# Math 5120: Complex analysis. Homework 1 Solutions

1.1.1.2 Omitting details of the computation, with $z = x + iy$ the answers are:

- $\Re z^4 = x^4 - 6x^2y^2 + y^4$, $\Im z^4 = 4xy(x^2 - y^2)$.
- $\Re z^{-1} = x/(x^2 + y^2)$, $\Im z^{-1} = -y/(x^2 + y^2)$.
- $\Re(z - 1)/(z + 1) = (x^2 + y^2 - 1)/((x + 1)^2 + y^2)$, $\Im(z - 1)/(z + 1) = 2y/((x + 1)^2 + y^2)$.
- $\Re z^{-2} = (x^2 - y^2)/(x^2 + y^2)^2$, $\Im z^{-2} = -2xy/(x^2 + y^2)^2$.

1.1.1.3 This is an easy direct computation, or can be done by noting that all of these numbers have modulus 1, the first pair have argument $\pi \pm \frac{\pi}{3}$ (so multiplication by 3 gives multiples of $2\pi$), while the other pair in the second set have arguments $\pm\frac{\pi}{3}$ (so multiplication by 6 gives multiples of $2\pi$).

1.1.2.4 One may apply the quadratic formula.

1.1.4.3 If $|a| = 1$ then

$$\left|\frac{a-b}{1-\bar{a}b}\right| = |a|\left|\frac{a-b}{a-|a|^2b}\right| = 1$$

and an analogous argument using $\bar{b}$ holds if $|b| = 1$. The only exception is when the denominator vanishes, which occurs iff $\bar{a}b = 1$. In the case that either $|a| = 1$ or $|b| = 1$ we conclude immediately that this occurs iff $a = b$.

1.1.4.4 An equation in complex variables is two equations in real variables, in this case two simultaneous linear equations. A fast way to extract them is to take the complex conjugate, so we have

$$\begin{aligned} az + b\bar{z} + c &= 0 \\ \bar{a}\bar{z} + \bar{b}z + \bar{c} &= 0. \end{aligned}$$

Multiplying by $\bar{a}$ and $b$ respectively,

$$\begin{aligned} |a|^2z + \bar{a}b\bar{z} + \bar{a}c &= 0 \\ \bar{a}b\bar{z} + |b|^2z + b\bar{c} &= 0 \end{aligned}$$

the difference eliminates $\bar{z}$ and we obtain

$$(|a|^2 - |b|^2)z + (\bar{a}c - b\bar{c}) = 0$$

so that there is a unique solution iff $|a| \neq |b|$, in which case the solution is

$$z = \frac{b\bar{c} - \bar{a}c}{|a|^2 - |b|^2}$$

1.1.5.1 Each of the following statements is equivalent.

$$\begin{gathered} \left|\frac{a-b}{1-\bar{a}b}\right|^2 < 1 \\ \frac{a-b}{1-\bar{a}b}\,\frac{\bar{a}-\bar{b}}{1-a\bar{b}} < 1 \\ |a|^2 + |b|^2 - b\bar{a} - a\bar{b} < 1 - \bar{a}b - a\bar{b} + |a|^2|b|^2 \\ |a|^2(1 - |b|^2) < 1 - |b|^2 \\ \Big(|a| < 1 \text{ and } |b| < 1\Big) \text{ or } \Big(|a| > 1 \text{ and } |b| > 1\Big). \end{gathered}$$

1.1.5.3 By induction and the triangle inequality

$$\Big|\sum_{j=1}^{n} \lambda_j a_j\Big| \le \sum_{j=1}^{n} |\lambda_j||a_j| < \sum_{j=1}^{n} |\lambda_j| = 1.$$

1.1.5.4 It is easiest to solve this problem if you recognize it as the equation of an ellipse with foci at $\pm a$, as then you know what to expect. In any case it is easy to see

$$2|a| = |a + a| = |a - z + a + z| \le |z - a| + |z + a| = 2|c|$$

so that $|a| \le |c|$ is necessary for the existence of $z$ satisfying the equation. Then for $|c| \ge |a|$ we may set $z = |c|a/|a|$ and find that $|z - a| = |c| - |a|$, $|z + a| = |c| + |a|$, so $|z - a| + |z + a| = 2|c|$, whence it is also sufficient.

To find the maximum of $|z|$ on the curve is easy, because repetition of our first argument shows $|z| \le |c|$ on the curve, but the point $z = |c|a/|a|$ has $|z| = |c|$, so this maximum is achieved. The minimum is a little trickier. One way is as follows:

$$4|z|^2 = 4z\bar{z} = ((z - a) + (z + a))((\bar{z} - \bar{a}) + (\bar{z} + \bar{a})) = |z - a|^2 + |z + a|^2 + 2|z|^2 - 2|a|^2$$

so $2(|z|^2 - |a|^2) = |z - a|^2 + |z + a|^2$. Then use the general inequality $2xy \le x^2 + y^2$ for $x, y \in \mathbb{R}$ to see $2|z - a||z + a| \le |z - a|^2 + |z + a|^2$ with equality iff $|z - a| = |z + a|$, and substitute into the square of the equation for $2|c|$ to obtain $2|c|^2 \le |z - a|^2 + |z + a|^2$. Combining these we have $2|c|^2 \le 2(|z|^2 - |a|^2)$, so $|z|^2 \ge |c|^2 - |a|^2$, with equality iff $|z - a| = |z + a|$. The latter fact allows us to guess that a point at distance $\sqrt{|c|^2 - |a|^2}$ from 0 in a direction perpendicular to $a$ will lie on the curve, a guess that is rapidly verified using Pythagoras' theorem.

1.2.1.1 The easy way to do this is to note that symmetry is preserved under rotation by $\pi/4$, such that the lines become the axes, then use that the symmetric points with respect to the axes are obtained by $\pm$ and conjugation, and rotate back. The points are then

$$\begin{aligned} e^{-i\pi/4}(\pm e^{i\pi/4} a) &= \pm a, \\ e^{-i\pi/4}\overline{(\pm e^{i\pi/4} a)} &= \pm e^{-i\pi/4} e^{-i\pi/4} \bar{a} = \mp i\bar{a}. \end{aligned}$$

1.2.1.2 The vertices of an equilateral triangle are mapped to the vertices of an equilateral triangle by any map of the form $z \mapsto \alpha(z + \beta)$, and for any given equilateral triangle there is a map of this form that takes it to the triangle with vertices 1, $e^{2\pi i/3}$, $e^{-2\pi i/3}$. The correct choice is $3\beta = a_1 + a_2 + a_3$ and $\alpha = 3/(2a_1 - a_2 - a_3)$. It is obvious that the equality

$$a_1^2 + a_2^2 + a_3^2 = a_1 a_2 + a_2 a_3 + a_3 a_1$$

is preserved by the map $z \mapsto \alpha z$, and readily verified that it is preserved by $z \mapsto z + \beta$. Thus it suffices to verify the equality for the triangle with vertices 1, $e^{2\pi i/3}$, $e^{-2\pi i/3}$, in which case both sides are easily checked to sum to 2.

1.2.2.2 If $z = e^{i\phi}$ then

$$\begin{aligned} 1 + \cos\phi + \ldots + \cos n\phi &= \Re(1 + z + \ldots + z^n) \\ &= \Re \frac{1 - z^{n+1}}{1 - z} \\ &= \Re \frac{1 - \bar{z} - z^{n+1} + z^n}{1 + |z|^2 - z - \bar{z}} \\ &= \frac{1 - \cos\phi - \cos(n+1)\phi + \cos n\phi}{2 - 2\cos\phi} \\ \sin\phi + \ldots + \sin n\phi &= \Im \frac{1 - \bar{z} - z^{n+1} + z^n}{1 + |z|^2 - z - \bar{z}} \\ &= \frac{\sin\phi - \sin(n+1)\phi + \sin n\phi}{2 - 2\cos\phi} \end{aligned}$$

1.2.2.4 Let $w = \cos(2\pi i/n) + i\sin(2\pi i/n) = e^{2\pi i/n}$. Provided $h$ is not a multiple of $n$ we have $w^{nh} \neq 1$ and thus

$$1 + w^h + \ldots + w^{(n-1)h} = \frac{1 - w^{nh}}{1 - w^h} = \frac{1 - e^{2\pi hi}}{1 - w^h} = 0.$$

1.2.3.1 To see when $az + b\bar{z} + c = 0$ represents a line, recall exercise 1.1.4.4. We must have infinitely many solutions for the simultaneous equations, so they must be the same, so $|a| = |b|$ and $\bar{a}c = b\bar{c}$. An equivalent form of the latter is that $c$ is a real multiple of $(a + b)$; we may find this by summing it with its complex conjugate to find $(a + b)\bar{c} = (\bar{a} + \bar{b})c$, from which the ratio $(a + b)/c$ is real. To be certain that the solution set is one dimensional not two dimensional we must insist that $|a| \neq 0$.

1.2.3.5 Rotation around the origin preserves the circle $|z| = 1$, preserves angles, and may be used to move $a$ to lie on the positive real axis. WLOG $a > 1$. Then the circle through $a$ and $\frac{1}{a}$ has center $\frac{1}{2}(a + \frac{1}{a})$ and radius $\frac{1}{2}(a - \frac{1}{a})$. The circles intersect at right angles iff the radii to the intersection point are at right angles iff (by Pythagoras' theorem) the sum of the squares of the radii of the circles is the square of the distance to the center of the the circle through $a$ and $\frac{1}{a}$. Written algebraically, this is equivalent to

$$1 + \frac{1}{4}(a - \frac{1}{a})^2 = 1 + \frac{1}{4}(a^2 + \frac{1}{a^2} - 2) = \frac{1}{4}(a^2 + \frac{1}{a^2} + 2) = \frac{1}{4}(a + \frac{1}{a})^2$$

1.2.4.5 I think it likely that this problem will have caused difficulties. The reason is that the spherical distance in equation (28) on page 20 of the book is not a geodesic metric! It is the metric corresponding to chords of the sphere, so the triangle inequality is always strict for three distinct points because they cannot all lie on a chord (the chord intersects the sphere at two points only). This and the fact that the center of the disc in the plane is not the same as the center of the circle on the Riemann sphere mean that you cannot do either of the following: (1) take the distance $d(a, a + Ra/|a|)$, or $d(a, a - Ra/|a|)$ (because $a$ is not the center in the spherical metric), (2) take half of $d(a - Ra/|a|, a + Ra/|a|)$ (because the metric is not geodesic, so half of the diameter is not the radius). You can do one thing to simplify your calculations, which is to rotate the plane (an isometry of the spherical metric) so $a \in \mathbb{R}$. At this point you have the option of finding the images on the sphere (using equations (25) and (26) on page 18), then find the center, then find the length of the chord corresponding to the radius. This latter can also be done by Pythagoras' theorem, which makes the problem much simpler.

Suppose that the length of the chord joining the images of $a \pm R$ on the sphere is $2s$. Then the distance from the center of the sphere to the midpoint of the chord is $\sqrt{1 - s^2}$ (by Pythagoras), so the distance radially from this midpoint to the sphere is $1 - \sqrt{1 - s^2}$. The landing point of the ray is the spherical center of the disc, and the spherical radius we seek is the hypotenuse of the right triangle with vertices the landing point, the midpoint of the chord, and the image of $a + R$. Since we have just seen the sides of this triangle have lengths $s$ and $1 - \sqrt{1 - s^2}$ the hypotenuse has length $2(1 - \sqrt{1 - s^2})$ (Pythagoras again). So it suffices to find $s$. By formula 28 on page 20,

$$2s = \frac{2|(a + R) - (a - R)|}{(1 + (a + R)^2)(1 + (a - R)^2)} = \frac{4R}{(1 + (a + R)^2)(1 + (a - R)^2)}.$$

We can then compute

$$\begin{aligned} 1 - s^2 &= \frac{(1 + (a + R)^2)(1 + (a - R)^2) - 4R^2}{(1 + (a + R)^2)(1 + (a - R)^2)} \\ &= \frac{1 + (a + R)^2 + (a - R)^2 + (a^2 - R^2)^2 - 4R^2}{(1 + (a + R)^2)(1 + (a - R)^2)} \\ &= \frac{1 + 2(a^2 - R^2) + (a^2 - R^2)^2}{(1 + (a + R)^2)(1 + (a - R)^2)} \\ &= \frac{(1 + a^2 - R^2)^2}{(1 + (a + R)^2)(1 + (a - R)^2)} \end{aligned}$$

so that

$$
\begin{aligned}
\text{Spherical radius of disc} &= 2(1 - \sqrt{1 - s^2}) \\
&= 2\left(1 - \frac{\left|1 + a^2 - R^2\right|}{\left(1 + (a + R)^2\right)^{1/2}\left(1 + (a - R)^2\right)^{1/2}}\right) \\
&= 2\left(\frac{\sqrt{(1 + a^2 - R^2)^2 + 4R^2} - |1 + a^2 - R^2|}{\sqrt{(1 + a^2 - R^2)^2 + 4R^2}}\right).
\end{aligned}
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