Math 5120: Complex analysis. Homework 5 Solutions

4.1.3.2 Using the parametrization, $z = re^{i\theta}$, so $z = x + iy$ with $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\begin{aligned}\int_{|z|=r} x\,dz &= \int_0^{2\pi} r\cos\theta i r e^{i\theta}\,d\theta \\ &= ir^2\int_0^{2\pi}\cos^2\theta + i\cos\theta\sin\theta\,d\theta \\ &= ir^2\int_0^{2\pi}\frac{1}{2}(\cos 2\theta + 1)\,d\theta - r^2\int_0^{2\pi}\frac{1}{2}\sin 2\theta\,d\theta \\ &= \pi i r^2.\end{aligned}$$

Using $x = \frac{1}{2}(z + r^2 z^{-1})$ as suggested in the text we use that $\int_{|z|=r} z\,dz = 0$ by equation (11) on pg 107 and $\int_{|z|=r} z^{-1}\,dz = 2\pi i$ by the following discussion to conclude $\int_{|z|=r} x\,dz = \pi i r^2$.

4.1.3.3 The function $(z^2-1)^{-1}$ does not have an antiderivative on the disc $|z| \le 2$, so we cannot directly apply the results from section 3.3. Since we do not yet have the Cauchy theorem, we must integrate directly using a parameter. This is messy, but the point is to get you to appreciate the Cauchy theorem and residue calculus once we have them. Take $z = 2e^{i\theta}$, so the integral is

$$\int_{-\pi}^{\pi}\frac{2ie^{i\theta}}{4e^{2i\theta}-1}\,d\theta = \int_{-\pi}^{\pi}\frac{2ie^{i\theta}(4e^{-2i\theta}-1)}{(4e^{2i\theta}-1)(4e^{-2i\theta}-1)}\,d\theta = \int_{-\pi}^{\pi}\frac{8ie^{-i\theta}-2ie^{i\theta}}{17-8\cos 2\theta}\,d\theta = \int_{-\pi}^{\pi}\frac{6i\cos\theta + 10\sin\theta}{17-8\cos 2\theta}\,d\theta.$$

The real part of the integral has an antisymmetric integrand on a symmetric interval so is zero. For the imaginary part we note from symmetry and periodicity of cosine that

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{\cos\theta}{17-8\cos 2\theta}\,d\theta &= 2\int_0^{\pi}\frac{\cos\theta}{17-8\cos 2\theta}\,d\theta \\ &= 2\int_0^{\pi/2}\frac{\cos\theta}{17-8\cos 2\theta}\,d\theta + 2\int_{pi/2}^{\pi}\frac{\cos\theta}{17-8\cos 2\theta}\,d\theta \\ &= 2\int_0^{\pi/2}\frac{\cos\theta}{17-8\cos 2\theta}\,d\theta + 2\int_{pi/2}^{0}\frac{\cos(\pi-\theta)}{17-8\cos(2\pi-2\theta)}\,(-d\theta) \\ &= 2\int_0^{\pi/2}\frac{\cos\theta}{17-8\cos 2\theta}\,d\theta + 2\int_0^{\pi/2}\frac{-\cos(\theta)}{17-8\cos(2\theta)}\,d\theta \\ &= 0\end{aligned}$$

so that the whole integral is zero.

4.1.3.6 We know that f'/f should have antiderivative $\log f$, provided that $\log f$ is well defined. Suppose then that $|f(z)-1| < 1$ for $z \in \Omega$. We know that log is a single-valued analytic function on $\mathbb{C} \setminus (-\infty, 0]$, with derivative z^{-1}. Since the values of f avoid $(-\infty, 0]$ it follows that $\log f$ is well defined and analytic, with derivative (by the chain rule) $f'(z)/f(z)$. Hence the integrand is the derivative of an analytic function on Ω and by the result on page 107 the integral around any closed curve in Ω is zero.

4.1.3.8 In order that $\int_\gamma \log z = 0$ is meaningful and true, we must first require that γ lies in a region where $\log z$ is a single-valued function (otherwise the integral is not well-defined). We know $\Omega = \mathbb{C} \setminus (-\infty, 0]$ is such a region (in fact, so is the complement of any connected set containing 0 and ∞). Now we might hope to find that $\int_\gamma \log z = 0$ for any closed curve in Ω by proving that $\log z$ is the derivative of an analytic function. Since we know that $(\log z)' = z^{-1}$ on Ω, it is easy to check that $(z \log z - z)' = \log z + z/z - 1 = \log z$ there. Hence a suitable condition is that γ is a closed curve in Ω.

4.2.2.1 Using Cauchy's integral formula we have

$$\int_{|z|=1} \frac{e^z}{z}\, dz = 2\pi i e^0 = 2\pi i.$$

4.2.2.2 Writing $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$ we can apply the Cauchy integral formula to each of the below integrals

$$\int_{|z|=2} \frac{1}{z^2+1}\, dz = \frac{1}{2i}\int_{|z|=2} \frac{1}{z-i}\, dz - \frac{1}{2i}\int_{|z|=2} \frac{1}{z+i}\, dz = \pi - \pi = 0.$$

Compare this to exercise 4.1.3.3 above!