Math 5120: Complex analysis. Homework 4 Solutions

3.3.1.1 The map $z \mapsto \bar{z}$ fixes 0,1 and ∞. Consider a fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ fixing these points. Fixing 0 implies $b = 0$, fixing ∞ implies $c = 0$, fixing 1 implies $a/d = 1$, so the map is the identity. Thus $z \mapsto \bar{z}$ is not a fractional linear transformation.

3.3.3.1 A reflection $z \mapsto z^* = g(z)$ is defined via $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$. Let ϕ be the fractional linear transformation so $\phi(z_1) = 1$, $\phi(z_2) = 0$ and $\phi(z_3) = \infty$, and let $f(z) = \bar{z}$. Then $\phi(z^*) = \overline{\phi(z)}$, so $g = \phi^{-1} \circ f \circ \phi$. Now we know from the book that ϕ maps circles to circles (on the Riemann sphere), so it suffices to see that the same is true of f. A circle not through ∞ has the form $|z - a| = r$, so is mapped to $r = |\bar{z} - a| = |z - \bar{a}|$, which is also a circle. A circle through ∞ is $z = x + iy$ with $ax + by + c = 0$, $a, b, c \in \mathbb{R}$, and is mapped to $z = x + iy$ with $ax - by + c = 0$, which is also a circle through ∞.

3.3.3.5 It is easy to think of some fractional linear transformations that fix $|z| = R$ as a set. For example, any rotation around the origin (i.e. $z \mapsto az$ with $|a| = 1$), the map $z \mapsto R^2/z$. In the case $R = 1$ we also know from prior homeworks that $z \mapsto \frac{z-a}{1-\bar{a}z}$ for $|a| \neq 1$ preserves $|z| = 1$, so $z \mapsto R\frac{(z/R)-a}{1-\bar{a}(z/R)} = \frac{Rz-aR^2}{R-\bar{a}z}$ preserves $|z| = R$. The more difficult question is whether we have found all of the maps or not. One way to approach the problem is to take an arbitrary fractional linear transformation that fixes $|z| = R$, compose it with some of the maps we know, and try to get the result to fix 3 points, so that the composition is the identity.

Suppose that ϕ is a fractional linear transformation that fixes $|z| = R$ as a set. Let $e^{i\theta} = \phi(R)/R$, so $z \mapsto e^{-i\theta}\phi(z)$ fixes the point $z = R$. Let $a = e^{-i\theta}\phi(0)/R$ and $\psi(z) = \frac{Rz-aR^2}{R-\bar{a}z}$. Then $z \mapsto \psi(e^{-i\theta}\phi(z))$ fixes 0 and R, and also fixes $|z| = R$ as a set. Since the mapping of reflection in a circle depends only on the circle, this map must preserve pairs of points that are symmetric under reflection in $|z| = R$. Now $\{0, \infty\}$ has such reflective symmetry and $\psi(e^{-i\theta}\phi(0)) = 0$, so we conclude $\psi(e^{-i\theta}\phi(\infty)) = \infty$. Thus we have a fractional linear transformation fixing three points, which we know is the identity, and so $\psi(e^{-i\theta}\phi(z)) = z$, from which $\phi(z) = e^{i\theta}\psi^{-1}(z)$. It is easy to compute that $\psi^{-1}(z) = \frac{Rz+aR^2}{R+\bar{a}z}$. We have thus shown that

$$\phi(z) = e^{i\theta}\frac{Rz + aR^2}{R + \bar{a}z}.$$

3.3.3.7 If we had a fractional linear transformation taking $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ to concentric circles, then by composing with a translation to move their common center to 0 and a dilation we could assume that the fractional linear transformation preserves the unit circle; note that neither operation affects the ratio of radii of the image circles. By exercise 3.3.3.5 with $R = 1$, such a map is of the form $z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z} = f(z)$, $|a| \neq 1$. The rotation preserves both the concentric property and the ratio of radii, so we are free to choose it, for example by setting $\theta = 0$. Now observe that the initial and final configurations are mapped to themselves by $z \mapsto \bar{z}$, so $g(z) = \overline{f(\bar{z})} = e^{-i\theta}\frac{z-\bar{a}}{1-az}$ also preserves $|z| = 1$ and takes $|z - \frac{1}{4}| = \frac{1}{4}$ to a circle with center at 0. If the image of $|z - \frac{1}{4}| = \frac{1}{4}$ under f is $|z| = r$, it follows from exercise 3.3.3.5 that

$$f \circ g^{-1}(z) = \frac{\frac{z+\bar{a}}{1+az} - a}{1 - \bar{a}\frac{z+\bar{a}}{1+az}} = \frac{(1 - a^2)z + (\bar{a} - a)}{(1 - |a|^2) + (a - \bar{a})z} = \frac{1 - a^2}{1 - |a|^2}\frac{z + \frac{\bar{a}-a}{1-a^2}}{1 + \frac{a-\bar{a}}{1-|a|^2}z}$$

is a fractional linear transformation fixing $|z| = r$, so is of the form $z \mapsto e^{i\phi} \frac{rz+r^2 b}{r+\bar{b}z}$. We conclude that $a \in \mathbb{R}$. Knowing this, we see that f preserves $\mathbb{R}$, and therefore that $f(0)$ and $f(\frac{1}{2})$ are the points $\pm r$. This gives us that simultaneously $-a = \pm r$ and $(\frac{1}{2} - a)/(1 - \frac{a}{2}) = \mp r$, which we reduce to $r^2 \pm 4r + 1 = 0$, or (finding only solutions with $r > 0$), $a = r = 2 \pm \sqrt{3}$. We have found necessary conditions for f to map $|z - \frac{1}{4}| = \frac{1}{4}$ to $|z| = r$, and it remains to see they are sufficient. But f is a fractional linear transformation so maps circles to circles; by construction the image of $|z - \frac{1}{4}| = \frac{1}{4}$ passes through $\pm r$, and since $a \in \mathbb{R}$ the image is mapped to itself by $z \mapsto \bar{z}$, so it is $|z| = r$. We have therefore determined that the map is one of $z \mapsto e^{i\theta} \frac{z-a}{1-az}$ for $a = 2 \pm \sqrt{3}$ and the ratio of the smaller to larger radius is $2 - \sqrt{3} : 1 = 1 : 2 + \sqrt{3}$.

3.3.3.8 We are now asked to repeat this problem for $|z| = 1$ and $x = 2$. All of the same arguments apply to say that the map is $z \mapsto e^{i\theta} \frac{z-a}{1-az}$ for $a \in \mathbb{R}$. The points $z = 2$ and $z = \infty$ are invariant under $z \mapsto \bar{z}$, so go to $\pm r$. Computing as in the previous problem we find again $r = 2 \pm \sqrt{3}$, but with $a = \frac{1}{r} = 2 \mp \sqrt{3}$. Alternatively we could notice that the map $z \mapsto \frac{1}{z}$ takes the circle in 3.3.3.7 to the line in this problem (to see this, note what happens to 0 and ∞ and that the symmetry under $z \mapsto \bar{z}$ is preserved), so the whole problem is obtained from 3.3.3.7 by composition with $z \mapsto \frac{1}{z}$.

3.4.2.2 The circles intersect at $z = 1$ and are parallel there, so any fractional linear transformation mapping 1 to ∞ will give us the region between two circles that intersect at ∞ with zero angle, meaning a strip between two parallel lines. There are many fractional linear transformations taking 1 to ∞, but it is convenient to take one that maps one of the circles to the real axis; for example $z \mapsto i\frac{z+1}{z-1}$ takes $|z| = 1$ to $\mathbb{R}$ because it takes -1 to 0 and i to 1. The image of the other line must have the form $z = x + ic$ for a constant c; by computing $0 \mapsto -i$ we find $c = -i$. Now we can multiply by $-\pi$ and exponentiate to get the upper half plane. Our final map is

$$z \mapsto \exp\Big(-\pi i \frac{z+1}{z-1}\Big).$$

3.4.2.3 The arc $|z| = 1$, $y \geq 0$ can be mapped to a segment on the real line by a fractional linear transformation, and we know how to map the complement of a segment on the complement of the unit disc (or rather we know how to map the complement of the unit disc on the complement of the segment $[-2, 2]$ using the map $z \mapsto z + z^{-1}$). For the first step we should take a point on $|z| = 1$ to ∞, and since it is convenient to make the image segment symmetric around 0 we may as well take i to 0 and $-i$ to ∞. After multiplying by 2 so our arc goes to $[-2, 2]$, the map is $z \mapsto 2i\frac{z-i}{z+i}$.

Now the trickier issue is how to invert $z \mapsto z + z^{-1}$. Formally using the quadratic equation the inverse is $z \mapsto \frac{1}{2}(z \pm \sqrt{z^2 - 4})$, but we have to make sense of the square root. If $z^2 - 4$ mapped the complement of $[-2, 2]$ to the complement of a ray connecting 0 and ∞ we could use a version of the usual square root, but it does not – the image in this case is the complement of the interval $[-4, 0]$. However, we can do whatever algebraic manipulation we desire to the expression for the formal inverse and still have a formal inverse. Knowing that we want the bit inside the square root to omit a ray from 0 to ∞ (preferably the negative real axis) and that right now what is in there omits $[-4, 0]$, tends to suggest we should divide by either z or $z - 2$ to move an endpoint to ∞. A little playing with the formula yields a different formal expression for the inverse branches

$$f_\pm(z) = \frac{z}{2}(1 \pm \sqrt{1 - 4/z^2})$$

in which $1 - 4/z^2$ is easily seen to omit the negative real axis (and the ray $(2, \infty)$ on the positive real axis). We can therefore use the usual definition of $\sqrt{\cdot}$ taking $\mathbb{C} \setminus (-\infty, 0]$ to the right half plane. The result is that the branches $f_\pm(z)$ are each well defined on the complement of $[-2, 2]$.

Now note that the two branches of the inverse map either to the interior or the exterior of the unit disc, and that we want the one which maps to the exterior. We can verify that the desired one is $\frac{z}{2}(z + \sqrt{1 - 4z^{-2}})$ by observing that it has image converging to ∞ as $z \to \infty$.

Our final result is therefore that

$$z \mapsto \frac{1}{2}\Big(2i\frac{z-i}{z+i}\Big)\Big(1 + \Big(1 - \frac{4(z+i)^2}{-4(z-i)^2}\Big)^{1/2}\Big) = \frac{1+iz}{z+i}\Big(1 + \Big(1 + \frac{(z+i)^2}{(z-i)^2}\Big)^{1/2}\Big)$$

where we readily determine that for z not on our arc, $1 + \frac{(z+i)^2}{(z-i)^2}$ is not on the negative real axis.

3.4.2.7 We wish to map the exterior of an ellipse on the interior of the unit disc with preservation of symmetries, those being $z \mapsto \bar{z}$ and $z \mapsto -z$. Recall that in studying the map $z \mapsto z + z^{-1} = f(z)$ we saw that the circle $z = re^{i\theta}$, is mapped to $z = x + iy$ with $x = (r + r^{-1})\cos\theta$ and $y = (r - r^{-1})\sin\theta$, which is an ellipse centered at 0 with semi-major axis along the real line and of length $a = r + r^{-1}$ and semi-minor axis along the imaginary axis and of length $b = |r - r^{-1}|$; this is the ellipse $(x/a)^2 + (y/b)^2 = 1$, $a > b$. The map also preserves the symmetries of the ellipse, in that $f(\bar{z}) = \overline{f(z)}$ and $f(-z) = -f(z)$.

The map $z \mapsto f(z)$ is a double covering; if $r > 1$ then both $|z| > r$ and $|z| < 1/r$ are mapped onto the the exterior of the ellipse. It is easy to see that when $r > 1$ we should have $r = (a - b)/2$ and thus $1/r = 2/(a - b)$. The inverse map has two branches $z \mapsto \frac{z}{2}(1 \pm \sqrt{1 - 4z^{-2}})$ which were discussed in the previous problem. Here we need the branch $z \mapsto \frac{z}{2}(1 - \sqrt{1 - 4z^{-2}})$ which maps the exterior of the ellipse to the interior of the disc of radius $2/(a - b)$. There are only a few more difficulties. The first is that we need the ellipse to have $a > b$, so that a preliminary rotation is needed if $b > a$. The second is that we need $r = (a - b)/2 > 1$ in the above analysis. To ensure that this is the case it is convenient to perform a preliminary dilation, for example one ensuring that $r = 2$. Then the inverse branch of f will map to the disc $|z| < 1/2$, and an additional dilation will bring us to $|z| < 1$. At this point it is simplest to split into cases.

Case 1: $a > b$. Then $(x/a)^2 + (y/b)^2 = 1$ has its longer axis along the real line. We perform the dilation $z \mapsto 4z/(a - b)$, so that the length of the semi-major axis becomes $4a/(a-b)$ and of the semi-minor axis becomes $4b/(a-b)$; the exterior of this is the image of $|z| < 1/2$ under $f(z)$, so applying $z \mapsto \frac{z}{2}(1 - \sqrt{1 - 4z^{-2}})$ takes it to $|z| < 1/2$ and dilating by a factor of 2 gives $|z| < 1$. The result is

$$z \mapsto 2\frac{4z}{2(a-b)}\Big(1 - \Big(1 - \frac{4(a-b)^2}{16z^2}\Big)^{1/2}\Big) = \frac{4z}{a-b}\Big(1 - \Big(1 - \frac{(a-b)^2}{4z^2}\Big)^{1/2}\Big)$$

Case 2: $b > a$. We can use the same argument as above, but must first rotate by $\pi/2$ to get the semi-major axis along the real line, and must undo this at the end. The maps are $z \mapsto iz \mapsto 4iz/(b-a)$, then the inverse branch, then dilation by 2 and rotation by $-i$, resulting in

$$z \mapsto -2i\frac{4iz}{2(b-a)}\Big(1-\Big(1-\frac{4(b-a)^2}{-16z^2}\Big)^{1/2}\Big) = \frac{4z}{b-a}\Big(1-\Big(1+\frac{(a-b)^2}{4z^2}\Big)^{1/2}\Big)$$

Case3: $b = a$. Here we have just the circle $|z| = a = b$, so we can scale by $z \mapsto z/a$ to the unit circle and invert to map the exterior to the interior. The resulting map is

$$z \mapsto \frac{a}{z}.$$