Math 5120: Complex analysis. Homework 10 Solutions

4.6.2.2 If $M(r) = 0$ for some $r > 0$ then f vanishes on a set containing a limit point, so $f \equiv 0$ and the result is trivial. Hence there is no loss of generality in assuming $M(r) > 0$ for $r > 0$, in which case the statement

$$M(r) \leq M(r_1)^{\alpha} M(r_2)^{(1-\alpha)}$$

for $\alpha = \frac{\log(r_2/r)}{\log(r_2/r_1)}$ is equivalent to

$$\log M(r) \leq \alpha \log M(r_1) + (1-\alpha) \log M(r_2) = \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2)$$

which is the same as

$$(\log r_2 - \log r_1) \log M(r) \leq \log r_2 \log M(r_1) - \log r_1 \log M(r_2) + (\log M(r_2) - \log M(r_1)) \log r$$

or

$$(\log M(r_1) - \log M(r_2)) \log r + (\log r_2 - \log r_1) \log M(r) \leq \log r_2 \log M(r_1) - \log r_1 \log M(r_2)$$

and it is this that we will prove.

It is suggested in the book that we apply the maximum principle (for harmonic functions) to a linear combination of $\log|z| + \log|f(z)|$. Of course we cannot do this directly if f has zeros, because $\log|f(z)|$ is not harmonic in any neighborhood of a zero of f (in fact it is subharmonic, and there is still a maximum principle for subharmonic functions, but we have not proved that). We will therefore need to do something about points where f is zero, but let us begin by assuming that no such points exist.

If f is analytic on the annulus $0 < r_1 < |z| < r_2$ then $A \log|z| + B \log|f(z)|$ is harmonic there, and the maximum principle for harmonic functions implies that the maximum occurs on the boundary. We obtain

(1)

$$A \log r + B \log M(r) = \max_{|z|=r} \Big(A \log|z| + B \log|f(z)| \Big) \leq \max\{A \log r_1 + B \log M(r_1), A \log r_2 + B \log M(r_2)\}$$

Taking $A = \log M(r_1) - \log M(r_2)$ and $B = \log r_2 - \log r_1$ we find that the terms on the right are both equal to $\log r_2 \log M(r_1) - \log r_1 \log M(r_2)$. Thus

$$\log r_2 \log M(r_1) - \log r_1 \log M(r_2) \geq (\log M(r_1) - \log M(r_2)) \log r + (\log r_2 - \log r_1) \log M(r)$$

which is what we needed to prove.

Now we deal with the points $\{z_j\}$ where $f(z) = 0$. Such points are can accumulate only at the boundary. Suppose that around each we place a small disc of radius δ_j (small enough that it is inside the annulus), and delete these discs from our domain. Then (1) must be modified so that for each j there is a term on the right side corresponding to the maximum of $A \log|z| + B \log|f(z)|$ on the new boundary circle $|z - z_j| = \delta_j$. However $B \log|f(z)| \to -\infty$ as $z \to z_j$, so we may choose δ_j so small that this new term is less than the right side of (1), and therefore need not be included. It follows that (1) is still valid when f has zeros, and therefore the result holds for general f.

Note that there is a degenerate case we did not consider, namely $r_1 = 0$. In this situation one should interpret the formula for α as corresponding to $\alpha = 0$, whereupon the result follows directly from the usual maximum principle for the harmonic function $|f(z)|$.

5.1.1.1 Let $K \subset \mathbb{C}$ be compact and $M = \max_{z\in K}|z|$. Observe that for $n > M$ we have $\left|\frac{z}{n}\right| < 1$ when $z \in K$. The principal branch of the logarithm is well-defined on $\{w : |1 + w| < 1\}$, so we conclude $\log(1 + \frac{z}{n})$ is well-defined on K for all $n > M$ and has Taylor expansion

$$\log\Big(1 + \frac{z}{n}\Big) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\Big(\frac{z}{n}\Big)^j.$$

The series is readily seen to be convergent on $\left|1+\frac{z}{n}\right| < 1$, thus uniformly convergent on compact subsets of this region, and in particular on K for $n > M$. Uniformity of the convergence implies we can exchange the limits in

$$\lim_{n\to\infty} n\log\Big(1 + \frac{z}{n}\Big) = \sum_{j=1}^{\infty} \lim_{n\to\infty} \frac{(-1)^{j+1}}{j}\frac{z^j}{n^{j-1}} = z$$

Exponentiating both sides and using continuity of the exponential we get that

$$e^z = \lim_{n\to\infty} \exp n\log\Big(1 + \frac{z}{n}\Big) = \lim_{n\to\infty}\Big(1 + \frac{z}{n}\Big)^n$$

uniformly on K, and since K was arbitrary the convergence is uniform on all compact sets in $\mathbb{C}$.

5.1.2.3 We wish to develop $\log(\frac{\sin z}{z})$ around 0 up to terms of order z^6. Since $\sin z$ has a simple zero at 0 the function $\frac{\sin z}{z}$ has a removable singularity at 0 and its extension (which is equal to 1 at 0) is entire. It is helpful to recall the series for $\sin z$ and divide by z to obtain a series convergent uniformly on all compact sets to $\frac{\sin z}{z}$

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{j=0}^{\infty}\frac{(-1)^j z^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty}\frac{(-1)^j z^{2j}}{(2j+1)!} = 1 + \sum_{j=1}^{\infty}\frac{(-1)^j z^{2j}}{(2j+1)!}.$$

Next we may compose with the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}w^k$ for $\log(1 + w)$, which is convergent for $|w| < 1$. This amounts to setting $w = \sum_{j=1}^{\infty}\frac{(-1)^j z^{2j}}{(2j+1)!}$, which we note satisfies $|w| < 1$ on a neighborhood of 0. Observe that since the lead z-term is z^2 it suffices to consider the 3$^{\text{rd}}$-order polynomial in w. We have

$$\begin{aligned}
\log\Big(\frac{\sin z}{z}\Big) &= \Big(\frac{\sin z}{z} - 1\Big) - \frac{1}{2}\Big(\frac{\sin z}{z} - 1\Big)^2 + \frac{1}{3}\Big(\frac{\sin z}{z} - 1\Big)^3 + [z^8] \\
&= \sum_{j=1}^{3}\frac{(-1)^j z^{2j}}{(2j+1)!} - \frac{1}{2}\left(\sum_{j=1}^{2}\frac{(-1)^j z^{2j}}{(2j+1)!}\right)^2 + \frac{1}{3}\left(\sum_{j=1}^{1}\frac{(-1)^j z^{2j}}{(2j+1)!}\right)^3 + [z^8] \\
&= -\frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} - \frac{1}{2}\left(-\frac{z^2}{3!} + \frac{z^4}{5!}\right)^2 + \frac{1}{3}\left(-\frac{z^2}{3!}\right)^3 + [z^8] \\
&= \Big(-\frac{1}{3!}\Big)z^2 + \Big(\frac{1}{5!} - \frac{1}{2(3!)^2}\Big)z^4 + \Big(-\frac{1}{7!} + \frac{1}{(3!)(5!)} - \frac{1}{3(3!)^3}\Big)z^6 + [z^8] \\
&= -\frac{z^3}{3!} + \frac{(3-5)z^4}{2^3 3^2 5} + \frac{(-9 + 63 - 70)z^6}{2^4 3^4 5\cdot 7} + [z^8] \\
&= -\frac{z^3}{2\cdot 3} - \frac{z^4}{2^2 3^2 5} - \frac{z^6}{3^4 5\cdot 7} + [z^8]
\end{aligned}$$