Math 5120: Complex analysis. Homework 9 Solutions

4.5.3.1.a

$$f(z) = \frac{1}{z^2 + 5z + 6} = \frac{1}{(z+2)(z+3)}$$

which has

- a pole of order 1 at $z = -2$ with residue $\lim_{z\to -2}(z+2)f(z) = 1$
- a pole of order 1 at $z = -3$ with residue $\lim_{z\to -3}(z+3)f(z) = -1$

4.5.3.1.b

$$f(z) = \frac{1}{(z^2-1)^2} = \frac{1}{(z-1)^2(z+1)^2}$$

which has

- a pole of order 2 at $z = 1$ with residue $\lim_{z\to 1} \frac{d}{dz}(z-1)^2 f(z) = -\frac{1}{4}$
- a pole of order 2 at $z = -1$ with residue $\lim_{z\to -1} \frac{d}{dz}(z+1)^2 f(z) = \frac{1}{4}$

4.5.3.1.c $f(z) = \frac{1}{\sin z}$ has poles at the zeros of $\sin z$, so at the points πk, $k \in \mathbb{Z}$. These zeros are simple, because $f'(z) = \cos z = \pm 1$ at these points, and consequently the poles are simple. The residue at πk may be computed by L'Hopital's rule

$$\lim_{z\to\pi k} \frac{z-\pi k}{\sin z} = \lim_{z\to\pi k} \frac{1}{\cos z} = (-1)^k$$

so that $f(z)$ has simple poles with residue $(-1)^k$ at each πk, $k \in \mathbb{Z}$.

4.5.3.1.d As $\cos z$ is entire, $f(z) = \cot z = \frac{\cos z}{\sin z}$ can only have poles at the zeros of $\sin z$, meaning the points $z = \pi k$, $k \in \mathbb{Z}$. Since $\cos z \neq 0$ at these points, there is a pole at each such point, and since the zeros of $\sin z$ are simple the poles are also simple. The residue at πk is

$$\lim_{z\to\pi k} \frac{(z-\pi k)\cos z}{\sin z} = \cos(\pi k) \lim_{z\to\pi k} \frac{1}{\cos z} = 1$$

so $\cot z$ has simple poles with residue 1 at each πk, $k \in \mathbb{Z}$.

4.5.3.1.e $f(z) = \frac{1}{\sin^2 z}$ has poles at each of the zeros $z = \pi k$, $k \in \mathbb{Z}$ of $\sin z$. These zeros are order 1, so the zeros of $\sin^2 z$ are order 2. The residues may be computed using L'Hopital

$$\begin{aligned}
\lim_{z\to\pi k} \frac{d}{dz}\Big(\frac{(z-\pi k)^2}{\sin^2 z}\Big) &= \lim_{z\to\pi k} \frac{2(z-\pi k)\sin z - 2(z-\pi k)^2\cos z}{\sin^3 z} \\
&= 2 \lim_{z\to\pi k}\Big(\frac{z-\pi k}{\sin z}\Big) \lim_{z\to\pi k}\Big(\frac{\sin z - (z-\pi k)\cos z}{\sin^2 z}\Big) \\
&= 2 \lim_{z\to\pi k}\Big(\frac{\cos z - \cos z + (z-\pi k)\sin z}{2\sin z}\Big) \\
&= \lim_{z\to\pi k}(z-\pi k) = 0.
\end{aligned}$$

We determine that $f(z)$ has poles of order 2 at each of the points πk, $k \in \mathbb{Z}$ and has residue zero at each pole.

4.5.3.1.f $f(z) = z^{-m}(1-z)^{-n}$, $m, n \in \mathbb{N}$ has a pole of order m at 0 and a pole of order n at 1. We may compute the residue at 0 using

$$\begin{aligned}
\lim_{z\to 0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(1-z)^{-n} &= \frac{1}{(m-1)!}\frac{(n+m-2)!}{(n-1)!}(1-0)^{-n-(m-1)} \\
&= \binom{(n-1)+(m-1)}{m-1} = \binom{(n-1)+(m-1)}{n-1}
\end{aligned}$$

The residue at 1 may be computed the same way

$$\lim_{z\to 1}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-1)^n f(z) = \frac{1}{(n-1)!}\lim_{z\to 1}\frac{d^{n-1}}{dz^{n-1}}(-1)^n z^{-m}$$
$$= (-1)^n(-1)^{n-1}\frac{(m+n-2)!}{(n-1)!(m-1)!}(1)^{-m-(n-1)}$$
$$= -\binom{(n-1)+(m-1)}{m-1} = -\binom{(n-1)+(m-1)}{n-1}$$

so we conclude that f has a pole of order m at zero with residue $\binom{(n-1)+(m-1)}{n-1}$ and a pole of order n at 1 with residue $-\binom{(n-1)+(m-1)}{n-1}$. It is worth noting that this is consistent with exercise 4.2.3.1(b).

4.5.3.3 There are a few things we will use repeatedly in computing these integrals. The curve γ_R will be the semicircle $|z| = R$ in the upper half-plane, with the usual (increasing angle) orientation. For $R, S, T \in (0,\infty)$ we also let $\Gamma_1 = \{S + iy : 0 \le y \le T\}$, $\Gamma_2 = \{x + iT : -R \le x \le S\}$, $\Gamma_3 = \{-R + iy : 0 \le y \le T\}$, oriented such that $\Gamma_1 + \Gamma_2 + \Gamma_3$ and the interval $[-R, S] \subset \mathbb{R}$ form a positively oriented closed curve. We will frequently use that (A) if the integrand $f(x)$ is bounded by $|x|^{-2}$ as $|x| \to \infty$ then $\int_{-\infty}^{\infty} = \lim_{R\to\infty}\int_{-R}^{R} f(x)$, and (B) if the integrand $f(z)$ is bounded by $|z|^{-2}$ as $|z| \to \infty$ then $\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0$

4.5.3.3.a Note that $\sin^2 x = \sin^2(-x) = \sin^2(\pi - x) = \sin^2(\pi + x)$ implies

$$\int_0^{\pi/2}\frac{dx}{a+\sin^2 x} = \frac{1}{4}\int_0^{2\pi}\frac{dx}{a+\sin^2 x}$$

and this may be seen as an integral on the unit circle with respect to the angle $dx = dz/iz$ where $\sin z = (z - z^{-1})/2i$. Thus

$$\int_0^{\pi/2}\frac{dx}{a+\sin^2 x} = \frac{1}{4}\int_{|z|=1}\frac{1}{a+((z-z^{-1})/2i)^2}\frac{dz}{iz}.$$

We simplify $((z-z^{-1})/2i)^2 = -(2z)^{-2}(z^4 - 2z^2 + 1)$ and find the integrand becomes $\frac{4iz}{z^4-(2+4a)z^2+1}$. At this point we can make our lives a little easier by making the substitution $w = z^2$. Notice that when z winds once around the unit circle, w winds around twice. We therefore find

$$\int_0^{\pi/2}\frac{dx}{a+\sin^2 x} = \frac{1}{2}\int_{|w|=1}\frac{2i}{w^2-(2+4a)w+1}\,dw = i\int_{|w|=1}\frac{dw}{w^2-(2+4a)w+1}.$$

Now we would like to say that the poles of $w^2 - (2 + 4a)w + 1$ are at $w_\pm = 1 + 2a \pm 2\sqrt{a^2 + a}$ by the quadratic formula, but this requires that we make sense of the square root. Fortunately, $|a| > 1$ by hypothesis, so $|1/a| < 1$ and $\sqrt{1 + (1/a)}$ is well-defined. We may therefore define an analytic branch of $\sqrt{a^2 + a}$ by $\sqrt{a^2 + a} = a\sqrt{1 + (1/a)}$, obtaining $w_\pm = 1 + 2a \pm 2a\sqrt{1 + (1/a)}$. By construction, each of $w_\pm$ is a branch of the inverse of the map $w \mapsto (w + w^{-1})^2/4$, evaluated at a. Since $w \mapsto (w + w^{-1})^2/4$ takes the unit circle to the interval $[-1, 1]$, and $|a| > 1$, we see that $|w_\pm| \ne 1$ (this is important for applying the residue theorem). It also follows that only one of $w_\pm$ can lie inside the unit disc, and that which one does so is independent of a. Taking $a \in (1,\infty)$ we readily see $|w_-| < 1$ and $|w_+| > 1$, so this must be true for all a. Thus the result of the integration can be computed from the residue at the simple pole w_-, which has value $1/(w_- - w_+) = -1/(4a\sqrt{1+(1/a)})$. Finally

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = \frac{2\pi i^2}{-(4a\sqrt{1+(1/a)})} = \frac{\pi}{2a\sqrt{1+(1/a)}}.$$

4.5.3.3.b Using that the integrand is even and (A), then the residue theorem, and then (B)

$$\begin{aligned}
\int_0^\infty \frac{x^2\,dx}{x^4+5x^2+6} &= \frac{1}{2}\int_{-\infty}^\infty \frac{z^2\,dz}{z^4+5z^2+6} \\
&= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \frac{z^2\,dz}{z^4+5z^2+6} \\
&= \pi i \sum_j \mathrm{Res}_{z_j} - \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} \frac{z^2\,dz}{z^4+5z^2+6} \\
&= \pi i \sum_j \mathrm{Res}_{z_j} \frac{z^2}{z^4+5z^2+6}
\end{aligned}$$

where the sum is over residues in the upper half-plane. Now $z^4 + 5z^2 + 6 = (z^2+2)(z^2+3)$, so the integrand has simple poles at $z = \pm i\sqrt{2}$ and $z = \pm i\sqrt{3}$. The residue at $i\sqrt{2}$ is $-2/(2\sqrt{2}i)(1)$, and at $i\sqrt{3}$ is $-3/(-1)(2\sqrt{3}i)$, so the result is

$$\int_0^\infty \frac{x^2\,dx}{x^4+5x^2+6} = \pi i\Big(\frac{-\sqrt{2}}{2i} + \frac{\sqrt{3}}{2i}\Big) = (\sqrt{3} - \sqrt{2})\frac{\pi}{2}.$$

4.5.3.3.c By (A), the residue theorem, and (B)

$$\begin{aligned}
\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9}\,dx & \\
&= \lim_{R\to\infty}\int_{-R}^R \frac{z^2-z+2}{z^4+10z^2+9}\,dz \\
&= 2\pi i \sum_j \mathrm{Res}_{z_j} - \lim_{R\to\infty}\int_{\gamma_R} \frac{z^2-z+2}{z^4+10z^2+9}\,dz \\
&= 2\pi i \sum_j \mathrm{Res}_{z_j} \frac{z^2-z+2}{z^4+10z^2+9}
\end{aligned}$$

where the sum is over residues in the upper half-plane. The zeros of $z^4+10z^2+9 = (z^2+1)(z^2+9)$ are at $\pm i$ and $\pm 3i$; each produces a simple pole in the integrand, and there are no others. The residue at i is $(-1-i+2)/(2i)(-1+9)$ and at $3i$ is $(-9-3i+2)/(-9+1)(6i)$, so the result is

$$\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9}\,dx = 2\pi i\Big(\frac{1-i}{16i} + \frac{7+3i}{48i}\Big) = (3-3i+7+3i)\frac{\pi}{24} = \frac{5\pi}{12}.$$

4.5.3.3.d Using that the integrand is even and (A), then the residue theorem, and then (B)

$$\begin{aligned}\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^3} &= \frac{1}{2}\int_{-\infty}^\infty \frac{z^2\,dz}{(z^2+a^2)^3}\\ &= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{z^2\,dz}{(z^2+a^2)^3}\\ &= \pi i \sum_j \mathrm{Res}_{z_j} - \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} \frac{z^2\,dz}{(z^2+a^2)^3}\\ &= \pi i \sum_j \mathrm{Res}_{z_j} \frac{z^2}{(z^2+a^2)^3}\end{aligned}$$

where the sum is over residues in the upper half-plane. Factoring $(z^2+a^2)^3 = (z+ai)^3(z-ai)^3$, $a\in\mathbb{R}$, we see that there is a single pole of order 3 in the upper half-plane, at $i|a|$. The residue there is

$$\lim_{z\to|a|i}\frac{d^2}{dz^2}\frac{z^2}{(z+|a|i)^3} = \frac{1}{8|a|^3 i}$$

so that the result is

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^3} = \frac{\pi}{8|a|^3}$$

4.5.3.3.e Using that the integrand is even and (A), that $\cos z = \Re e^{iz}$, then the residue theorem for $R = S$ and T sufficiently large,

$$\begin{aligned}\int_0^\infty \frac{\cos x\,dx}{x^2+a^2} &= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos z\,dz}{z^2+a^2}\\ &= \frac{1}{2}\lim_{R\to\infty}\Re\int_{-R}^{R}\frac{e^{iz}\,dz}{z^2+a^2}\\ &= \pi i\sum_j \mathrm{Res}_{z_j}\frac{e^{iz}}{z^2+a^2} - \frac{1}{2}\lim_{R,T\to\infty}\Re\left(\int_{\Gamma_1}\frac{e^{iz}\,dz}{z^2+a^2}\int_{\Gamma_2}\frac{e^{iz}\,dz}{z^2+a^2} + \int_{\Gamma_3}\frac{e^{iz}\,dz}{z^2+a^2}\right)\end{aligned}$$

where the sum is over residues in the upper half-plane. However the integrand is bounded by a constant multiple of $e^{-y}/|z|^2$ for $z = x+iy$ and $|z|$ sufficiently large. Writing $f(z)$ for the integrand, and taking $R = S$ and T large enough we find that $\left|\int_{\Gamma_1} f(z)\,dz\right| \le R^{-2}\int_0^T e^{-y}\,dy \le R^{-2}$, and similarly for Γ_3. Now on Γ_2 we have that $(z^2+a^2)^{-1}$ is integrable (with integral bounded by constant C) if T is large enough, and therefore $\left|\int_{\Gamma_2} f(z)\,dz\right| \le Ce^{-T}$. Sending R and T to ∞ we find

$$\int_0^\infty \frac{\cos x\,dx}{x^2+a^2} = \Re\pi i\,\mathrm{Res}_{|a|i} f(z) = \frac{\pi}{2|a|}$$

where at the last step we computed that $z^2+a^2 = (z+ai)(z-ai)$, has one simple pole in the upper half-plane, at $i|a|$, with residue $\lim_{z\to|a|i}\frac{\cos z}{z+|a|i} = \frac{\cos a}{2|a|i}$.

4.5.3.3.f We use that the integrand is even and $x\sin x/(x^2+a^2) = \Im ze^{iz}/(z^2+a^2)$. Taking R, S, T large enough that the curve $(-R,S)\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3$ encloses the simple pole at $|a|i$, where the residue is $\lim_{z\to|a|i}\frac{ze^{iz}}{z+|a|i} = \frac{|a|ie^{-|a|}}{2|a|i} = \frac{e^{-|a|}}{2}$ we obtain

$$\begin{aligned}\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx &= \frac{1}{2}\lim_{R,S\to\infty}\Im\int_{-R}^{S}\frac{ze^{iz}}{z^2+a^2}\,dz \\ &= \Im\pi i\frac{e^{-|a|}}{2} - \lim_{R,S\to\infty}\Im\left(\int_{\Gamma_1}\frac{ze^{iz}}{z^2+a^2}\,dz + \int_{\Gamma_2}\frac{ze^{iz}}{z^2+a^2}\,dz + \int_{\Gamma_3}\frac{ze^{iz}}{z^2+a^2}\,dz\right)\end{aligned}$$

valid for all sufficiently large T. However, the integrand $f(z)$ satisfies $|f(z)| \le \frac{|z|e^{-y}}{|z|^2-|a|^2}$ for $z = x+iy$. The integral for Γ_1 can be bounded by $\frac{R}{R^2-|a|^2}\int_0^T e^{-y}dy = \frac{R}{R^2-|a|^2}$ and similarly that for Γ_3 can be bounded by $\frac{S}{S^2-|a|^2}$. The integral for Γ_2 can be bounded by $\frac{Se^{-T}}{S^2-|a|^2}(R+S)$. If we first send $T\to\infty$ so the Γ_2 integral goes to 0, and then send $R,S\to\infty$ we find that they make no contribution to the result, and therefore

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx = \Im\pi i\frac{e^{-|a|}}{2} = \frac{\pi}{2e^{|a|}}$$

4.5.3.3.g We will do this for general $\beta\in(-1,1)$, as it will be useful later. Take $\delta>0$, $\epsilon>0$, and $R>2$. Let $\Gamma_\pm = \{re^{i\pm\delta}, r\in(\epsilon,R)\}$ be rays at angle $\pm\delta$, and also take arcs $\Gamma_R = \{Re^{i\theta} : \theta\in(\delta,2\pi-\delta)\}$ and $\Gamma_\epsilon = \{\epsilon e^{i\theta} : \theta\in(\delta,2\pi-\delta)\}$. Let z^β be a branch on $\mathbb{C}\setminus[0,\infty)$, so it is well-defined and analytic in a simply connected neighborhood of the closed curve $\Gamma_+ + \Gamma_R - \Gamma_- - \Gamma_\epsilon$. Provided δ and ϵ are sufficiently small, this curve winds once around the simple poles of $f(z) = z^\beta(1+z^2)^{-1}$, which are at $\pm i$, and where there are residues $i^\beta/2i$ and $(-i)^\beta/-2i$ respectively.

$$\int_{\Gamma_++\Gamma_R-\Gamma_--\Gamma_\epsilon}\frac{z^\beta}{z^2+1}\,dz = 2\pi i(i^\beta-(-i)^\beta)/2i = \pi(e^{i\pi\beta/2}-e^{i3\pi\beta/2}).$$

Now on Γ_+ we have $z^\beta = r^\beta e^{i\delta\beta}$, while on Γ_-, $z^\beta = r^\beta e^{i(2\pi-\delta)\beta}$. It follows that

$$\lim\delta\to 0\int_{\Gamma_+-\Gamma_-}\frac{z^\beta}{z^2+1}\,dz = (1-e^{i2\pi\beta})\int_\epsilon^R\frac{x^\beta}{x^2+1}\,dx.$$

At the same time, we see that on Γ_ϵ the integrand has the bound $|f(z)|\le 2\epsilon^\beta$, and the length of the curve is less than $2\pi\epsilon$, so the integral is bounded by $4\pi\epsilon^{1+\beta}\to 0$ as $\epsilon\to 0$, provided $\beta>-1$. On Γ_R we have $|f(z)|\le R^\beta/(R^2-1)$, and the curve has length less than $2\pi R$, so the integral is bounded by $2\pi R^{1+\beta}/(R^2-1)\to 0$ as $R\to\infty$ provided $\beta<1$. We conclude that if $\beta\in(-1,1)$ then

$$\begin{aligned}\int_0^\infty\frac{x^\beta}{x^2+1}\,dx &= \lim_{\epsilon\to0,R\to\infty,\delta\to0}\frac{1}{1-e^{i2\pi\beta}}\int_{\Gamma_++\Gamma_R-\Gamma_--\Gamma_\epsilon}\frac{z^\beta}{z^2+1}\,dz \\ &= \pi\frac{e^{i\pi\beta/2}-e^{i3\pi\beta/2}}{1-e^{i2\pi\beta}} \\ &= \pi\frac{e^{i\pi\beta}(e^{-i\pi\beta/2}-e^{i\pi\beta/2})}{e^{i\pi\beta}(e^{-i\pi\beta}-e^{i\pi\beta})} \\ &= \pi\frac{\sin(\pi\beta/2)}{\sin\pi\beta} = \pi\frac{\sin(\pi\beta/2)}{2\sin(\pi\beta/2)\cos(\pi\beta/2)} = \frac{\pi}{2}\sec\Big(\frac{\pi\beta}{2}\Big)\end{aligned}$$

In the special case $\beta=\frac{1}{3}$, we have $\sin(\pi/6)/\sin(\pi/3) = 1/\sqrt{3}$, so that

$$\int_0^\infty\frac{x^{1/3}}{x^2+1}\,dx = \frac{\pi}{\sqrt{3}}.$$

4.5.3.3.h For this problem, let $\Gamma_+ = (\epsilon, R) \subset \mathbb{R}$ and $\Gamma_- = (-R, -\epsilon) \subset \mathbb{R}$, Γ_ϵ and Γ_R be the semicircles of radius ϵ and R (respectively) in the upper half-plane. Define $\log z$ to be the branch of the logarithm on the complement of the negative imaginary axis. Taking $\Gamma_+ + \Gamma_R - \Gamma_- - \Gamma_\epsilon$ arranged to form a curve winding once around the simple pole of $\frac{\log z}{(z^2+1)}$ at $z = i$, we find from the residue theorem that

$$\int_{\Gamma_+ + \Gamma_R + \Gamma_- - \Gamma_\epsilon} \frac{\log z}{z^2+1}\, dz = 2\pi i (\log i)/2i = \frac{i\pi^2}{2}.$$

The computations showing that the contributions from Γ_ϵ and Γ_R vanish in the limit are essentially the same as in exercise 4.5.3.3.g. All that is different is we use the bound $|\log z| \le (\log|z| + 2\pi)$. Since $\log z$ is $\log|z|$ on Γ_+ and $\log|z| + \pi i$ on Γ_- we find

$$\int_{\Gamma_+ + \Gamma_-} \frac{\log z}{z^2+1}\, dz = \int_\epsilon^R \frac{\log x}{x^2+1}\, dx + \int_{-R}^{-\epsilon} \frac{\log x + \pi i}{x^2+1}\, dx = 2\int_\epsilon^R \frac{\log x}{x^2+1}\, dx + \int_\epsilon^R \frac{1}{x^2+1}\, dx.$$

Combining these facts we see

$$\begin{aligned} \frac{i\pi^2}{2} &= \lim_{\epsilon\to 0, R\to\infty} \int_{\Gamma_+ + \Gamma_R + \Gamma_- - \Gamma_\epsilon} \frac{\log z}{z^2+1}\, dz \\ &= \lim_{\epsilon\to 0, R\to\infty} \int_{\Gamma_+ + \Gamma_-} \frac{\log z}{z^2+1}\, dz \\ &= 2\int_0^\infty \frac{\log x}{x^2+1}\, dx + \int_0^\infty \frac{i\pi}{x^2+1}\, dx \\ &= 2\int_0^\infty \frac{\log x}{x^2+1}\, dx + \frac{i\pi^2}{2} \end{aligned}$$

so that

$$\int_0^\infty \frac{\log x}{x^2+1}\, dx = 0.$$

4.5.3.3.i Let us first observe that $f(x) = x^{(-1-\alpha)}\log(1+x^2)$ is integrable on $[0,\infty)$ because it is bounded by $C_\alpha x^{-1-(\alpha/2)}$ as $x \to \infty$ and $\alpha > 0$, while as $x \downarrow 0$ one has $|\log(1+x^2)| \le 2x^2$ so $|f(x)| \le x^{1-\alpha}$ and $\alpha < 2$. It follows that we can write the integral as a limit and can integrate by parts

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}\, dx = \lim_{R\to\infty}\int_{\frac{1}{R}}^R \frac{\log(1+x^2)}{x^{1+\alpha}}\, dx = \lim_{R\to\infty}\left[\frac{x^{-\alpha}}{-\alpha}\log(1+x^2)\right]_{\frac{1}{R}}^R + \frac{1}{\alpha}\lim_{R\to\infty}\int_{\frac{1}{R}}^R \frac{2x^{1-\alpha}}{1+x^2}\, dx.$$

We observe that as $R \to \infty$, $R^{-\alpha}\log(1+R^2) \to 0$, and also $|R^\alpha \log(1+R^{-2})| \le 2R^{2-\alpha} \to 0$, so the boundary term from the integration makes no contribution in the limit. The remaining term may be dealt with by the computation in 4.5.3.3.g. Indeed, from that problem with $\beta = (1-\alpha) \in (-1,1)$, we have

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}\, dx = \frac{1}{\alpha}\int_0^\infty \frac{2x^{1-\alpha}}{1+x^2}\, dx = \frac{\pi}{\alpha}\sec\Big(\frac{(1-\alpha)\pi}{2}\Big) = \frac{\pi}{\alpha}\csc\Big(\frac{\alpha\pi}{2}\Big)$$

4.5.3.4 Parameterizing $|z| = \rho$ by $z = \rho e^{i\theta}$ we have $dz = iz d\theta$ and $|dz| = \rho d\theta$, so $|dz| = \rho dz/iz$. Also $|z-a|^2 = (z-a)(\bar{z}-\bar{a}) = (z-a)(\frac{\rho^2}{z} - \bar{a})$. Hence we find

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_{|z|=\rho} \frac{\rho}{iz(z-a)(\frac{\rho^2}{z} - \bar{a})}\, dz = \int_{|z|=\rho} \frac{\rho}{i(z-a)(\rho^2 - \bar{a}z)}\, dz$$

which can be computed by the residue theorem. There are simple poles at a and $\rho^2/\bar{a}$. By hypothesis, $|a| \neq \rho$; if $|a| < \rho$ then a is inside $|z| = \rho$ and $\rho^2/\bar{a}$ is not, and the reverse is true if $|a| > \rho$.

The residue at $z = a$ is $\frac{\rho}{i(\rho^2 - |a|^2)}$ and that at $\rho^2/\bar{a}$ is $\frac{-\rho}{i(\rho^2 - |a|^2)}$. We conclude from the residue theorem that

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \begin{cases} \frac{2\pi\rho}{\rho^2 - |a|^2} & \text{if } |a| < \rho \\ \frac{-2\pi\rho}{\rho^2 - |a|^2} & \text{if } |a| > \rho \end{cases} = \frac{2\pi\rho}{\left|\rho^2 - |a|^2\right|}.$$