Math 5120: Complex analysis. Homework 6 Solutions

4.2.3.1(a) By the Cauchy formula for derivatives, $\int_{|z|=1} z^{-n} f(z)\,dz = 2\pi i f^{(n-1)}(z)/(n-1)!$ provided f is analytic in the unit disc. Applying this to $f(z) = e^z$ gives $\int_{|z|=1} z^{-n} e^z\,dz = 2\pi i/(n-1)!$.

4.2.3.1(b) If n and m are both non-negative then the integrand in $\int_{|z|=2} z^n(1-z)^m\,dz$ is a polynomial, hence analytic, and so the integral is zero by the Cauchy formula. If n is negative and m non-negative then our previous reasoning says the result is $2\pi i f^{(|n|-1)}(0)/(|n|-1)!$ for $f(z) = (1-z)^m$, which is 0 if $m < |n|-1$ and $2\pi i(-1)^{|n|-1}\binom{m}{|n|-1}$ otherwise. Similarly, if n is non-negative and m negative then we get 0 if $n < |m|-1$ and $2\pi i g^{(|m|-1)}(1)/(|m|-1)! = 2\pi i\binom{n}{|m|-1}$ otherwise (with $g(z) = z^n$). If both m and n are negative it is a little more work, basically because we do not have the Cauchy formula for general curves, so cannot reduce to a circle around 0 and another around 1. However there is a still a relatively quick argument of the same type as above. We may use partial fractions to write $z^n(1-z)^m = P(z)z^n + Q(z)(1-z)^m$, where P has degree at most $|n|-1$ and Q has degree at most $|m|-1$; note that $P(z)(1-z)^{-m} + Q(z)z^{-n} = 1$. Integrating the $P(z)z^n$ term gives $2\pi i P^{(|n|-1)}(0)/(|n|-1)!$, which is just $2\pi i$ times the leading coefficient p of $P(z)$. Similarly integrating the $Q(z)(1-z)^m$ term gives $(-1)^m 2\pi i$ times the leading coefficient q of Q. These leading terms give the power of z^{-m-n-1} in $P(z)(1-z)^{-m} + Q(z)z^{-n}$ to be $(-1)^m(p+q)$, and this must be zero because $m, n < 0$ implies $-m-n \geq 2$, so $-m-n-1 \geq 1$. We may summarize our results as

$$\int_{|z|=2} z^n(1-z)^m\,dz = \begin{cases} 2\pi i(-1)^{-n-1}\binom{m}{-n-1} & \text{if } n < 0 \text{ and } m \geq -n-1 \\ 2\pi i\binom{n}{-m-1} & \text{if } m < 0 \text{ and } n \geq -m-1 \\ 0 & \text{otherwise} \end{cases}$$

4.2.3.1(c) We write $|z-a|^2 = (z-a)(\bar{z}-\bar{a}) = (z-a)((\rho^2/z)-\bar{a})$ on $|z| = \rho$ and use the fact that on $z = \rho e^{i\theta}$ we have $|dz| = \rho d\theta$ and $dz = ire^{i\theta}d\theta = izd\theta$, so $|dz| = \rho dz/iz$. Thus the integral is

$$\int_{|z|=\rho} |z-a|^{-4}\,|dz| = \int_{|z|=\rho} \frac{1}{(z-a)^2}\Big(\frac{\rho^2}{z}-\bar{a}\Big)^{-2}\frac{\rho}{iz}\,dz = \frac{\rho}{i}\int_{|z|=\rho}\frac{z}{(z-a)^2(\rho^2-\bar{a}z)^2}\,dz.$$

Now by assumption, $|a| \neq \rho$. The integrand has singularities at $z = a$ and $z = \rho^2/\bar{a}$, which are reflection-symmetric in the circle $|z| = \rho$, so only one is inside the circle. The Cauchy formula then says that the integral is $2\pi\rho f'(a)$ with $f(z) = z/(\rho^2-\bar{a}z)^2$ if $|a| < \rho$, which evaluates to $2\pi\rho(\rho^2+|a|^2)/(\rho^2-|a|^2)^3$. Similarly if $|a| > \rho$ it becomes $2\pi\rho g'(\rho^2/\bar{a})$ with $g(z) = z/(\bar{a})^2(z-a)^2$, which is the negative of the previous answer. Summarizing

$$\int_{|z|=\rho} |z-a|^{-4}\,|dz| = \begin{cases} \frac{2\pi\rho(\rho^2+|a|^2)}{(\rho^2-|a|^2)^3} & \text{if } |a| < \rho \\ -\frac{2\pi\rho(\rho^2+|a|^2)}{(\rho^2-|a|^2)^3} & \text{if } |a| > \rho. \end{cases}$$

4.2.3.2 We assume f is globally analytic and there is C so $|f(z)| \leq C|z|^n$ for all sufficiently large $|z|$. Let $P(z)$ be the $(n-1)^{\text{th}}$ order Taylor polynomial of $f(z)$ at 0, so $f(z)-P(z)$ has a zero of order n at 0 and therefore $f(z) - P(z) = z^n g(z)$ for some globally analytic $g(z)$. Since P is an order $n-1$ polynomial we have $|P(z)|/|z|^n \leq 1$ for all sufficiently large $|z|$; using the hypothesis we get

$$|g(z)| \leq \frac{|f(z)|}{|z|^n} + \frac{|P(z)|}{|z|^n} \leq C + 1$$

for all sufficiently large $|z|$, whence continuity of g implies it is bounded and Liouville's theorem implies it is a constant a. Thus $f(z) = az^n + P(z)$ is a polynomial of degree at most n.

4.2.3.5 Suppose f is analytic at z_0. Let $g(z) = f(z) - f(z_0)$. If $n \geq 1$ is an integer and $r > 0$ is sufficiently small (so f is analytic on $|z| \leq r$), then by the Cauchy formula

$$\left|f^{(n)}(z_0)\right| = \left|g^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{|z-z_0|=r}\frac{g(z)}{(z-z_0)^{n+1}}\,dz\right| \leq \frac{n!}{r^n}\max_{|z-z_0|=r}|g(z)| = \frac{n!}{r^n}\max_{|z-z_0|=r}|f(z)-f(z_0)|.$$

Putting $r = 1/n$ we see that $n!n^n\left|f^{(n)}(z_0)\right| \leq \max_{|z-z_0|=1/n}|f(z) - f(z_0)| \to 0$ as $n \to \infty$. This is strictly stronger than the statement that $\left|f^{(n)}(z_0)\right| > n!n^n$ cannot occur for an unbounded sequence of $n \in \mathbb{N}$.

4.3.2.1 The algebraic order of f at a is defined to be that h such that

$$\lim_{z\to a}|z-a|^\alpha|f(z)| = \begin{cases}\infty & \alpha < h\\ 0 & \alpha > h.\end{cases}$$

If f has order h and g has order k at $z = a$, then it is immediate that $|z-a|^\alpha|f(z)g(z)| = \infty$ for $\alpha < h+k$ and $= 0$ for $\alpha > h+k$, so the order of $f(z)g(z)$ is $h+k$. It is also easy to see that the order of $1/g$ is $-k$; combining these results we see that the order of $f(z)/g(z)$ is $h-k$.

To see that the order of $f+g$ cannot exceed $\max\{h,k\}$ it suffices to note that $|z-a|^\alpha|f(z)+g(z)| \leq |z-a|^\alpha|f(z)| + |z-a|^\alpha|g(z)| \to 0$ as $z \to a$, by the triangle inequality.

4.3.2.3 The map e^z takes any infinite horizontal strip of width 2π to $\mathbb{C}\setminus\{0\}$, so it takes any punctured neighborhood $|z| > R$ of ∞ to $\mathbb{C}\setminus\{0\}$. It follows that e^z does not have a limit in $\bar{\mathbb{C}}$ as $z \to \infty$, and therefore that the isolated singularity of e^z at ∞ is essential. Similar arguments apply to $\cos z$ and $\sin z$. All we need verify is that the image of $|z| > R$ cannot converge in $\bar{\mathbb{C}}$ as $R \to \infty$, which is true for cos because it maps all lines $z = 2\pi k + iy$ onto $[0,\infty) \subset \mathbb{R}$ and for sin because it maps the same lines onto the imaginary axis.

4.3.2.5 Suppose f has an isolated singularity at a, so f is analytic on $0 < |z-a| < \delta$. If either $\Re f$ or $\Im f$ is bounded on this punctured disc, then the image of the punctured disc cannot be dense in $\mathbb{C}$, so the Casorati-Weierstrass theorem implies the singularity cannot be essential. The singularity is therefore removable or a pole. However, if it is a pole then $f(z) = (z-a)^{-k}g(z)$ for some $k \in \mathbb{N}$ and g analytic on $|z-a| < \delta$ with $g(a) \neq 0$. Then we can easily verify that both $\Re f$ and $\Im f$ are unbounded on $0 < |z-a| < \delta$ as follows. Take $z = a + re^{i\theta}$, so $f(z) = r^{-k}e^{-ik\theta}g(z)$. For $r > 0$ so small that $|g(z) - g(a)| < |g(a)|/2$ and θ chosen so $e^{-ik\theta}g(a)$ is real we have $\Re f(z) \geq r^{-k}|g(a)|/2$. A similar argument works for the imaginary part. It follows that the singularity cannot be a pole, so it must be removable.

4.3.2.6 Let f have an isolated singularity at a, so f is analytic on $0 < |z-a| < \delta$. Consider $g(z) = e^{f(z)}$, which also has an isolated singularity at a. Suppose the singularity of $g(z)$ is a pole. Then $\lim_{z\to a}|g(z)| = \infty$, and $|g(z)| = e^{\Re f(z)}$, so $\lim_{z\to a}\Re f(z) = \infty$. In particular, $f(z)$ is unbounded on $0 < |z-a| < \delta$, so the singularity of f is not removable. Also, $f(z)$ omits a left half-plane, so the Casorati-Weierstrass theorem implies the singularity of f cannot be essential. Thus f has a pole at a. Reasoning as in the previous question we see that for any sufficiently small $r > 0$, the image of any $0 < |z-a| < r$ under f covers a set of the form $|z| > R$. However the exponential map takes any horizontal infinite strip of height 2π to $\mathbb{C} \setminus \{0\}$. Since a region of the form $|z| > R$ contains such a horizontal strip we conclude that g maps any $0 < |z - a| < r$ to all of $\mathbb{C} \setminus \{0\}$, contradicting the assumption that g has a pole at a. We conclude that if $f(z)$ has an isolated singularity at a then $e^{f(z)}$ cannot have a pole at a.