Math 5120: Complex analysis. Homework 2 Solutions

2.1.2.4 Suppose $f = u + iv$ is analytic and $|f|$ is constant, so $|f|^2 = u^2 + v^2 = c$. Analyticity gives existence of the partial derivatives u_x, u_y, v_x, v_y and validity of the Cauchy-Riemann equations. Differentiating $|f|^2 = c$ gives $uu_x + vv_x = uu_y + vv_y = 0$; Cauchy-Riemann lets us rewrite these as

$$\begin{pmatrix} u & v \\ v & -u \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant is $|f|^2 = c$, we discover that either $c = 0$, whence $f = 0$ everywhere, or the matrix is invertible and $u_x = v_x = 0$, from which also $u_y = v_y = 0$ by Cauchy-Riemann and thus u and v, and hence f, are constant.

2.1.2.5 Each of the following statements is equivalent.

$$f(z) \text{ is analytic at } z = z_0.$$

$$\lim_{z\to 0} \frac{f(z+z_0) - f(z_0)}{z} \text{ exists.}$$

$$\lim_{z\to 0} \frac{\overline{f(z+z_0)} - \overline{f(z_0)}}{\bar{z}} \text{ exists.}$$

$$\lim_{w\to 0} \frac{\overline{f(\bar{w}+\bar{w}_0)} - \overline{f(\bar{w}_0)}}{w} \text{ exists. (Set } w = \bar{z},\ w_0 = \bar{z}_0\text{)}$$

$$\overline{f(\bar{w})} \text{ is analytic at } w = w_0.$$

2.1.4.1(a) Let $R(z) = z^4(z^3-1)^{-1}$. It has poles at ∞, 1, $e^{2\pi i/3}$, $e^{2\pi i/3}$. Set $\tau = 2\pi i/3$. We need to expand at each pole, for which purpose (using the notation in the book) perform each of the following expansions. We include only the relevant terms, noting the presence of those with lower order using an ellipsis.

$$R(z) = z + \frac{z}{z^3-1} \text{ so } G(z) = z$$

$$R(1+z^{-1}) = \frac{(1+z^{-1})^4}{(1+z^{-1})^3 - 1} = \frac{(z+1)^4}{z(z+1)^3 - z^4} = \frac{z^4 + \cdots}{3z^3 + \cdots} = \frac{z}{3} + \cdots$$

$$G_1\Big(\frac{1}{z-1}\Big) = \frac{1}{3(z-1)}$$

$$R(e^\tau + z^{-1}) = \frac{(e^\tau + z^{-1})^4}{(e^\tau + z^{-1})^3 - 1} = \frac{(e^\tau z + 1)^4}{z(e^\tau z+1)^3 - z^4} = \frac{e^{4\tau}z^4 + \cdots}{3e^{2\tau}z^3 + \cdots} = \frac{e^{2\tau}z}{3} + \cdots$$

$$G_{e^\tau}\Big(\frac{1}{z-e^\tau}\Big) = \frac{e^{2\tau}}{3(z-e^\tau)}$$

$$R(e^{2\tau} + z^{-1}) = \frac{(e^{2\tau} + z^{-1})^4}{(e^{2\tau} + z^{-1})^3 - 1} = \frac{(e^{2\tau} z + 1)^4}{z(e^{2\tau} z+1)^3 - z^4} = \frac{e^{8\tau}z^4 + \cdots}{3e^{4\tau}z^3 + \cdots} = \frac{e^{4\tau}z}{3} + \cdots = \frac{e^{\tau}z}{3} + \cdots$$

$$G_{e^{2\tau}}\Big(\frac{1}{z-e^{2\tau}}\Big) = \frac{e^{\tau}}{3(z-e^{2\tau})}$$

Thus

$$R(z) = z + \frac{1}{3(z-1)} + \frac{e^{4\pi i/3}}{3(z-e^{2\pi i/3})} + \frac{e^{2\pi i/3}}{3(z-e^{4\pi i/3})}.$$

2.1.4.1(b) In a similar but more tedious calculation, $R(z) = z^{-1}(z+1)^{-2}(z+2)^{-3}$ has poles at 0, -1 and -2. We perform the following expansions, including only the relevant terms.

$$R(z^{-1}) = \frac{1}{z^{-1}(z^{-1}+1)^2(z^{-1}+2)^3} = \frac{z^6}{(1+z)^2(1+2z)^3} = \frac{z^6}{8z^5+\cdots} = \frac{z}{8}+\cdots$$
$$G_0\Big(\frac{1}{z}\Big) = \frac{1}{8z}$$

$$R(-1+z^{-1}) = \frac{1}{(-1+z^{-1})z^{-2}(z^{-1}+1)^3} = \frac{z^6}{(z-1)(1+z)^3} = \frac{z^6}{-z^4-2z^3+\cdots} = -z^2+2z+\cdots$$
$$G_{-1}\Big(\frac{1}{(z+1)}\Big) = \frac{-1}{(z+1)^2} + \frac{2}{(z+1)}$$

$$\begin{aligned} R(-2+z^{-1}) &= \frac{1}{(-2+z^{-1})(-1+z^{-1})^2z^{-3}} = \frac{z^6}{(-2z+1)(-z+1)^2} \\ &= \frac{z^6}{-2z^3+5z^2-4z+\cdots} = \frac{-z^3}{2} - \frac{5z^2}{4} - \frac{17z}{8} + \cdots \end{aligned}$$
$$G_{-2}\Big(\frac{1}{(z+2)}\Big) = \frac{-1}{2(z+2)^3} - \frac{5}{4(z+2)^2} - \frac{17}{8(z+2)}$$

Thus
$$R(z) = \frac{1}{8z} + \frac{-1}{(z+1)^2} + \frac{2}{(z+1)} - \frac{1}{2(z+2)^3} - \frac{5}{4(z+2)^2} - \frac{17}{8(z+2)}.$$

2.1.4.4 Suppose $R(z)$ is a rational function with $|R(z)| = 1$ on $|z| = 1$. Then
$$S(z) = \frac{1}{\overline{R(\frac{1}{\bar z})}}$$
is also a rational function, and since $w = \frac{1}{\bar w}$ whenever $|W| = 1$ we see that $R(z) = S(z)$ on $|z| = 1$. But then $R(z) = S(z)$, because their difference is a rational function with zeros at every point of the unit circle. Now observe that if $R(z)$ has a root at a of order α then $S(z)$ has a pole at $z = (\bar a)^{-1}$ with the same order, so $R(z) = S(z)$ has a pole of this order at this location. This argument is reversible; if $R(z)$ has a pole at b of order β then $S(z)$ has a zero at $z = (\bar b)^{-1}$ with the same order, so $R(z) = S(z)$ has a zero of this order at this location. Thus the poles and zeros of R are bijectively paired by the map $z \mapsto (\bar z)^{-1}$. (Note that this implies there are neither poles nor zeros at points of $|z| = 1$, because for these points $z = (\bar z)^{-1}$ and there cannot simultaneously be a zero and a pole at a single point.) An equivalent formulation is that $R(z)$ is necessarily a product of factors of the form $\frac{z-a}{1-\bar a z}$ with $|a| \neq 1$. Since we proved in Exercise 1.1.4.3 that such factors have modulus 1 on the unit circle, this condition is also sufficient. Grouping the factors and using negative powers α_j when a zero is outside the unit disc we see that the general form of the rational function we seek is
$$R(z) = cz^k \prod_{j=1}^{m} \Big(\frac{z-a_j}{1-\bar a_j z}\Big)^{\alpha_j}$$
with k and α_j in $\mathbb{Z}$ and $|a_j| < 1$ for all j.

2.1.4.6 Suppose $R(z) = cP(z)/Q(z)$ is a rational function, where P and Q are monic polynomials without common zeros. Let $m = \deg(P)$ and $n = \deg(Q)$. It will be convenient at first to assume neither P nor Q is constant. We have

$$R'(z) = c\frac{P'(z)Q(z) - P(z)Q'(z)}{Q^2(z)}.$$

It is an easy observation that $\deg(P') = m-1$ and $\deg(Q') = n-1$, thus $\deg(P'Q) = \deg(PQ') \le m+n-1$. Observe that the lead term in $P'Q$ has coefficient n while that in PQ' has coefficient m. If we assume $m \ne n$ then these terms cannot cancel, so $\deg(P'Q - PQ') \le m+n-1$ with equality at least when $m \ne n$. To find the degree of $R'(z)$ we must then cancel common factors. Any root of Q with order α is a root of $P'Q$ with the same order, so to be a root of $P'Q - PQ'$ it must divide P or Q'. By assumption P and Q have no common roots, so it must be a root of Q'. An easy computation shows that Q and Q' have common roots if and only if Q has multiple roots, and that the order of the root in Q' is $\alpha - 1$. We conclude that Q'/Q is of the form S/T, where T is the (monic) product of the distinct factors of Q and has order $k \le n$, and $\deg S = k-1$. It is then apparent that

$$R'(z) = c\frac{P'(z)T(z) - P(z)S(z)}{Q(z)T(z)}$$

with $\deg(P'T - PS) \le m+k-1$ (with equality if $m \ne n$), $\deg(QT) = n+k$, and the numerator and denominator having no common roots. It follows that

$$\deg(R') \le \max\{m+k-1, n+k\} = \max\{m-1, n\}+k \le \max\{m-1, n\}+n \le \deg R+n \le 2\deg(R)$$

and that equality holds at the first inequality if $m \ne n$, at the second if $k = n$ (i.e. all poles of R are distinct), at the third and fourth if $\deg R = n \ge m$. Thus we have $\deg(R') \le 2\deg(R)$ and conditions under which equality holds.

We may also get lower bounds. If $m \ne n$ then we have equality in the first inequality above, so $\deg(R') = \max\{m+k-1, n+k\} \ge \max m, n+1 \ge \deg(R)$ because we assumed Q non-constant and thus $k \ge 1$. If $m = n$ then $\deg(P'T - PS) \le m+k-1 < n+k = \deg(QT)$, so $\deg(R') = n+k \ge \deg(R)+1$. (Note this also implies that $\deg(R') = 2\deg(R)$ when $m = n = k$

What remains are the special cases where either P or Q is constant. If Q is constant and P is not then R is a polynomial, so $\deg(R') = \deg(R) - 1$. If P is constant and Q is not then the above reasoning implies $R' = S/QT$, so $\deg(R') = \deg(QT) = k+n > n = \deg(R)$. If both are constant then $\deg(R') = -\infty$.

We may summarize our results as follows. If $R(z)$ is a non-constant rational function, then $\deg(R) - 1 \le \deg(R) \le 2\deg(R)$, with equality on the left iff R is a polynomial and equality on the right at least when the poles of R are distinct and there is no pole at infinity.

2.2.3.3 Suppose $\sum |a_j|$ converges. A re-ordering of the sum is given by a bijection $\eta : \mathbb{N} \to \mathbb{N}$, where the new sum is $\sum_j a_{\eta(j)}$. Both sums converge absolutely; set $S = \sum_j a_j$ and $T = \sum_j a_{\eta(j)}$. Given $\epsilon > 0$ let N be so large that for any $K \ge N$, each of $\sum_{j>K} |a_j| \le \epsilon$, $\left|S - \sum_{j\le K} a_j\right| \le \epsilon$ and $\left|T - \sum_{j\le K} a_{\eta(j)}\right| \le \epsilon$. Let M be so large that $\{\eta(j) : j \le M\} \supset \{j \le N\}$ (Note that then $M \ge N$.)

$$|S - T| \le \left|\sum_{j\le M} a_j - \sum_{j\le M} a_{\eta(j)}\right| + 2\epsilon \le \sum_{j>N} |a_j| + 2\epsilon \le 3\epsilon$$

where middle inequality used that the sum is over those j in $(\{j \le M\} \setminus \{\eta(j) : j \le M\}) \cup (\{\eta(j) : j \le M\} \setminus \{j \le M\})$, which does not include any $j \le N$.

2.2.3.4 For fixed z with $|z| \geq 1$ we have $|nz^n| = n|z|^n \to \infty$, while if $|z| < 1$ we have $n|z|^n = \exp(\log n + n\log|z|) \to 0$ because $\log|z| < 0$ and $(n\log|z|)/\log n \to -\infty$ as $n \to \infty$ (using, for example, L'Hopital's rule). Hence $\{nz^n\}$ converges pointwise on $|z| < 1$. A slight improvement is that on $|z| \leq r < 1$ we have $n|z|^n \leq nr^n \to 0$, so that the convergence is uniform on this disc. Any compact set that lies in $|z| < 1$ is in such a disc, so we have $\{nz^n\}$ converges uniformly to the zero function on any compact subset of the open unit disc. Moreover, this is the largest collection of sets on which the convergence is uniform. To see this, suppose S is a set on which $\{nz^n\}$ converges uniformly, from which we see S is in the unit disc and $n|z|^n \to 0$. If S is not a compact subset of the disc then it contains points z_k such that $|z_k| \to 1$. If n is so large that $n|z|^n < 1/2$ on S then the fact that $n|z_k|^n \to n$ as $k \to \infty$ provides a contradiction.

2.2.3.6 There are a number of ways of doing this, the most usual being to use summation by parts. I will show a method that is messier, but perhaps easier to visualize. Let $U = \sum_j u_j$, $V = \sum_j v_j$; suppose WLOG that $\sum_j u_j$ is the absolutely convergent series and set $W = \sum_j |u_j|$. It is helpful to arrange the terms in a doubly-infinite list.

$$(u_j v_k)_{j,k} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \dots \\ u_2v_1 & u_2v_2 & u_2v_3 & \dots \\ u_3v_1 & u_3v_2 & u_3v_3 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

We see immediately that $\sum_{j=1}^{n-1} u_j v_{n-j}$ is the sum along the n^{th} upward diagonal, so $\sum_{n=1}^{m}\sum_{j=1}^{n-1} u_j v_{n-j}$ is the sum over the upper left triangle $j + k \leq m$ in the list. We can also see that UV may be approximated by the sum over a square $j \leq p$, $k \leq p$, because for fixed $\epsilon > 0$ we can take N so $p \geq N$ implies $\left|U - \sum_{j\leq p} u_j\right| \leq \epsilon$, $\left|V - \sum_{k\leq p} v_k\right| \leq \epsilon$, and so

$$\left|UV - \Bigl(\sum_{j\leq p} u_j\Bigr)\Bigl(\sum_{k\leq p} v_k\Bigr)\right| = \left|UV - V\sum_{j\leq p} u_j + V\sum_{j\leq p} u_j - \Bigl(\sum_{j\leq p} u_j\Bigr)\Bigl(\sum_{k\leq p} v_k\Bigr)\right| \leq |V|\epsilon + \left|\sum_{j\leq p} u_j\right|\epsilon \leq \epsilon\Bigl(|V| + \sum_{j\leq p} |u_j|\Bigr) \leq \epsilon(|V| + W).$$

Thus the difference between UV and the sum we are considering is controlled by a small term plus the sum over a region inside an upper left triangle and outside a square, i.e. $j + k \leq m$ but also $j \geq p$, $k \geq p$.

$$\left|UV - \sum_{n=1}^{m}\sum_{j=1}^{n-1} u_j v_{n-j}\right| \leq \epsilon(|V| + W) + \left|\sum_{n=1}^{m}\sum_{j=1}^{n-1} u_j v_{n-j} - \Bigl(\sum_{j\leq p} u_j\Bigr)\Bigl(\sum_{k\leq p} v_k\Bigr)\right| \leq \epsilon(|V| + W) + \left|\sum_{\{j\geq p, k\geq p, j+k\leq m\}} u_j v_k\right|.$$

We split this sum into two pieces, one with $k \geq p$ and one with $k \leq p$. We want a bound independent of m, and you should think that $m \gg p$. Since it is a finite sum it can be rearranged any way we like; we will sum first along the rows and then down the columns. For the piece with $k \geq p$ the sum along the j^{th} row is bounded as follows:

$$\left|u_j \sum_{k=p+1}^{m-j} v_k\right| \leq |u_j|\epsilon$$

provided p is so large that $\left|\sum_{p+1}^{q} v_k\right| \leq \epsilon$ for all $q \geq p$; this can be achieved by (if necessary) increasing N, because $\sum v_k$ is convergent. Summing over all relevant rows we have a contribution bounded as follows:

$$\left|\sum_{j=1}^{m-p} u_j \sum_{k=p+1}^{m-j} v_k\right| \le \epsilon \sum_{j=1}^{m-p} |u_j| \le \epsilon \sum_{j=1}^{\infty} |u_j| = W\epsilon.$$

Now for the piece with $k < p$ we reason as follows. Since $\sum_k v_k$ converges, $\sum_{k\le r} v_k$ is a convergent sequence in r, so has absolute value bounded by a constant X. Now the sum across each row in the second piece is of size

$$\left|u_j \sum_{1}^{\min\{p-1,m-j\}} v_k\right| \le X|u_j|$$

and summing over the rows we have

$$\left|\sum_{j=p+1}^{m} u_j \sum_{1}^{\min\{p-1,m-j\}} v_k\right| \le X \sum_{j=p+1}^{m} |u_j| \le X \sum_{j=p+1}^{\infty} |u_j| \le X\epsilon$$

provided p is large enough that $\sum_{j\ge p+1} |u_j| \le \epsilon$; again this can be achieved by increasing N. Combining our estimates we now have

$$\left|UV - \sum_{n=1}^{m} \sum_{j=1}^{n-1} u_j v_{n-j}\right| \le \epsilon(|V| + 2W + X)$$

and the result follows.

By the way, if you are wondering how this is connected to the material in this chapter, the following may be of interest. Consider the functions $U(z) = \sum_j u_j z^j$ and $V(z) = \sum_j v_j z^j$. These converge at $z = 1$, so have radius of convergence at least 1; moreover Abel's limit theorem says that $U(r) \to U(1)$ and $V(r) \to V(1)$ as $r \to 1$, where $r \in (0, 1)$. On the disc $|z| < 1$ they are absolutely convergent, and their product $U(z)V(z)$ has a power series expansion $\sum_j c_n z^n$. We can compute by the Leibniz rule

$$c_n = \frac{1}{n!}\frac{d^n}{dz^n}(U(z)V(z))\Big|_{z=0} = \frac{1}{n!}\sum_{j=0}^{n} \frac{n!}{j!(n-j)!}\frac{d^j U(z)}{dz^j}\Big|_{z=0}\frac{d^{n-j}V(z)}{dz^{n-j}} = \sum_{j=1}^{n-1} u_j v_{n-j}.$$

If we now knew that $\sum_n c_n = \sum_n \sum_{j=1}^{n-1} u_j v_{n-j}$ was convergent, then Abel's limit theorem would imply $\sum_n c_n = \lim_{r\to 1} U(r)V(r) = \lim_{r\to 1} U(r) \lim_{r\to 1} V(r) = U(1)V(1)$, which is the theorem we seek. This suggests that there is a proof essentially by the proof of Abel's limit theorem, which is the standard summation by parts proof.

2.2.4.2

$$\begin{aligned}\frac{2z+3}{z+1} &= \frac{2(z-1)+5}{(z-1)+2} = 2 + \frac{1}{2}\frac{1}{1+(z-1)/2}\\ &= 2 + \frac{1}{2}\sum_{j=0}^{\infty}\frac{(-1)^j}{2^j}(z-1)^j \text{ if } |z-1| < 2.\end{aligned}$$

The radius of convergence is 2.

2.2.4.4 The following statements are equivalent.

$$\sum_n a_n z^n \text{ has radius of convergence } R$$
$$\limsup |a_n|^{1/n} = R^{-1}$$
$$\limsup |a_n|^{1/2n} = R^{-1/2} \text{ and } \limsup |a_n|^{2/n} = R^{-2}$$
$$\sum_n a_n z^{2n} \text{ has radius of convergence } \sqrt{R}, \text{ and}$$
$$\sum_n a_n^2 z^n \text{ has radius of convergence } R^2$$

2.2.4.8 The series $\sum_0^\infty z^n(1+z)^{-n}$ converges if and only if $|z(1+z)^{-1}| < 1$, which is if and only if $|z| < |1+z|$, if and only if $\Re(z) > \frac{1}{2}$.

2.3.2.2

$$\cos iz = \frac{e^{-z} + e^z}{2} = \cosh z$$
$$\sin iz = \frac{e^{-z} - e^z}{2i} = i \sinh z$$

We can get addition formulae for cosh and sinh by computing

$$\cosh a \cosh b = \frac{e^a + e^{-a}}{2} \frac{e^b + e^{-b}}{4} = \frac{e^{a+b} + e^{-(a+b)} + e^{a-b} + e^{b-a}}{4}$$
$$\sinh a \sinh b = \frac{e^a - e^{-a}}{2} \frac{e^b - e^{-b}}{4} = \frac{e^{a+b} + e^{-(a+b)} - e^{a-b} - e^{b-a}}{4}$$
$$\cosh a \sinh b = \frac{e^a + e^{-a}}{2} \frac{e^b - e^{-b}}{4} = \frac{e^{a+b} - e^{-(a+b)} - e^{a-b} + e^{b-a}}{4}$$

elementary algebra implies

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b \qquad \cosh 2a = \cosh^2 a - \sinh^2 a$$
$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b \qquad \sinh 2a = 2 \sinh a \cosh a$$

and substitution of the above expressions for cos and sin retrieves the usual trigonometric addition formulae, which could alternatively be used to obtain the above.

2.3.2.3

$$\cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y$$
$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

2.3.4.4

$$e^z = 2 \text{ when } z = \log 2 + 2k\pi i, \quad k \in \mathbb{Z}$$
$$e^z = -1 \text{ when } z = (1 + 2k)\pi i, \quad k \in \mathbb{Z}$$
$$e^z = i \text{ when } z = \Big(\frac{1}{2} + 2k\Big)\pi i, \quad k \in \mathbb{Z}$$
$$e^z = \frac{-i}{2} \text{ when } z = -\log 2 + \Big(\frac{3}{2} + 2k\Big)\pi i, \quad k \in \mathbb{Z}$$
$$e^z = -1 - i \text{ when } z = \frac{1}{2}\log 2 + \Big(\frac{5}{4} + 2k\Big)\pi i, \quad k \in \mathbb{Z}$$
$$e^z = 1 + 2i \text{ when } z = \frac{1}{2}\log 5 + (\arctan 2 + 2k)\pi i, \quad k \in \mathbb{Z} \text{ and arctan in } (0, \pi/2).$$

2.3.4.5 Let $z = x + iy$, so $e^z = e^x \cos y + ie^x \sin y$. Then the real and imaginary parts of $\exp(e^z)$ are obtained from

$$\exp(e^z) = \exp(e^x \cos y + ie^x \sin y) = \exp(e^x \cos y)\cos(e^x \sin y) + i\exp(e^x \cos y)\sin(e^x \sin y)$$

2.3.4.6

$$2^i = \exp(i \log 2) = \cos(\log 2) + i\sin(\log 2) \text{ single valued because } 2 \in \mathbb{R}$$
$$i^i = \exp(i \log i) = \exp\Big(\frac{-\pi}{2} - 2k\pi\Big), \quad k \in \mathbb{Z}$$
$$(-1)^{2i} = \exp(2i\log(-1)) = \exp(-2\pi - 4\pi k), \quad k \in \mathbb{Z}.$$

Note that the last example shows something we have lost in making the convention that the logarithm is single valued on the positive reals and multivalued elsewhere, because $(-1)^{2i} \neq ((-1)^2)^i = 1$, but simply contains 1 as one of its values.