \in A \Leftrightarrow \exists k P(X, k)$ where P is a Π^0_{n+1} predicate. For $k \in \omega$, let $A_k = \{X : \neg P(X, k)\}$. Apply the induction hypothesis to A_k to fix indices (uniformly in k) for the Δ^0_{n+1} codes U_k and $\langle D_{i,k} : i \in \omega\rangle$ and for the Δ^0_{n+2} code V_k so that if $p \in \cap_{i\in\omega} D_{i,k}$, then $p \in U_k$ implies $\neg P(k, p)$ and $p \in V_k$ implies $P(k, p)$. Let

$$U = \{\langle\langle k, m\rangle, \sigma\rangle : \langle m, \sigma\rangle \in V_k\} \text{ and}$$
$$V = \{\langle 0, \sigma\rangle : \forall k\, \forall \tau \succeq \sigma\, \exists m\, \exists \rho \succeq \tau\, \langle m, \rho\rangle \in U_k\}.$$

U is a Δ^0_{n+2} code for $\cup_k V_k$, and V is a Δ^0_{n+3} code such that $\langle m, \sigma\rangle \in V$ if and only if every U_k is dense in $[\sigma]$. We claim that $U \cup V$ is dense. Fix σ and assume $U \cap [\sigma] = \emptyset$, so $V_k \cap [\sigma] = \emptyset$ for all k. Since $U_k \cup V_k$ is dense, $U_k \cap [\tau] \neq \emptyset$ for all $\tau \succeq \sigma$ and all k, so $\langle 0, \sigma\rangle \in V$.

For $i = \langle a_i, b_i\rangle$, define $D_i = D_{a_i,b_i} \cap (U_i \cup V_i)$. D_i has a Δ^0_{n+2} code as a dense open set and the index can be uniformly computed from the indices for U_i, V_i and D_{a_i,b_i}. Furthermore, if $p \in \cap_i D_i$ then $p \in \cap_{i,k} D_{i,k}$ and $p \in \cap_k (U_k \cup V_k)$.

Assume that $p \in \cap_i D_i$. First, we show that if $p \in U$, then $p \in A$. Suppose $p \in U = \cup_k V_k$ and fix k such that $p \in V_k$. Since $p \in \cap_i D_{i,k}$ for this fixed k, $p \notin A_k$ by the induction hypothesis. Therefore, $P(k, p)$ holds and hence $p \in A$.

Second, we show that if $p \in V$ then $p \notin A$. Assume $p \in V$ and fix $\langle 0, \sigma\rangle \in V$ such that $\sigma \prec p$. It suffices to show $\neg P(k, p)$ holds for an arbitrary $k \in \omega$. Since $p \in \cap_i D_i$, we have $p \in U_k \cup V_k$ and $p \in \cap_i D_{i,k}$. If $p \in U_k$, then $\neg P(k, p)$ holds by induction and we are done. Therefore, suppose for a contradiction that $p \in V_k$. Fix

$\langle 0, \tau \rangle \in V_k$ such that $\sigma \preceq \tau$ and $\tau \prec p$. Since $\langle 0, \sigma \rangle \in V$ and $\sigma \preceq \tau$, there are $\rho \succeq \tau$ and m such that $\langle m, \rho \rangle \in U_k$, and therefore $[\rho] \subseteq U_k \cap V_k$. This containment is the desired contradiction because $q \in [\rho] \cap \cap_i D_{i,k}$ would satisfy $q \in A_k$ and $q \notin A_k$. $\square$

Theorem 5.4. *For every coloring $(\omega)^2 = R \cup \overline{R}$ such that R is a computable code for a Σ^0_{n+2} set, there is either a $\emptyset^{(n)}$-computable $x \in (\omega)^\omega$ which is homogeneous for $\overline{R}$ or a $\emptyset^{(n+1)}$-computable $x \in (\omega)^\omega$ which is homogeneous for R.*

Proof. Fix R and fix a Π^0_{n+1} predicate $P(k, y)$ such that for $y \in (\omega)^2$, $y \in R \Leftrightarrow \exists k\, P(k, y)$. Let U_k, V_k and $\langle D_{i,k} : i \in \omega \rangle$ be the codes from Proposition 5.3 for $R_k = \{y : \neg P(y, k)\}$. Let $U = \cup_k V_k$, $V = \cup\{[\sigma] : \forall k\ U_k \text{ is dense in } [\sigma]\}$ and D_i, $i \in \omega$, be the corresponding codes for R. We split non-uniformly into cases.

Case 1: Assume V is dense in $[0^\ell]$ for some fixed ℓ. We make two observations. First, U is disjoint from $[0^\ell]$. Therefore, each V_k is disjoint from $[0^\ell]$ and hence each U_k is dense in $[0^\ell]$. Second, suppose $y \in (\bigcap_{i,k} D_{i,k}) \cap (\bigcap_k U_k)$. For each k we have $y \in \cap_i D_{i,k}$ and $y \in U_k$, so $\forall k\, \neg P(k, y)$ holds and hence $y \in \overline{R}$.

We apply Theorem 5.2 relativized to $\emptyset^{(n)}$ to the computable open set $O = [0^\ell]$ (which has nonempty intersection with $[0^j]$ for every j) and the $\emptyset^{(n)}$-computable sequence of codes $D_{i,k}$ and U_k for $i, k < \omega$. By the first observation, each coded set in this sequence is dense in O. Therefore, there is a $\emptyset^{(n)}$-computable $x \in (\omega)^\omega$ such that $(x)^2 \subseteq [0^\ell] \cap (\bigcap_{i,k} D_{i,k}) \cap (\bigcap_k U_k)$. By the second observation, $(x)^2 \subseteq \overline{R}$ as required.

Case 2: Assume V is not dense in $[0^m]$ for any m. In this case, since $U \cup V$ is dense, we have $U \cap [0^m] \neq \emptyset$ for all m. We apply Theorem 5.2 relativized to $\emptyset^{(n+1)}$ to the $\emptyset^{(n+1)}$-computable open set U and the $\emptyset^{(n+1)}$-computable sequence of dense sets D_i for $i \in \omega$ to obtain an $\emptyset^{(n+1)}$-computable x with $(x)^2 \subseteq U \cap (\bigcap_i D_i) \subseteq R$ as required. $\square$

We end this section by showing that the non-uniformity in the proof of Theorem 5.1 is necessary.

Theorem 5.5. *For every Turing functional Δ, there are computable codes R_0 and R_1 for complementary open sets in $(\omega)^2$ such that $\Delta^{R_0 \oplus R_1}$ is not an infinite homogeneous partition for the reduced coloring $(\omega)^2 = R_0 \cup R_1$.*

Proof. Fix Δ. We define R_0 and R_1 in stages as $R_{0,s}$ and $R_{1,s}$. Our construction proceeds in a basic module while we wait for $\Delta_s^{R_{0,s} \oplus R_{1,s}}$ to provide appropriate computations. If these computations appear, we immediately diagonalize and complete the construction.

For the basic module at stage s, put $0^{2s+1}1 \in R_{0,s}$ and $0^{2s+2}1 \in R_{1,s}$. Check whether there is a $0 < k < s$ such that $\Delta_s^{R_{0,s} \oplus R_{1,s}}(i) = 0$ for all $i < k$ and $\Delta_s^{R_{0,s} \oplus R_{1,s}}(k) = 1$. If there is no such k, then we proceed to stage $s + 1$ and continue with the basic module.

If there is such a k, then we stop the basic module and fix $i < 2$ such that $0^k 1 \in R_{i,s}$. (Since $k < s$, we have already enumerated $0^k 1$ into one of $B_{0,s}$ or $B_{1,s}$ depending on whether k is even or odd.) We end the construction at this stage and define $R_i = R_{i,s}$ and $R_{1-i} = R_{1-i,s} \cup \{0^t 1 \mid 2s + 2 < t\}$.

This completes the construction. It is clear that R_0 and R_1 are computable codes for complementary open sets and $(\omega)^2 = R_0 \cup R_1$ is a reduced coloring. If the construction never finds an appropriate value k, then $\Delta^{R_0 \oplus R_1}$ is not an element of

$(\omega)^\omega$ and we are done. Therefore, assume we find an appropriate value k at stage s in the construction. Fix i such that $0^k 1 \in R_{i,s}$ and assume that $p = \Delta^{R_0 \oplus R_1}$ is a element of $(\omega)^\omega$. We show p is not homogeneous by giving elements $q_0, q_1 \in (p)^2$ such that $q_0 \in R_i$ and $q_1 \in R_{1-i}$.

By construction, $[0,k) \subseteq B_0^p$ and $k = \mu^p(1) \in B_1^p$. Let $q_0 \in (p)^2$ be any coarsening such that $\mu^{q_0}(1) = k$. Since $[0,k) \subseteq B_0^{q_0}$ and $k \in B_1^{q_0}$, we have $0^k 1 \prec q_0$ and so $q_0 \in [0^k 1] \subseteq R_i$ as required.

On the other hand, since $p \in (\omega)^\omega$, there are infinitely many p-blocks. Let B_n^p be the first p-block with $\mu^p(n) > 2s+2$. Let $q_1 \in (p)^2$ be any coarsening for which $\mu^{q_1}(1) = \mu^p(n)$ and hence $q_1 \in [0^{\mu^p(n)} 1]$. Since $\mu^p(n) > 2s+2$, we put $0^{\mu^p(n)} 1 \in R_{1-i}$, so $q_1 \in R_{1-i}$ as required. □

5.2. **Strong reductions for reduced colorings.** A reduced coloring $(\omega)^2 = R_0 \cup R_1$ is classically open and the color of $p \in (\omega)^2$ depends only on $\mu^p(1)$. When R_0 and R_1 are codes for open sets, there is a homogeneous partition computable in $R_0 \oplus R_1$, although by Theorem 5.5, not uniformly. We consider the case when the open sets R_0 and R_1 are represented by Borel codes for Σ_n^0 sets with $n \geq 2$.

Δ_n^0-rDRT_2^2 is the statement that for each reduced coloring $(\omega)^2 = R_0 \cup R_1$ where R_0 and R_1 are Borel codes for Σ_n^0 sets, there exists an $x \in (\omega)^\omega$ and an $i < 2$ such that $(x)^2 \subseteq R_i$. In effective algebra, this statement is clear, but in RCA_0, we need to specify how to handle these codes.

A Borel code for a Σ_n^0 set is a labelled subtree of $\omega^{<n+1}$ which, in this section, we write as (B, ℓ) to specify the labeling function ℓ. The labels come from the set $S = \{\cup, \cap\} \cup L$ where $\cup$ is the label for an interior node to denote a union, $\cap$ is the label for an interior node to denote an intersection and L is the set of labels for the leaves, namely our fixed codes for the clopen sets $\emptyset$, $(\omega)^2$ $[\tau]$ and $\overline{[\tau]}$ for $\tau \in (\omega)^2_{\text{fin}}$. For a leaf σ and a partition p, we write $p \in \ell(\sigma)$ if p is an element of the clopen set coded by $\ell(\sigma)$. Similarly, we write $\ell(\sigma) = [\tau]$ to avoid specifying a coding scheme. Since this code is for a Σ_n^0 set, we require that $\ell(\lambda) = \cup$.

We construct a Σ_n^0 formula $\eta(B, \ell, p)$ such that if (B, ℓ) is a Borel code for a Σ_n^0 set and $p \in (\omega)^2$, then $\eta(B, \ell, p)$ says p is in the set coded by (B, ℓ). We begin by defining formulas $\beta_k(\sigma, B, \ell, p)$ for $1 \leq k \leq n$ by downward induction on k. For $\sigma \in B$ with $|\sigma| = k$, $\beta_k(\sigma, B, \ell, p)$ says that p is in the set coded by the labeled subtree of (B, ℓ) above σ. Since any $\sigma \in B$ with $|\sigma| = n$ is a leaf, $\beta_n(\sigma, B, \ell, p)$ is the formula $p \in \ell(\sigma)$. For $1 \leq k < n$, $\beta_k(\sigma, B, \ell, p)$ is the formula

$$(\ell(\sigma) = \cup \to \alpha_k^\cup) \wedge (\ell(\sigma) = \cap \to \alpha_k^\cap) \wedge (\ell(\sigma) \in L \to \alpha_k^L), \text{ where}$$
$$\alpha_k^\cup(\sigma, B, \ell, p) \text{ is } \exists \tau \in B \big(\sigma \prec \tau \wedge |\tau| = k+1 \wedge \beta_{k+1}(\tau, B, \ell, p)\big)$$
$$\alpha_k^\cap(\sigma, B, \ell, p) \text{ is } \forall \tau \in B \big((\sigma \prec \tau \wedge |\tau| = k+1) \to \beta_{k+1}(\tau, B, \ell, p)\big)$$
$$\text{and } \alpha_k^L(\sigma, B, \ell, p) \text{ is } p \in \ell(\sigma).$$

The formula $\eta(B, \ell, p)$ is $\exists \sigma \in B\, (|\sigma| = 1 \wedge \beta_1(\sigma, B, \ell, p))$. In RCA_0, we write $p \in B$ for $\eta(B, \ell, p)$. The statement Δ_n^0-rDRT_2^2 now has the obvious translation in RCA_0.

A Borel code (B, ℓ) for a Σ_n^0 set is *in normal form* if $B = \omega^{<n+1}$ and for every σ with $|\sigma| < n$, if $|\sigma|$ is even, then $\ell(\sigma) = \cup$, and if $|\sigma|$ is odd, then $\ell(\sigma) = \cap$. In RCA_0, for every (B, ℓ), there is a $(\widehat{B}, \widehat{\ell})$ in normal form such that for all $p \in (\omega)^2$, $p \in B$ if and only if $p \in \widehat{B}$. Moreover, the transformation from (B, ℓ) to $(\widehat{B}, \widehat{\ell})$ is uniformly computable in (B, ℓ). We describe the transformation when (B, ℓ) is a Borel code for a Σ_2^0 set. The case for a Σ_n^0 set is similar.

Let (B, ℓ) be a Borel code for a Σ^0_2 set. By definition, $\lambda \in B$ with $\ell(\lambda) = \cup$. Each $\sigma \in B$ with $|\sigma| = 1$ is the root of a subtree coding a Σ^0_0 set (if $\ell(\sigma) \in L$), a Σ^0_1 set (if $\ell(\sigma) = \cup$) or a Π^0_1 set (if $\ell(\sigma) = \cap$). Consider the following sequence of transformations.

- To form (B_1, ℓ_1), for each $\sigma \in B$ with $|\sigma| = 1$ and $\ell(\sigma) = \cup$, remove the subtree of B above σ (including σ). For each $\tau \in B$ with $\tau \succ \sigma$, add a new node τ' to B_1 with $|\tau'| = 1$ and $\ell_1(\tau') = \ell(\tau) \in L$.
- To form (B_2, ℓ_2), for each leaf $\sigma \in B_1$ with $|\sigma| = 1$, relabel σ by $\ell_2(\sigma) = \cap$ and add a new successor τ to σ with label $\ell_2(\tau) = \ell_1(\sigma)$.
- To form (B_3, ℓ_3), for each $\sigma \in B_2$ with $|\sigma| = 1$, let $\tau_\sigma \in B_1$ be the first successor of σ. Add infinite many new nodes $\delta \succ \sigma$ to B_3 with $\ell_3(\delta) = \ell_2(\tau_\sigma)$.
- To form (B_4, ℓ_4), let σ be the first node of B_3 at level 1. Add infinitely many copies of the subtree above σ to B_4 with the same labels as in B_3.

In (B_4, ℓ_4), the leaves are at level 2, every interior node is infinitely branching and $\ell_4(\sigma) = \cap$ when $|\sigma| = 1$. There is a uniform procedure to define a bijection $f : B_4 \to \omega^{<3}$. We define $(\widehat{B}, \widehat{\ell})$ by $\widehat{B} = \omega^{<3}$ and $\widehat{\ell}(\sigma) = \ell_4(f^{-1}(\sigma))$. In RCA_0, for all $p \in (\omega)^2$, $\eta(B, \ell, p)$ holds if and only if $\eta(\widehat{B}, \widehat{\ell}, p)$ holds.

When (B, ℓ) is a Borel code for a Σ^0_n set in normal form, $\eta(B, \ell, p)$ is equivalent to $\exists x_0\, \forall x_1 \cdots \mathsf{Q}_{n-1} x_{n-1}\, (p \in \ell(\langle x_0, x_1, \ldots, x_{n-1}\rangle))$ where Q_{n-1} is $\forall$ or $\exists$ depending on whether $n-1$ is odd or even. We have analogous definitions for Borel codes for Π^0_n sets in normal form.

To define D^n_2, let $[\omega]^n$ denote the set of n element subsets of ω. We view the elements of $[\omega]^n$ as strictly increasing sequences $s_0 < s_1 < \cdots < s_{n-1}$.

Definition 5.6. A coloring $c : [\omega]^n \to 2$ is *stable* if for all k, the limit

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1})$$

exists. $L \subseteq \omega$ is *limit-homogeneous* for a stable coloring c if there is an $i < 2$ such that for each $k \in L$,

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = i.$$

D^n_2 is the statement that each stable coloring $c : [\omega]^n \to 2$ has an infinite limit-homogeneous set.

Below, the proof of Theorem 5.7(2) is a formalization of the proof of Theorem 5.7(1), and the additional induction used is a consequence of this formalization. We do not know if its use is necessary; that is, we do now if $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$ can be replaced simply by RCA_0 when $n > 2$.

Theorem 5.7. *Fix $n \geq 2$.*

1. Δ^0_n-$\mathsf{rDRT}^2_2 \equiv_{\mathrm{sW}} \mathsf{D}^n_2$.
2. *Over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$, Δ^0_n-rDRT^2_2 is equivalent to D^n_2.*

Corollary 5.8. *Δ^0_2-rDRT^2_2 is equivalent to SRT^2_2 over RCA_0.*

Proof. D^2_2 is equivalent to SRT^2_2 over RCA_0 by Chong, Lempp, and Yang [4]. □

Corollary 5.9. Δ^0_2-$\mathsf{rDRT}^2_2 <_{\mathrm{sW}} \mathsf{SRT}^2_2$.

Proof. $\mathsf{D}^2_2 <_{\mathrm{sW}} \mathsf{SRT}^2_2$ by Dzhafarov [5, Corollary 3.3]. (It also follows immediately that Δ^0_2-$\mathsf{rDRT}^2_2 \equiv_{\mathrm{W}} \mathsf{D}^2_2 <_{\mathrm{W}} \mathsf{SRT}^2_2$.) □

Proof of Theorem 5.7. We prove the two parts simultaneously, remarking, where needed, how to formalize the argument in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$.

To show that Δ^0_n-$\mathsf{rDRT}^2_2 \leq_{\mathrm{sW}} \mathsf{D}^n_2$, and that Δ^0_n-rDRT^2_2 is implied by D^n_2 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$, fix an instance $(\omega)^2 = R_0 \cup R_1$ of Δ^0_n-rDRT^2_2 where each R_i is a Borel code for a Σ^0_n set. Without loss of generality, R_0 and R_1 are in normal form. For each $k \geq 1$, fix the partition p_k defined by $B^{p_k}_0 = \omega - \{k\}$ and $B^{p_k}_1 = \{k\}$.

For $m < n$, we let $R_i(t_0, \ldots, t_m)$ denote the Borel set coded by the subtree of R_i above $\langle t_0, \ldots, t_m\rangle$. Since $\langle t_0, \ldots, t_{n-1}\rangle$ is a leaf, $R_i(t_0, \ldots, t_{n-1})$ is the clopen set $\ell_i(\langle t_0, \ldots, t_{n-1}\rangle)$. If $m < n-1$, then $R_i(t_0, \ldots, t_m)$ is a code for a $\Sigma^0_{n-(m+1)}$ set (if m is odd) or a $\Pi^0_{n-(m+1)}$ set (if m is even) in normal form.

We define a coloring $c : [\omega]^n \to 2$ as follows. Let $c(0, s_1, \ldots, s_{n-1}) = 0$ for all $s_1 < \cdots < s_{n-1}$. For $m \leq n$, let Q_m stand for $\exists$ or $\forall$, depending as m is even or odd, respectively. Given $1 \leq k < s_1 < \ldots < s_{n-1}$, define

$$c(k, s_1, \ldots, s_{n-1}) = 1$$

if and only if there is a $t_0 \leq s_1$ such that

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m) \cdots (\mathsf{Q}_{n-1} t_{n-1} \leq s_{n-1})\ p_k \in \ell_0(\langle t_0, \ldots, t_{n-1}\rangle)$$

and for which there is no $u_0 < t_0$ such that

$$(\forall u_1 \leq s_1) \cdots (\mathsf{Q}_m u_m \leq s_m) \cdots (\mathsf{Q}_{n-1} u_{n-1} \leq s_{n-1})\ p_k \in \ell_1(\langle u_0, \ldots u_{n-1}\rangle).$$

(Note that s_1 bounds t_0, t_1 and u_1, whereas the other s_m bound only t_m and u_m.) The coloring c is uniformly computable in (R_0, ℓ_0) and (R_1, ℓ_1) and is definable in RCA_0 as a total function since all the quantification is bounded.

We claim that for each $k \geq 1$,

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1})$$

exists. Furthermore, if this limit equals 1, then $p_k \in R_0$, and if this limit equals 0, then $p_k \in R_1$. We break this claim into two halves.

First, for $1 \leq m \leq n-1$, we claim that for all fixed $1 \leq k < s_1 < \ldots < s_m$,

$$\lim_{s_{m+1}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_m, s_{m+1}, \ldots, s_{n-1})$$

exists, and the limit equals 1 if and only if there is a $t_0 \leq s_1$ such that

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m)\ p_k \in R_0(t_0, \ldots, t_m) \tag{1}$$

and there is no $u_0 < t_0$ such that

$$(\forall u_1 \leq s_1) \cdots (\mathsf{Q}_m u_m \leq s_m)\ p_k \in R_1(u_0, \ldots, u_m). \tag{2}$$

The proof is by downward induction on m. (In RCA_0, the induction is performed externally, so we do not need to consider its complexity.) For $m = n-1$, there are no limits involved and the values of c are correct by definition.

Assume the result is true for $m+1$ and we show it remains true for m. By the definition of $R_0(t_0, \ldots, t_m)$, t_0 satisfies (1) if and only if

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m)(\mathsf{Q}_{m+1} t_{m+1})\ p_k \in R_0(t_0, \ldots, t_m, t_{m+1}),$$

which in turn holds if and only if there is a bound v such that for all $s_{m+1} \geq v$,

$$(\forall t_1 \leq s_1) \cdots (\mathsf{Q}_m t_m \leq s_m)(\mathsf{Q}_{m+1} t_{m+1} \leq s_{m+1})\ p_k \in R_0(t_0, \ldots, t_m, t_{m+1}).$$

If Q_{m+1} is $\exists$, then over RCA_0, this equivalence requires a bounding principle. Since $p_k \in R_0(t_0, \ldots, t_{m+1})$ is a $\Pi^0_{n-(m+2)}$ predicate and $m+2 \geq 3$, we need at most

$\mathsf{B}\Pi^0_{n-3}$ which follows from $\mathsf{I}\Sigma^0_{n-1}$. An analogous analysis applies to numbers u_0 satisfying (2). Thus, we can fix a common bound v that works for all $t_0 \leq s_1$ in (1) and all $u_0 < t_0 \leq s_1$ in (2).

Suppose there is a $t_0 \leq s_1$ satisfying (1) for which there is no $u_0 < t_0$ satisfying (2). Then, for all $s_{m+1} \geq v$, t_0 satisfies the version of (1) for $m+1$, and there is no $u_0 < t_0$ satisfying the version of (2) for $m+1$. Therefore, by induction

$$\exists v \forall s_{m+1} \geq v \big(\lim_{s_{m+2}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 1 \big)$$

and hence $\lim_{s_{m+1}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 1$ as required.

On the other hand, suppose that there is no $t_0 \leq s_1$ satisfying (1), or that for every $t_0 \leq s_1$ satisfying (1), there is a $u_0 < t_0$ satsifying (2). Then, for all $s_{m+1} \geq v$, we have the analogous condition for $m+1$ and the induction hypothesis gives $\lim_{s_{m+1}} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 0$. This completes the first part of the claim.

We can now prove the rest of the claim. For each $k \geq 1$, we have $p_k \in R_0$ or $p_k \in R_1$. Let t_0 be least such that $p_k \in R_0(t_0)$ or $p_k \in R_1(t_0)$. Since $p_k \in R_i(t)$ is a Π^0_{n-1} statement, we use $\mathsf{I}\Sigma^0_{n-1}$ to fix this value in RCA_0.

Suppose $p_k \in R_0(t_0)$, so for all $u_0 < t_0$, it is not the case that $p_k \in R_1(u_0)$. By the first half of the claim with $m = 1$, we have for every $s_1 \geq t_0$

$$\lim_{s_2} \cdots \lim_{s_{n-1}} c(k, s_1, s_2, \ldots, s_{n-1}) = 1,$$

and therefore $\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 1$.

Suppose $p_k \notin R_0(t_0)$, and hence $p_k \in R_1(t_0)$. Again, by the first half of the claim with $m = 1$, we have for every $s_1 \geq t_0$

$$\lim_{s_2} \cdots \lim_{s_{n-1}} c(k, s_1, s_2, \ldots, s_{n-1}) = 0,$$

so $\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = 0$. This completes the proof of the claim.

Since c is an instance of D^n_2, fix $i < 2$ and an infinite limit-homogeneous set L for c with color i. By the claim, $p_k \in R_{1-i}$ for all $k \in L$. List the non-zero elements of L as $k_0 < k_1 < \cdots$, and let $p \in (\omega)^\omega$ be the partition whose blocks are $[0, k_0)$ and $[k_m, k_{m+1})$ for $m \in \omega$. Each $x \in (p)^2$ satisfies $\mu^x(1) = k_m$ for some m. Since $R_0 \cup R_1$ is a reduced coloring, x and p_{k_m} have the same color, which is R_{1-i}. Since x was arbitrary, $(p)^2 \subseteq R_{1-i}$ as required to complete this half of the theorem.

Next, we show that $\mathsf{D}^n_2 \leq_{\mathrm{sW}} \Delta^0_n\text{-}\mathsf{rDRT}^2_2$, and that D^n_2 is implied by $\Delta^0_n\text{-}\mathsf{rDRT}^2_2$ over RCA_0. (No extra induction is necessary for this implication.) Fix an instance $c : [\omega]^n \to 2$ of D^n_2, and define a partition $R_0 \cup R_1$ of $(\omega)^2$ as follows. For $x \in (\omega)^2$ with $\mu^x(1) = k$, $x \in R_i$ for the unique i such that

$$\lim_{s_1} \cdots \lim_{s_{n-1}} c(k, s_1, \ldots, s_{n-1}) = i.$$

Since each of the iterated limits is assumed to exist over what follows on the right, we may express these limits by alternating Σ^0_2 and Π^0_2 definitions, as

$$(\exists t_1 \forall s_1 \geq t_1)(\forall t_2 \geq s_1 \exists s_2 \geq t_2) \cdots c(k, s_1, \ldots, s_{n-1}) = i.$$

Thus, R_0 and R_1 are Σ^0_n-definable open subsets of $(\omega)^2$. By standard techniques, there are Borel codes for R_0 and R_1 as Σ^0_n sets uniformly computable in c and in RCA_0. (Below, we illustrate this process for D^3_2.)

By definition, $(\omega)^2 = R_0 \cup R_1$ is a reduced coloring and hence is an instance of Δ^0_n-rDRT^2_2. Let $p \in (\omega)^\omega$ be a solution to this instance, say with color $i < 2$. Thus, for every $x \in (p)^2$, the limit color of $k = \mu^x(1)$ is i. Define $L = \{\mu^p(m) : m \geq 1\}$. Since for each $k \in L$, there is an $x \in (p)^2$ such that $\mu^x(1) = k$, L is limit-homogeneous for c with color i.

We end this proof by illustrating how to define the Borel codes for R_0 and R_1 as Σ^0_3 sets from a stable coloring $c(k, s_1, s_2)$. In this case, we have

$$\lim_{s_1} \lim_{s_2} c(k, s_1, s_2) = i \Leftrightarrow \exists t_1 (\forall s_1 \geq t_1 \, \forall t_2 \geq s_1)(\exists s_2 \geq t_2) \; c(k, s_1, s_2) = i.$$

The nodes in each R_i are the initial segments of the strings $\langle \langle k, t_1 \rangle, \langle s_1, t_2 \rangle, s_2 \rangle$ for $k \leq t_1 < s_1 \leq t_2 < s_2$ and the labeling functions are $\ell_i(\sigma) = \cup$ if $|\sigma| \in \{0, 2\}$, $\ell_i(\sigma) = \cap$ if $|\sigma| = 1$ and $\ell_i(\langle \langle k, t_1 \rangle, \langle s_1, t_2 \rangle, s_2 \rangle) = [0^k 1]$ if $c(k, s_1, s_2) = i$ and is equal to $\emptyset$ if $c(k, s_1, s_2) = 1 - i$. It is straightforward to check in RCA_0 that R_i represents the union of clopen sets $[0^k 1]$ such that the limit color of k is i. $\square$

6. Reverse math and Borel codes

In this section, we define Borel codes in second order arithmetic and prove Theorem 3.14. Although we give definitions specific to the setting of $(\omega)^k$, we assume the reader is familiar with ATR_0 as well as the Turing jump and the hyperarithmetic hierarchy in second order arithmetic from Simpson [16] Chapters V and VIII. We begin by defining a Borel code for a subset of $(\omega)^k$ in RCA_0. Although the tree structure is similar to Definition 4.1, these codes are not defined inductively and the label for each node is coded by its last numerical entry. Let $\tau_0, \tau_1, \cdots$ be a fixed enumeration of $(\omega)^k_{\text{fin}}$ and let λ denote the empty string.

Definition 6.1 (RCA_0)**.** A *Borel code for a subset of* $(\omega)^k$ is a tree $B \subseteq \omega^{<\omega}$ with no infinite path such that there is exactly one $m \in \omega$ (denoted m_B) with $\langle m \rangle \in B$. A Borel code is *trivial* if $\sigma(|\sigma| - 1) \in \{0, 1\}$ for every leaf $\sigma \in B$.

Definition 6.2 (RCA_0)**.** Let B be a Borel code for a subset of $(\omega)^k$ and $p \in (\omega)^k$. An *evaluation map for* B *at* p is a function $f : B \to \{0, 1\}$ such that for all $\sigma \in B$ and $n = |\sigma| - 1$

- if σ is a leaf, then
$$f(\sigma) = \begin{cases} \sigma(n) & \text{if } \sigma(n) \in \{0,1\} \\ 1 & \text{if } \sigma(n) = 2m+2 \text{ and } \tau_m \prec p \\ 1 & \text{if } \sigma(n) = 2m+3 \text{ and } \tau_m \not\prec p \\ 0 & \text{otherwise} \end{cases}$$
- if $\sigma \neq \lambda$ is not a leaf, then
$$f(\sigma) = \begin{cases} 1 & \text{if } \sigma(n) \text{ is even and } \exists m \, (\sigma^\frown m \in B \wedge f(\sigma^\frown m) = 1) \\ 1 & \text{if } \sigma(n) \text{ is odd and } \forall m \, (\sigma^\frown m \in B \to f(\sigma^\frown m) = 1) \\ 0 & \text{otherwise} \end{cases}$$
- and $f(\lambda) = f(\langle m_B \rangle)$.

We write $p \in B$ if there is an evaluation map for B at p with $f(\lambda) = 1$ and $p \notin B$ if there is an evaluation map f for B at p with $f(\lambda) = 0$.

The leaf nodes of a Borel code B code the basic clopen sets $\emptyset$, $(\omega)^k$, $[\tau_m]$ or $\overline{[\tau_m]}$ depending on the last entry in σ. The interior nodes of B code either a union (if the last entry is even) or an intersection (if the last entry is odd). A trivial Borel code represents a set built from $\emptyset$ and $(\omega)^k$ using unions and intersections. Since ACA_0 suffices to prove that the Kleene-Brouwer order on B is a well order and to prove arithmetic transfinite induction, ACA_0 proves that an evaluation map for B at p is unique, provided it exists.

For a binary string σ, let $\overline{\sigma}$ be defined by $|\overline{\sigma}| = |\sigma|$, $\overline{\sigma}(n) = 2m$ if $\sigma(n) = 2m+1$, and $\overline{\sigma}(n) = 2m+1$ if $\sigma(n) = 2m$. For a Borel code B, let $\overline{B} = \{\overline{\sigma} : \sigma \in B\}$. RCA_0 proves that if B is a Borel code, then $\overline{B}$ is a Borel code for the complement of B in the sense that for every $p \in (\omega)^k$, f is an evaluation map for B at p if and only if $\overline{f}(\overline{\sigma}) = 1 - f(\sigma)$ is an evaluation map for $\overline{B}$ at p. In particular, $\overline{f}(\lambda) = 1 - f(\lambda)$.

Lemma 6.3 (RCA_0)**.** *For every code O for an open set, there is a Borel code B such that $(\omega)^k = B \cup \overline{B}$ and for all $x \in (\omega)^k$, $x \in B$ if and only if $x \in O$.*

Proof. Fix O. Let B contain λ, $\langle 0\rangle$ and, for all $\langle s, \tau_m\rangle \in O$, both $\langle 0, \langle s, \tau_m\rangle\rangle$ and $\langle 0, \langle s, \tau_m\rangle, 2m+2\rangle$. We claim that for every $x \in (\omega)^k$, there is a unique evaluation map f for B at x, and $f(\lambda) = 1$ if and only if $x \in O$. To prove this claim, we define two potential evaluation functions, f_0 and f_1, and show that one of them is correct.

For each $i < 2$ and leaf $\tau = \langle 0, \langle s, \tau_m\rangle, 2m+2\rangle$, let $f_i(\langle 0, \langle s, \tau_m\rangle\rangle) = f_i(\tau) = 1$ if $\tau_m \prec x$ and have value 0 otherwise. Note that $f_i(\langle 0, \langle s, \tau_m\rangle\rangle)$ is correctly defined because τ is the unique successor of $\langle 0, \langle s, \tau_m\rangle\rangle$ and therefore $f_i(\langle 0, \langle s, \tau_m\rangle\rangle) = f_i(\tau)$ regardless of whether $\langle s, \tau_m\rangle$ is coded by an even or odd number. Set $f_i(\lambda) = f_i(\langle 0\rangle) = i$. If there is a pair $\langle s, \tau_m\rangle \in O$ with $\tau_m \prec x$, then f_1 satisfies the conditions for an evaluation function and hence $x \in B$. Otherwise, f_0 is an evaluation function and $x \notin B$. In either case, the corresponding f_i is the unique evaluation function for B at x and it agrees with whether $x \in O$ or $x \in \overline{O}$. □

If B is a trivial Borel code, then an evaluation map for B at p is independent of p, so we can refer to an evaluation map f for B. Below, we show the statement "every trivial Borel code has an evaluation map" implies ACA_0 over RCA_0. We prove a form of effective transfinite recursion in ACA_0 and use this recursion method to show "every trivial Borel code has an evaluation map" implies ATR_0. The main ideas in the effective transfinite recursion are similar to those in Section 7.7 of Ash and Knight [1]. Since "for every Borel code B, there is a p such that $p \in B$ or $p \notin B$" implies "every trivial Borel code has an evaluation map" these results show (2) implies (1) in Theorem 3.14. Because we work with trivial Borel codes, the underlying topological space does not matter as long as Borel codes are defined in a manner similar to Definitions 6.1 and 6.2. For example, Theorem 3.14 holds for Borel codes of subsets of 2^ω or ω^ω as defined in Simpson [16].

Proposition 6.4 (RCA_0)**.** *The statement "every trivial Borel code has an evaluation map" implies* ACA_0*.*

Proof. Fix $g : \omega \to \omega$ and we show range(g) exists. Let B be the trivial Borel code consisting of the initial segments of $\langle 0, 2n, m, 1\rangle$ for $g(m) = n$ and $\langle 0, 2n, m, 0\rangle$ for $g(m) \neq n$. Let f be an evaluation function for B.

Assume $g(m) = n$ and we show $f(\langle 0, 2n\rangle) = 1$. By definition, $\langle 0, 2n, m, 1\rangle \in B$ is a leaf and $f(\langle 0, 2n, m, 1\rangle) = 1$. Since $\langle 0, 2n, m\rangle$ has only one successor in B, $f(\langle 0, 2n, m\rangle) = 1$ regardless of whether m is even or odd. Since $2n$ is even, it follows that $f(\langle 0, 2n\rangle) = 1$.

Similarly, if $n \notin$ range(g), then $f(\langle 0, 2n\rangle) = 0$ because all the leaves extending $\langle 0, 2n\rangle$ have the form $\langle 0, 2n, m, 0\rangle$ and $f(\langle 0, 2n, m, 0\rangle) = 0$. Therefore, range$(g) = \{n : f(\langle 0, 2n\rangle) = 1\}$. □

Let $LO(X)$ and $WO(X)$ be the standard formulas in second order arithmetic saying X is a linear order and X is a well order. We abuse notation and write $x \in X$ in place of $x \in \text{field}(X)$. For a formula $\varphi(n, X)$, $H_\varphi(X, Y)$ is the formula stating $LO(X)$ and $Y = \{\langle n, j\rangle : j \in X \wedge \varphi(n, Y^j)\}$ where $Y^j = \{\langle m, a\rangle : a <_X j \wedge \langle m, a\rangle \in Y\}$. When φ is arithmetic, $H_\varphi(X, Y)$ is arithmetic and ACA_0 proves that if $WO(X)$, then there is at most one Y such that $H_\varphi(X, Y)$. We define our formal version of effective transfinite recursion.

Definition 6.5. ETR is the axiom scheme

$$\forall X \left[\big(WO(X) \wedge \forall Y \, \forall n \, (\varphi(n, Y) \leftrightarrow \neg \psi(n, Y)) \big) \rightarrow \exists Y \, H_\varphi(X, Y) \right]$$

where φ and ψ range over Σ^0_1 formulas.

We show that ETR is provable in ACA_0. Following Simpson [16], we avoid using the recursion theorem and note that the only place the proof goes beyond RCA_0 is in the use of transfinite induction for Π^0_2 formulas, which holds is ACA_0 and is equivalent to transfinite induction for Σ^0_1 formulas. Greenberg and Montalbán [7] point out that ETR can also be proved using the recursion theorem, although this proof also uses Σ^0_1 transfinite induction.

Proposition 6.6. ETR *is provable in* ACA_0.

Proof. Fix a well order X and Σ^0_1 formulas φ and ψ. Throughout this proof, we let f, g and h be variables denoting finite partial functions from ω to $\{0, 1\}$ coded in the canonical way as finite sets of ordered pairs. We write $f \preceq g$ (or $f \prec X$) if $f \subseteq g$ (or $f \subseteq \chi_X$) as sets of ordered pairs. By the usual normal form results (e.g. Theorem II.2.7 in Simpson), we fix a Σ^0_0 formula φ_0 such that

$$\forall Y \, \forall n \, \big(\varphi(n, Y) \leftrightarrow \exists f \, (f \prec Y \wedge \varphi_0(n, f)) \big)$$

and such that if $\varphi_0(n, f)$ and $f \prec g$, then $\varphi_0(n, g)$. We fix a formula ψ_0 related to ψ in the same manner. Since $\varphi(n, Y) \leftrightarrow \neg\psi(n, Y)$, we cannot have compatible f and g such that $\varphi_0(n, f)$ and $\psi_0(n, g)$.

Our goal is to use partial functions f as approximations to a set Y such that $H_\varphi(X, Y)$. Therefore, we view $\text{dom}(f)$ as consisting of coded pairs $\langle n, a\rangle$. For f to be a suitable approximation to Y, we need that if $\langle n, a\rangle \in \text{dom}(f)$ and $a \notin X$, then $f(\langle n, a\rangle) = 0$. Similarly, if f is an approximation to Y^j, we need that $f(\langle n, a\rangle) = 0$ whenever $\langle n, a\rangle \in \text{dom}(f)$ and $a \geq_X j$. These observations motivate the following definitions.

Let f be a finite partial function and let $i \in X$. We define

$$f^i = f \restriction \{\langle n, a\rangle : n \in \omega \wedge a <_X i\}.$$

We say $g \succeq f$ is an *i-extension of* f if for all $\langle n, a\rangle \in \text{dom}(g) - \text{dom}(f)$, $g(\langle n, a\rangle) = 0$ and either $a \notin X$ or $i \leq_X a$.

For $j \in X$, f is a *j-approximation* if the following conditions hold.

- If $\langle n, a\rangle \in \text{dom}(f)$ with $a \notin X$ or $j \leq_X a$, then $f(\langle n, a\rangle) = 0$.
- If $\langle n, a\rangle \in \text{dom}(f)$ and $a <_X j$, then
  - if $f(\langle n, a\rangle) = 1$, then there is an a-extension h of f^a such that $\varphi_0(n, h)$, and
  - if $f(\langle n, a\rangle) = 0$, then there is an a-extension h of f^a such that $\psi_0(n, h)$.

Note that if f is a j-approximation and $i <_X j$, then f^i is an i-approximation. Also, if f is a j-approximation and g is a j-extension of f, then g is a j-approximation.

Claim. For all $j \in X$, there do not exist $m \in \omega$ and j-approximations f and g such that $\varphi_0(m, f)$ and $\psi_0(m, g)$.

The proof is by transfinite induction on j. Fix the least $j \in X$ for which this property fails and fix witnesses m, f and g. To derive a contradiction, it suffices to show that f and g are compatible. Fix $\langle k, a\rangle$ such that both $f(\langle k, a\rangle)$ and $g(\langle k, a\rangle)$ are defined. If $a \notin X$ or $j \leq_X a$, then $f(\langle k, a\rangle) = g(\langle k, a\rangle) = 0$.

Suppose for a contradiction that $a <_X j$ and $f(\langle k, a\rangle) \neq g(\langle k, a\rangle)$. Without loss of generality, $f(\langle k, a\rangle) = 1$ and $g(\langle k, a\rangle) = 0$. Fix a-extensions h and h' of f^a and g^a respectively such that $\varphi_0(k, h)$ and $\psi_0(k, h')$. Since f is a j-approximation, f^a is an a-approximation, and since h is an a-extension of f^a, h is also an a-approximation. Similarly, h' is an a-approximation. Therefore, we have $k \in \omega$, $a <_X j$ and a-approximation h and h' such that $\varphi_0(k, h)$ and $\psi_0(k, h')$ contradicting the minimality of j.

Claim. For any j-approximation f and any $m \in \omega$, there is a j-approximation $g \succeq f$ such that either $\varphi_0(m, g)$ or $\psi_0(m, g)$.

The proof is again by transfinite induction on j. Fix the least j for which this property fails and fix witnesses f and m. Let $\langle n_s, i_s\rangle$ enumerate the pairs not in the domain of f. Below, we define a sequence $f = f_0 \preceq f_1 \preceq \cdots$ of j-approximations such that $f_{s+1}(\langle n_s, i_s\rangle)$ is defined. Let Y be the set with $\chi_Y = \cup_s f_s$. Either $\varphi(m, Y)$ or $\psi(m, Y)$ holds, and so there is a $g \prec Y$ such that $\varphi_0(m, g)$ or $\psi_0(m, g)$ holds. Fixing s such that $g \preceq f_s$ shows that either $\varphi_0(m, f_s)$ or $\psi_0(m, f_s)$ holds for the desired contradiction.

To define f_{s+1}, we need to extend f_s to a j-approximation f_{s+1} with $\langle n_s, i_s\rangle \in \mathrm{dom}(f_{s+1})$. We break into several cases. If $f_s(\langle n_s, i_s\rangle)$ is already defined, let $f_{s+1} = f_s$. Otherwise, if $i_s \notin X$ or $j \leq_X i_s$, set $f_{s+1}(\langle n_s, i_s\rangle) = 0$ and leave the remaining values as in f_s. In both cases, it is clear that f_{s+1} is a j-approximation.

Finally, if $i_s <_X j$ and $f_s(\langle n_s, i_s\rangle)$ is undefined, we apply the induction hypothesis to the i_s-approximation $f_s^{i_s}$ to get an i_s-approximation $g \succeq f_s^{i_s}$ such that either $\varphi_0(n_s, g)$ holds or $\psi_0(n_s, g)$ holds. Define f_{s+1} as follows.

- For $\langle m, a\rangle \in \mathrm{dom}(g)$ with $a <_X i_s$, set $f_{s+1}(\langle m, a\rangle) = g(\langle m, a\rangle)$.
- For $\langle m, a\rangle \in \mathrm{dom}(f_s)$ with $i_s \leq_X a$ or $a \notin X$, set $f_{s+1}(\langle m, a\rangle) = f_s(\langle m, a\rangle)$.
- Set $f_{s+1}(\langle n_s, i_s\rangle) = 1$ if $\varphi_0(n_s, g)$ holds and $f_{s+1}(\langle n_s, i_s\rangle) = 0$ if $\psi_0(n_s, g)$ holds.

It is straightforward to verify that $f_s \prec f_{s+1}$, g is an i_s-extension of $f_{s+1}^{i_s}$ and f_{s+1} is a j-approximation, completing the proof of the claim.

We define the set Y for which we will show $H_\varphi(X, Y)$ holds by $\langle m, j\rangle \in Y$ if and only if $j \in X$ and there is a j-approximation f such that $\varphi_0(m, f)$. It follows from the claims above that $\langle m, j\rangle \notin Y$ if and only if either $j \notin X$ or there is a j-approximation f such that $\psi_0(m, f)$. Therefore, Y has a Δ^0_1 definition. The next two claims show that $H_\varphi(X, Y)$ holds, completing our proof.

Claim. If f is a j-approximation, then $f \prec Y^j$.

Consider $\langle m, a\rangle \in \mathrm{dom}(f)$. If $a \notin X$ or $j \leq_X a$, then $f(\langle m, a\rangle) = Y^j(\langle m, a\rangle) = 0$. Suppose $a <_X j$. If $f(\langle m, a\rangle) = 1$, then there is an a-extension g of f^a such that $\varphi_0(m, g)$. Since f^a is an a-approximation and g is an a-extension of f^a, g is an a-approximation. Therefore, $\langle m, a\rangle \in Y$ by definition and hence $\langle m, a\rangle \in Y^j$. By similar reasoning, if $f(\langle m, a\rangle) = 0$, then $\langle m, a\rangle \notin Y$ and hence $\langle m, a\rangle \notin Y^j$.

Claim. $\langle m, j\rangle \in Y$ if and only if $\varphi(m, Y^j)$.

Assume that $\langle m, j\rangle \in Y$ and fix a j-approximation f such that $\varphi_0(m, f)$. Since $f \prec Y^j$, $\varphi(m, Y^j)$. For the other direction, assume that $\varphi(m, Y^j)$. Fix a j-approximation f such that either $\varphi_0(m, f)$ or $\psi_0(m, f)$. Since $f \prec Y^j$ and $\varphi(m, Y^j)$, we must have $\varphi_0(m, f)$ and therefore $\langle m, j\rangle \in Y$ by definition. □

We recall some notation and facts from Simpson [16] to state the equivalence of ATR_0 we will prove. We let $TJ(X)$ denote the Turing jump in ACA_0 given by fixing a universal Π^0_1 formula. We use the standard recursion theoretic notations Φ^X_e and $\Phi^X_{e,s}$ with the understanding that they are defined by this fixed universal formula.

$\mathcal{O}_+(a, X)$ is the arithmetic statement that $a = \langle e, i\rangle$, e is an X-recursive index of an X-recursive linear order $\leq^X_e$ and $i \in \text{field}(\leq^X_e)$. $\mathcal{O}^X_+ = \{a : \mathcal{O}_+(a, X)\}$ exists in ACA_0. For $a, b \in \mathcal{O}^X_+$, we write $b <^X_{\mathcal{O}} a$ if $a = \langle e, i\rangle$, $b = \langle e, j\rangle$ and $j <^X_e i$. For $a \in \mathcal{O}^X_+$, the set $\{b : b <^X_{\mathcal{O}} a\}$ exists in ACA_0.

$\mathcal{O}(a, X)$ is the Π^1_1 statement $\mathcal{O}_+(a, X) \wedge WO(\{b : b <^X_{\mathcal{O}} a\})$. Intuitively, $\mathcal{O}(a, X)$ says that $a = \langle e, i\rangle$ is an X-recursive ordinal notation for the well ordering given by the restriction of $\leq^X_e$ to $\{j : j <^X_e i\}$. In ATR_0, if $\mathcal{O}(a, X)$, then the set

$$H^X_a = \{\langle y, 0\rangle : y \in X\} \cup \{\langle y, b+1\rangle : b <^X_{\mathcal{O}} a \wedge y \in TJ(H^X_b)\}$$

exists. In ACA_0, there is an arithmetic formula $H(a, X, Y)$ which, under the assumption that $\mathcal{O}(a, X)$, holds if and only if $Y = H^X_a$.

By Theorem VIII.3.15 in Simpson [16], ATR_0 is equivalent over ACA_0 to

$$\forall X\, \forall a\, (\mathcal{O}(a, X) \to H^X_a \text{ exists}).$$

If $\mathcal{O}(a, X)$ with $a = \langle e, i\rangle$, then we can assume without loss of generality that there are a' and a'' such that $\mathcal{O}(a', X)$, $\mathcal{O}(a'', X)$ and $a <^X_{\mathcal{O}} a' <^X_{\mathcal{O}} a''$ by adding two new successors of i in $\leq^X_e$ if necessary. Therefore, to prove ATR_0, it suffices to fix a and X such that $\mathcal{O}(a, X)$ and prove $\forall c <^X_{\mathcal{O}} b\, (H^X_c \text{ exists})$ for each $b <^X_{\mathcal{O}} a$.

Theorem 6.7 (ACA_0)**.** *The statement "every trivial Borel code has an evaluation function" implies* ATR_0.

Proof. Fix a and X such that $\mathcal{O}(a, X)$, so the restriction of $<^X_{\mathcal{O}}$ to $\{b : b <^X_{\mathcal{O}} a\}$ is a well order. Using ETR, we define trivial Borel codes $B_{x,b}$ for $x \in \omega$ by transfinite recursion on $b <^X_{\mathcal{O}} a$. We explain the intuitive construction before the formal definition.

Let $b <^X_{\mathcal{O}} a$ and $x \in \omega$. We want to define a trivial Borel code $B_{x,b}$ such that if f is an evaluation map for $B_{x,b}$, then $f(\lambda) = 1$ if and only if $x \in TJ(H^X_b)$. We put $\langle 0\rangle \in B_{x,b}$, so $B_{x,b}$ codes a union of sets. For each binary string σ such that $\Phi^\sigma_{x,|\sigma|}(x)$ converges, we add a successor $\langle 0, \sigma\rangle$ with a unique extension $\langle 0, \sigma, 1\rangle$. Therefore, regardless of whether the code for σ is even or odd, we have $f(\langle 0, \sigma\rangle) = f(\langle 0, \sigma, 1\rangle)$. It follows that $f(\lambda) = f(\langle 0\rangle) = 1$ if and only if there is a σ such that $\Phi^\sigma_{x,|\sigma|}(x)$ converges and $f(\langle 0, \sigma, 1\rangle) = 1$. (If $\Phi^\sigma_{x,|\sigma|}(x)$ always diverges, then $\langle 0\rangle$ is a leaf. In this case, $f(\lambda) = f(\langle 0\rangle) = 0$ and $x \notin TJ(H^X_b)$ which is what we want.)

Next, we want to ensure $f(\langle 0, \sigma, 1\rangle) = 1$ if and only if $\sigma \prec H^X_b$. For each $k < |\sigma|$, we add a successor $\langle 0, \sigma, 1, k\rangle$. Since $\langle 0, \sigma, 1\rangle$ codes the intersection of the sets coded by $\langle 0, \sigma, 1, k\rangle$, we want $f(\langle 0, \sigma, 1, k\rangle) = 1$ if and only if $\sigma(k) = H^X_b(k)$. We break into cases to determine the extensions of $\langle 0, \sigma, 1, k\rangle$.

For the first case, suppose $k = \langle y, 0\rangle$. We want $f(\langle 0, \sigma, 1, k\rangle) = 1$ if and only if $y \in X$. If $\sigma(k) = X(y)$, we add $\langle 0, \sigma, 1, k, 1\rangle$ to $B_{x,b}$ as the unique successor of

$\langle 0, \sigma, 1, k\rangle$, and if $\sigma(k) \neq X(y)$, we add $\langle 0, \sigma, 1, k, 0\rangle$ as the unique successor. In either case, the successor nodes will be leaves so we have $f(\langle 0, \sigma, 1, k\rangle) = 1$ if and only if $k \in H_b^X$.

For the second case, suppose $k = \langle y, c+1\rangle$ and $c <_{\mathcal{O}}^X b$. By the induction hypothesis, we have defined the trivial Borel code $B_{y,c}$ already. If $\sigma(k) = 1$, then we extend $\langle 0, \sigma, 1, k\rangle$ by a copy of $B_{y,c}$, treating $\langle 0, \sigma, 1, k\rangle$ as the root of this tree. The map f restricted to the subtree above $\langle 0, \sigma, 1, k\rangle$ is an evaluation map for $B_{y,c}$ and hence by the inductive hypothesis

$$f(\langle 0, \sigma, 1, k\rangle) = 1 \Leftrightarrow y \in TJ(H_c^X) \Leftrightarrow k \in H_b^X \Leftrightarrow \sigma(k) = H_b^X(k).$$

On the other hand, if $\sigma(k) = 0$, then we extend $\langle 0, \sigma, 1, k\rangle$ by a copy of $\overline{B}_{y,c}$. By the inductive hypothesis, we have

$$f(\langle 0, \sigma, 1, k\rangle) = 1 \Leftrightarrow y \notin TJ(H_c^X) \Leftrightarrow k \notin H_b^X \Leftrightarrow \sigma(k) = H_b^X(k).$$

For the third case, suppose that $k = \langle y, c+1\rangle$ and $c \not<_{\mathcal{O}}^X b$. In this case, we know $H_b^X(k) = 0$. If $\sigma(k) = 0$, then we add a unique successor $\langle 0, \sigma, 1, k, 1\rangle$ to $\langle 0, \sigma, 1, k\rangle$ as a leaf. We have $f(\langle 0, \sigma, 1, k, 1\rangle) = 1$ (since this node is a leaf) and hence $f(\langle 0, \sigma, 1, k\rangle) = 1$ which is what we want since $\sigma(k) = H_b^X(k) = 0$. On the other hand, if $\sigma(k) = 1$, then we add a unique successor $\langle 0, \sigma, 1, k, 0\rangle$ as a leaf, so $f(\langle 0, \sigma, 1, k\rangle) = f(\langle 0, \sigma, 1, k, 0\rangle) = 0$. Since $\langle 0, \sigma, 1\rangle$ codes an intersection, we get $f(\langle 0, \sigma, 1\rangle) = 0$ which is what we want since $\sigma(k) \neq H_b^X(k)$ and hence $\sigma \not\prec H_b^X$.

The formal construction follows this outline. To simplify the notation, for a trivial Borel code B, we let $B^1 = B$ and $B^0 = \overline{B}$. Since "$\Phi_{x,|\sigma|}^{\sigma}(x)$ converges" is a bounded quantifier statement and $c <_{\mathcal{O}}^X b$ is a Δ_1^0 statement with parameter X, the following recursion on $b <_{\mathcal{O}}^X a$ can be done with ETR. For each $x \in \omega$, we put λ and $\langle 0\rangle$ in $B_{x,b}$. For each σ such that $\Phi_{x,|\sigma|}^{\sigma}(x)$ converges, we put $\langle 0, \sigma\rangle$, $\langle 0, \sigma, 1\rangle$ and $\langle 0, \sigma, 1, k\rangle$ in $B_{x,b}$ for all $k < |\sigma|$. We extend $\langle 0, \sigma, 1, k\rangle$ as follows.

- For $k = \langle y, 0\rangle$: if $\sigma(k) = X(y)$, then $\langle 0, \sigma, 1, k, 1\rangle \in B_{x,b}$ and if $\sigma(k) \neq X(y)$, then $\langle 0, \sigma, 1, k, 0\rangle \in B_{x,b}$.
- For $k = \langle y, c+1\rangle$ with $c <_{\mathcal{O}}^X b$, $\langle 0, \sigma, 1, k\rangle^\frown\tau \in B_{x,b}$ for all $\tau \in B_{y,c}^{\sigma(k)}$.
- For $k = \langle y, c+1\rangle$ with $c \not<_{\mathcal{O}}^X b$, $\langle 0, \sigma, 1, k, 1-\sigma(k)\rangle \in B_{x,b}$.

This completes the construction of the trivial Borel codes $B_{x,b}$ for $b <_{\mathcal{O}}^X a$ by ETR. To complete the proof, we fix an arbitrary $b <_{\mathcal{O}}^X a$ and show that $\forall c <_{\mathcal{O}}^X b\,(H_c^X$ exists).

Fix an index x and $s \in \omega$ such that $\Phi_{x,s}^{1^s}(x)$ converges. Let N be the least value of s witnessing this convergence so $\Phi_{x,s}^{1^s}(x)$ converges for all $s \geq N$. Let f be an evaluation map for $B_{x,b}$.

For $c <_{\mathcal{O}}^X b$ and $y \in \omega$, let $\sigma = 1^{N+k}$ where $k = \langle y, c+1\rangle$. Define $f_{y,c}(\tau) = f(\langle 0, \sigma, 1, k\rangle^\frown\tau)$. We claim $f_{y,c}$ is an evaluation map for $B_{y,c}$. By the choice of x, $\Phi_{x,|\sigma|}^{\sigma}(x)$ converges. Since $c <_{\mathcal{O}}^X b$ and $\sigma(k) = 1$, we have $\langle 0, \sigma, 1, k\rangle^\frown\tau \in B_{x,b}$ if and only if $\tau \in B_{y,c}$. Therefore, $f_{y,c}$ is defined on $B_{y,c}$ and it satisfies the conditions for an evaluation map because f does.

Recall that $H(x, X, Y)$ is a fixed arithmetic formula such that if $\mathcal{O}(x, X)$, then $H(x, X, Y)$ holds if and only if $Y = H_x^X$. Define

$$Z = \{\langle y, 0\rangle : y \in X\} \cup \{k : k = \langle y, c+1\rangle \wedge c <_{\mathcal{O}}^X b \wedge f(\langle 0, 1^{N+k}, 1, k\rangle) = 1\}.$$

For $c <_{\mathcal{O}}^X b$, let $Z^c = \{\langle y, r\rangle \in Z : r = 0 \vee r - 1 <_{\mathcal{O}}^X c\}$. We show the following properties by simultaneous arithmetic induction on $c <_{\mathcal{O}}^X b$.

(1) $H(c, X, Z^c)$ holds. That is, $Z^c = H^X_c$.
(2) For all y, $f_{y,c}(\lambda) = 1$ if and only if $y \in TJ(Z^c) = TJ(H^X_c)$.

These properties imply $\forall c <^X_{\mathcal{O}} b\,(H^X_c$ exists) completing our proof.

Fix $c <^X_{\mathcal{O}} b$ and assume (1) and (2) hold for $d <^X_{\mathcal{O}} c$. To see (1) holds for c, fix k. If $k = \langle y, 0\rangle$, then $k \in Z^c \Leftrightarrow y \in X \Leftrightarrow k \in H^X_c$. Suppose $k = \langle y, d+1\rangle$. If $d \not<^X_{\mathcal{O}} c$, then $k \notin H^X_c$ and $k \notin Z^c$. If $d <^X_{\mathcal{O}} c$, then

$$k \in Z^c \Leftrightarrow f(\langle 0, 1^{N+k}, 1, k\rangle) = 1 \Leftrightarrow f_{y,d}(\lambda) = 1.$$

By the induction hypothesis, $k \in Z^c$ if and only if $y \in TJ(Z^d) = TJ(H^X_d)$, which holds if and only if $k \in H^X_c$, completing the proof of (1).

To prove (2), fix y and let $k = \langle y, c+1\rangle$. By definition,

$$k \in Z^c \Leftrightarrow f_{y,c}(\lambda) = f(\langle 0, 1^{N+k}, 1, k\rangle) = 1,$$

and $y \in TJ(Z^c) = TJ(H^X_c)$ if and only if there is a σ such that $\Phi^\sigma_{y,|\sigma|}(y)$ converges and $\sigma \prec Z^c = H^X_c$.

Suppose there are no σ such that $\Phi^\sigma_{y,|\sigma|}(y)$ converges. In this case, $y \notin TJ(H^X_c)$ and (since $B_{y,c}$ consists of λ and $\langle 0\rangle$) $f_{y,c}(\lambda) = 0$. Therefore $f_{y,c}(\lambda) = 1$ if and only if $y \in TJ(H^X_c)$ as required.

Suppose $\Phi^\sigma_{y,|\sigma|}(y)$ converges for some σ. For any such σ, $\langle 0, \sigma, 1, \ell\rangle \in B_{y,c}$ for all $\ell < |\sigma|$. By the induction hypothesis and the case analysis in the intuitive explanation of the construction, we have $f_{y,c}(\langle 0, \sigma\rangle) = f_{y,c}(\langle 0, \sigma, 1\rangle) = 1$ if and only if $\sigma \prec H^X_c = Z^c$, and therefore, $f_{y,c}(\lambda) = 1$ if and only if there is a σ such that $\Phi^\sigma_{y,|\sigma|}(y)$ converges and $\sigma \prec H^X_c$, completing the proof of (2) and of the theorem. $\square$

Proposition 6.8 (ATR_0). *Every Borel code for a subset of $(\omega)^k$ has a Baire code.*

Proof. Fix a Borel code B. For $\sigma \in B$, let $B_\sigma = \{\tau \in B : \tau \text{ is comparable to } \sigma\}$. B_σ is a Borel code for the set coded coded by the subtree of B above σ in the following sense. Let f be an evaluation map for B at x. The function $g : B_\sigma \to 2$ defined by $g(\tau) = f(\tau)$ for $\tau \succeq \sigma$ and $g(\tau) = f(\sigma)$ for $\tau \prec \sigma$ is an evaluation function for B_σ at x which witnesses $x \in B_\sigma$ if and only if $f(\sigma) = 1$. We denote this function g by $f_{\sigma,x}$.

Formally, our proof proceeds in two steps. First, by arithmetic transfinite recursion on the Kleene-Brouwer order $KB(B)$, we construct open sets U_σ, V_σ and $D_{n,\sigma}$, $n \in \omega$, which are intended to form a Baire code for B_σ. This construction is essentially identical to the proof of Proposition 5.3. Second, for any $x \in (\omega)^k$ and evaluation map f for B at x, we show by arithmetic transfinite induction on $KB(B)$ that if $x \in \cap_{n\in\omega} D_{n,\sigma}$, then $x \in U_\sigma$ implies $x \in B_\sigma$ via $f_{\sigma,x}$ and $x \in V_\sigma$ implies $x \notin B_\sigma$ via $f_{\sigma,x}$. For ease of presentation, we combine these two steps. Since ATR_0 suffices to construct evaluation maps, we treat Borel codes as sets in a naive manner and suppress explicit mention of the evaluation maps.

If σ is a leaf coding a basic clopen set $[\tau]$, we set $U_\sigma = [\tau]$, $V_\sigma = \overline{[\tau]}$ and $D_{n,\sigma} = (\omega)^k$. Similarly, if σ codes $\overline{[\tau]}$, we switch the values of U_σ and V_σ. In either case, it is clear that these open sets form a Baire code for B_σ.

Suppose σ is an internal node coding a union, so B_σ is the union of $B_{\sigma^\frown k}$ for $\sigma^\frown k \in B$. We define U_σ to be the union of $U_{\sigma^\frown k}$ for $\sigma^\frown k \in B$ and V_σ to be the union of $[\tau]$ such that $V_{\sigma^\frown k}$ is dense in $[\tau]$ for all $\sigma^\frown k \in B$. The sequence $D_{n,\sigma}$ is the sequence of all open sets of the form $D_{n,\sigma^\frown k} \cap (U_{\sigma^\frown k} \cup V_{\sigma^\frown k})$ for $n \in \omega$ and $\sigma^\frown k \in B$. As in the proof of Proposition 5.3, $U_\sigma \cup V_\sigma$ and each $D_{n,\sigma}$ are dense.

Let $x \in \cap_{n\in\omega} D_{n,\sigma}$. Suppose $x \in U_\sigma$ and we show $x \in B_\sigma$. By the definition of U_σ, fix $\sigma^\frown k \in B$ such that $x \in U_{\sigma^\frown k}$. Since $x \in \cap_{n\in\omega} D_{n,\sigma^\frown k}$, we have by induction that $x \in B_{\sigma^\frown k}$ and hence $x \in B_\sigma$. On the other hand, suppose $x \in V_\sigma$ and we show $x \notin B_\sigma$. Fix τ such that $\tau \prec x$ and $[\tau] \subseteq V_\sigma$, and fix k such that $\sigma^\frown k \in B$. Since $x \in \cap_{n\in\omega} D_{n,\sigma}$, $x \in U_{\sigma^\frown k} \cup V_{\sigma^\frown k}$. However, $V_{\sigma^\frown k}$ is dense in $[\tau]$. Therefore, $x \notin U_{\sigma^\frown k}$ (because $U_{\sigma^\frown k}$ and $V_{\sigma^\frown k}$ must be disjoint as in the proof of Proposition 5.3), so $x \in V_{\sigma^\frown k}$. Since $x \in \cap_{n\in\omega} D_{n,\sigma^\frown k}$, we have by induction that $x \notin B_{\sigma^\frown k}$. Because this holds for every $\sigma^\frown k \in B$, it follows that $x \notin B_\sigma$, completing the case for unions.

The case for an interior node coding an intersection is similar with the roles of U_σ and V_σ switched. Finally, the Baire codes for the unique $\langle m \rangle \in B$ satisfy the conditions to be Baire codes for B itself. □

We conclude with a proof of Theorem 3.14.

Proof. Lemma V.3.3 in Simpson [16] shows (1) implies (2) in the space 2^ω and the proof translates immediately to $(\omega)^k$. By Proposition 6.8, (1) implies (3). It follows from Theorem 6.7 that (2) implies (1). We show (3) implies (2). Let B be a Borel code. Fix a Baire code U, V and D_n for B. Since each D_n and $U \cup V$ is a dense open set, there is an $x \in (U \cup V) \cap \cap_{n\in\omega} D_n$. If $x \in U$, then by the definition of a Baire code, $x \in B$, and similarly, if $x \in V$, then $x \notin B$. Therefore, we have a partition x such that $x \in B$ or $x \notin B$ as required. □

7. Open Questions

While Figure 1 summarizes the known implications among the studied principles, in most cases it is not known whether the results are optimal. It is particularly dissatisfying that the best upper bound for these principles remains $\Pi^1_1 - \mathsf{CA}_0$. Observe that, on the basis of the proof of CDRT^k_ℓ given in Theorem 3.30, any upper bound on the strength of the Carlson-Simpson Lemma $\mathsf{CSL}(k-1,\ell)$ would also imply a related upper bound on the strength of CDRT^k_ℓ. Therefore, it would be interesting to know the following:

Question 7.1. For any $k \geq 3$, does $\mathsf{CSL}(k,\ell)$ follow from ATR_0?

The best known upper bound for $\mathsf{CSL}(2,\ell)$ is ACA_0; it is shown in [10] that the stronger principle $\mathsf{OVW}(2,\ell)$ follows from ACA_0. In the more general case, [12] have obtained upper bounds below ATR_0 for two other combinatorial theorems similar to CSL. While their methods do not currently apply to CSL, their approach may be extendable.

Turning attention now to lower bounds, the principles CDRT^k_ℓ for $k \geq 4$ are not obviously implied by HT or ACA_0^+. We wonder whether an implication might go the other way.

Question 7.2. For any $k \geq 4$, does CDRT^k_ℓ imply HT or ACA_0^+?

When $k \geq 4$, it is known that CDRT^k_ℓ implies ACA_0 (this was proved for ODRT^k_ℓ in [11]). On the other hand, while CDRT^3_2 is provable from Hindman's Theorem, the best lower bound we have on CDRT^3_2 is RT^2_2. Furthermore, nothing about the relationship of CDRT^3_2 and ACA_0 is known.

Question 7.3. Is CDRT^3_2 comparable to ACA_0?

These are just a few of the many questions that remain concerning these principles.

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