Math 5120: Complex analysis. Homework 3 Solutions

3.1.2.6 $\mathbb{R}$ and $\mathbb{C}$ are separable metric spaces, so see Exercise 3.1.3.7 below.

3.1.3.4 Let $A = \{(0, y) : |y| \leq 1\}$ and $B = \{(x, \sin(1/x) : x > 0\}$ in $\mathbb{R}^2$. Note that both A and B are continuous images of connected subsets of $\mathbb{R}$, so are connected. If there is a disconnection of $A \cup B$, then the set containing a point of A must contain all of A because A is connected, and similarly for B. So the disconnection must consist of open sets $U \supset A$ and $V \supset B$. But then U contains a disc around 0, and any such disc contains points from B, for example those of the form $((2\pi n)^{-1}, \sin(2\pi n))$ for all sufficiently large $n \in \mathbb{N}$, so $U \cap V \neq \emptyset$ and there is no such disconnection. We conclude that $A \cup B$ is connected. (Note: $A \cup B$ is not path connected).

3.1.3.5 Let $E_n = \{(x, 1/n) : 0 \leq x \leq 1\}$ and $E_\infty = \{(x, 0) : 0 \leq x \leq 1\}$ in $\mathbb{R}^2$. Let $E = E_\infty \cup (\cup_n E_n)$. For any n let U_n be the open set consisting of all points within distance $(n + 1)^{-2}$ of E_n, and V_n be the interior of the complement of U_n. Then $U_n \cap V_n = \emptyset$, $U_n \supset E_n$ and $V_n \supset E \setminus E_n$. We conclude that the maximal connected set containing a point of E_n is contained in E_n. Since E_n is connected (because it is a curve), we conclude that E_n is a component of E for each n. For a point in E_∞ we see that the component containing this point must contain E_∞, because E_∞ is connected; from the above it does not contain any point of E_n for any n, so E_∞ is a component. We have therefore found the decomposition of E into its components. We have also seen that the components of the form E_n are closed in $\mathbb{R}^2$ and relatively open in E; the component E_∞ is closed in $\mathbb{R}^2$, but not relatively open in E because any open neighborhood of E_∞ in $\mathbb{R}^2$ intersects some E_n.

Finally, a set is locally connected if every neighborhood of a point in the set contains a connected neighborhood of the point. Consider $(0, 0) \in E_\infty$. Any neighborhood of this point is the intersection of E with an open set in $\mathbb{R}^2$, and therefore cannot be connected because it contains points from another component E_n. Since this point has no connected neighborhoods it is not locally connected. (Remark: local connectivity of a set is equivalent to all components being open.)

3.1.3.7 S is discrete in a metric space (M, d), so for all $x \in S$ there is r_x such that the ball around x of radius r_x, denoted $B(x, r_x)$, has $B(x, r_x) \cap S = \{x\}$. Observe that then $B(x, r_x/2) \cap B(y, r_y/2) = \emptyset$ if $x \neq y$, because if there is z in the intersection we would have

$$|x - y| \leq |x - z| + |y - z| < (r_x + r_y)/2 \leq \max\{r_x, r_y\}$$

contradicting either $x \ni B(y, r_y)$ or $y \ni B(x, r_x)$.

Now if $A \subset M$ is a dense set then for each x there is $a_x \in B(x, r_x/2) \cap A$, and $a_x \neq a_y$ if $x \neq y$ because $B(x, r_x/2) \cap B(y, r_y/2) = \emptyset$. The map $x \mapsto a_x$ is then an injection from S to A, so the cardinality of S cannot exceed that of A. In particular if M is separable we may take A to be a countable dense subset and conclude S is countable.

3.2.2.1 We want to give a domain and a definition of $\sqrt{\cdot}$ so that $\sqrt{1 + z} + \sqrt{1 - z}$ is a single-valued analytic function. In the book there is a definition of a single valued analytic branch of $\sqrt{w}$ for $w \in \mathbb{C} \setminus (-\infty, 0]$. It suffices then that we ensure $1 + z$ and $1 - z$ are in this set, which is true if $z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Nothing more need be done, as on this set $\sqrt{1 + z} + \sqrt{1 - z}$ is a sum of single-valued analytic functions.