Math 5120: Complex analysis. Homework 8 Solutions

4.3.4.1 Suppose $|f(z)| \leq 1$ on $|z| \leq 1$ and f analytic. Fix z_0 in the open unit disc and let $w_0 = f(z_0)$. Using a composition of fractional linear transformations and the Schwarz lemma, it is proved in the book that

$$\left|\frac{f(z) - w_0}{1 - \bar{w_0} f(z)}\right| \leq \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|. \tag{1}$$

With a little rewriting we can extract $f'(z_0)$:

$$\frac{|f'(z_0)|}{1 - |f(z)|^2} = \lim_{z \to z_0} \left|\frac{f(z) - w_0}{(z - z_0)(1 - \bar{w_0} f(z))}\right| \leq \lim_{z \to z_0} \left|\frac{1}{1 - \bar{z}_0 z}\right| = \frac{1}{1 - |z_0|^2} \tag{2}$$

and z_0 was an arbitrary point in the disc, so the estimate is valid for all $|z_0| < 1$.

4.3.4.2 Suppose f is non-constant analytic and $\Im f(z) \geq 0$ if $\Im z \geq 0$. Composition with a fractional linear map gives a map from the unit disc to itself to which we can apply the Schwarz lemma. Specifically, for any z_0 with $\Im z_0 > 0$ we set $w_0 = f(z_0)$ and have $\Im w_0 > 0$ by the maximum principle. For α in the upper half-plane let $g_\alpha(z) = \frac{z-\alpha}{z-\bar{\alpha}}$ so $g_{z_0}^{-1}$ maps the unit disc to the upper half-plane with $0 \mapsto z_0$ and g_{w_0} maps the upper half-plane to the unit disc with $w_0 \mapsto 0$. Thus $g_{w_0} \circ f \circ g_{z_0}^{-1}$ takes the unit disc to itself and fixes 0, whence $|g_{w_0} \circ f \circ g_{z_0}^{-1}(z)| \leq |z|$ by the Schwarz lemma, and so $|g_{w_0} \circ f(z)| \leq |g_{z_0}(z)|$. Substituting gives

$$\left|\frac{f(z) - f(z_0)}{f(z) + \overline{f(z_0)}}\right| \leq \left|\frac{z - z_0}{z + \bar{z}_0}\right|. \tag{3}$$

As in the previous question, we can obtain a bound for $|f'(z_0)|$ by dividing both sides of the previous equation by $|z - z_0|$ and sending $z \to z_0$. Since $z + \bar{z} = 2\Im z$, we obtain

$$\frac{|f'(z_0)|}{\Im f(z_0)} \leq \frac{1}{\Im z_0} \tag{4}$$

valid at all points in the upper half-plane.

4.3.4.3 Let

$$F(z) = \frac{(f(z) - w_0)(1 - \bar{z}_0 z)}{(z - z_0)(1 - \bar{w}_0 f(z))}.$$

Then F is analytic on the open unit disc except for a removable singularity at z_0, and (1) says $|F(z)| \leq 1$. If equality holds in (2) then $|F(z_0)| = 1$, so the maximum principle implies F is constant of modulus 1, which we may write as $e^{i\theta}$, $\theta \in [0, 2\pi)$. Multiplying out gives that $f(z) = h_{w_0}^{-1}(e^{i\theta} h_{z_0}(w))$ where $h_\alpha = \frac{z-\alpha}{1-\bar{\alpha}z}$.

Similarly, if we let

$$G(z) = \frac{(f(z) - w_0)(z + \bar{z}_0)}{(z - z_0)(f(z) + \bar{w}_0)}$$

then we obtain a function analytic on $\Im z > 0$ except for the removable singularity at z_0, and $|G(z)| \leq 1$ by (3). Equality in (4) implies $G \equiv e^{i\theta}$ for some $\theta \in [0, 2\pi)$ by the same maximum principle argument as before, and inverting the maps we have $f(w) = g_{w_0}^{-1}(e^{i\theta} g_{z_0}(w))$ with g as in the previous exercise.

In either of these two cases we see that f is a composition of fractional linear maps, so is fractional linear.

4.4.7.3 Let γ be a closed curve. Its complement is open, so the connected components of the complement are open, and there are thus at most countably many of them. Label them $\{U_j\}_{j=0}^{\infty}$, and let U_0 be the unique unbounded one; it is unique because compactness of γ implies the complement of some large disc is in $\mathbb{C} \setminus \gamma$, and this is a connected set so is contained in a single component.

For each j, let $V_j = \bar{\mathbb{C}} \setminus U_j$ and observe $V_j = \cup_{k \neq j} U_j \cup \gamma \cup \{\infty\}$. As these sets are connected, each must lie in exactly one component of V_j. However connectivity of U_k implies connectivity of its closure, which intersects γ, so all U_k and γ must lie in a single component of V_j. Moreover if $j \neq 0$ then $U_0 \subset V_j$ and ∞ is its closure, so is in this same component of V_j. We conclude that if $j \neq 0$ then V_j has only one component, so U_j is simply connected by definition. Also V_0 has at most two components, one being $\{\infty\}$ and the other $\cup j \geq 1 U_j \cup \gamma$. Taking r so $|z| = r$ does not intersect γ, we see that the disjoint open sets $|z| > r$ and $|z| < r$ separate V_0 into these two components, so V_0 is doubly connected.

4.4.7.5 Let Ω be a region not intersecting a connected set E with $\pm 1 \in E$. If $\phi(z)$ is a globally analytic function such that $f(z) = \frac{1-z^2}{(\phi(z))^2}$ maps E to a connected region containing 0, then f has a well-defined logarithm and therefore well-defined roots on on $\mathbb{C} \setminus f(E)$. Provided $f(\Omega)$ does not intersect $f(E)$ we may define a single-valued analytic function $\sqrt{1-z^2}$ on Ω by

$$\sqrt{1-z^2} = \phi(z)\sqrt{\frac{1-z^2}{(\phi(z))^2}}$$

and it is a legitimate branch of the square root because squaring both sides leads to an equality.

It remains to find such a $\phi(z)$, but writing $1 - z^2 = (1-z)(1+z)$ immediately suggests $\phi(z) = (1+z)$, as then $f(z) = \frac{1-z}{1+z}$, which is a fractional linear transformation with $1 \mapsto \infty$ and $1 \mapsto 0$, from which $f(E)$ is a connected set containing 0 and ∞. Our definition is then

$$\sqrt{1-z^2} = (1+z)\sqrt{\frac{1-z}{1+z}}.$$

Now consider the integral

$$\int_\gamma \frac{dz}{\sqrt{1-z^2}} = \int_\gamma \frac{1}{(1+z)\sqrt{\frac{1-z}{1+z}}}\,dz$$

where γ is a closed curve in a component of $\mathbb{C} \setminus E$. The integrand has no singularities in this component, so if γ does not wind around any point of E then the integral is zero. In particular if $\infty \in E$ then γ cannot wind around E, so the integral is zero in this case.

We therefore suppose that E is bounded and take $r > 0$ so large that $E \subset \{z : |z| < r\}$. In this case the winding number $n(\gamma, z)$ is a constant $2\pi i N$ on E and γ is homologous in the unbounded component of $\mathbb{C} \setminus E$ to $N\gamma_r$, where γ_r is the circle of radius r around 1. The Cauchy theorem then implies

$$\int_\gamma \frac{dz}{\sqrt{1-z^2}} = N\int_{\gamma_r} \frac{1}{(1+z)\sqrt{\frac{1-z}{1+z}}}\,dz. = N\int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{re^{i\theta}\sqrt{\frac{2}{re^{i\theta}}-1}} = iN\int_0^{2\pi} \frac{d\theta}{\sqrt{\frac{2}{re^{i\theta}}-1}}$$

It is tempting at this point to attempt to use the residue theorem, but that theorem is valid only for isolated singularities, and our square root has singularities along a connected set joining ± 1. Instead we observe that we may take $r \to \infty$ without changing the value of the integral. We find that the integrand converges to $1/\sqrt{-1}$, which is one of $\pm i$ (we cannot determine which without knowing more about the set E; draw some pictures to see why). Therefore the possible values of the integral are $\pm 2\pi N$, or any element of $2\pi\mathbb{Z}$.

4.5.2.1 Let $f(z) = z^7 - 2z^6 - z + 1$. Then $|f(z)| \leq 5$ on $|z| = 1$, so Rouché's theorem implies $6z^3 + f(z)$ has the same number of roots as $6z^3$ in the unit disc, namely three.

4.5.2.2 We use Rouché's theorem twice. For $|z| = 2$, $|z^4| = 16 > |-6z + 3|$, so $z^4 - 6z + 3$ has 4 roots in $|z| \leq 2$. For $|z| = 1$, $|6z| = 6 > |z^4 + 3|$, so there is one root in $|z| \leq 1$. We conclude that there are 3 roots in $1 < |z| < 2$.