Math 5120: Complex analysis. Homework 7 Solutions

4.3.3.1 Observe that $f(0) = 0$ and $f'(0) \neq 0$, so that f is injective on some neighborhood of 0. Also $f(z)$ is not injective on $|z| < 2$, because $f(0) = f(-1) = 0$. It follows that $R = \sup\{r : f \text{ is injective on } |z| < r\}$ is positive and finite.

One easy way to find R is by direct computation. The roots of $f(z) - \zeta$ are $z = -\frac{1}{2} \pm \frac{\sqrt{1+4\zeta^2}}{2}$, so they are at opposite points of a diameter of a circle radius $s = \left|\frac{\sqrt{1+4\zeta^2}}{2}\right|$ around $-\frac{1}{2}$. One of these points has real part less than $-\frac{1}{2}$, so is distance at least $\frac{1}{2}$ from 0. We conclude that no open disc $|z| < r$ with $r \leq \frac{1}{2}$ contains two roots of $f(z) - \zeta$. Moreover if $r > \frac{1}{2}$ we may choose ζ to make s so small that $|z - \frac{1}{2}| < s$ is inside $|z| < r$, at which point there will be two roots of $f(z) - \zeta$ in $|z| < r$. This proves that $R = \frac{1}{2}$.

A more general approach is to first recognize that an analytic f cannot be injective on an open set containing a critical point. The reason is that if $f'(z_0) = 0$, then $f(z) - f(z_0)$ has a zero of some order $n \geq 2$ at z_0. The argument principle then says that if w is sufficiently close to $f(z_0)$, the function $f(z) - w$ must have n simple roots in a neighborhood of z_0, so f cannot be injective. (See book, top of page 132.) For our problem, the critical point is at $2z + 1 = f'(z) = 0$, so $z = -\frac{1}{2}$ and we discover $R \leq \frac{1}{2}$.

Now if f is a polynomial, then we know that all critical points lie in the convex hull of the roots (see Theorem 1 on page 29 of the book). So if we have a polynomial f of degree n and a w so $f(z)-w$ has n roots (counting multiplicity) in $|z| < r$, then f also has a critical point in $|z| < r$. In the situation we face, with $n = 2$, we discover that if there are 2 roots of $f(z) - \zeta$ in $|z| < r$ then $r > \frac{1}{2}$. Thus $R \geq \frac{1}{2}$, and we conclude $R = \frac{1}{2}$.

It is interesting to think how, if at all, this idea could be generalized.

4.3.3.4 We are given f analytic at 0 and $f'(0) \neq 0$. Then $f(z) - f(0) = zh(z)$ with h analytic at 0 and $h(0) = f'(0) \neq 0$. We may then take an open disc U around 0 which is so small that h is analytic on this disc and $|h(z) - h(0)| < |h(0)|$ for $z \in U$. This condition on h implies $\log h(z)$ and hence $k(z) = h(z)^{1/n}$ are well-defined (single-valued) analytic functions on U, from which $f(z^n) = f(0) + z^n h(z^n) = f(0) + \left(zk(z^n)\right)^{1/n}$ on U, so taking $g(z) = zk(z^n)$ completes the proof.