5.2.9 Sps ϕ of period π is given by $\sum_1^\infty a_n \sin nx$ for all $x \in \mathbb{R}$. Then

$$\sum_1^\infty a_n \sin nx = \phi(x) = \phi(x-\pi) = \sum_1^\infty a_n \sin n(x-\pi)$$

$$= \sum_1^\infty a_n \left(\sin(nx)\cos(n\pi) - \cos(nx)\sin(n\pi)\right)$$

$$= \sum_1^\infty a_n (-1)^n \sin nx$$

because $\sin n\pi = 0$, $\cos n\pi = (-1)^n$.

But the sine series expansion is unique and we have two expansions for ϕ. Equivalently, taking the difference of the two sides we find

$$\sum_1^\infty a_n \left(1 - (-1)^n\right) \sin(nx) = 0$$

$$\Rightarrow \quad 2 \sum_{m=1}^\infty a_{2m+1} \sin((2m+1)x) = 0$$

So that all $a_{2m+1} = 0$. These are the odd coeffs in the series for ϕ.

5.3.2 a) Show x is $\perp$ constants on $[-1,1]$

Well the inner product is $\langle 1,x\rangle = \int_{-1}^{1} x\,dx = 0$.

b) Find a quadratic $\perp$ to 1 and x.

Take $f = ax^2+bx+c$ and compute

$$\langle 1,f\rangle = \int_{-1}^{1} f\,dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx\Big|_{-1}^{1} = \frac{2}{3}a + 2c$$

$$\langle x,f\rangle = \int_{-1}^{1} (ax^3+bx^2+cx)\,dx = a\frac{x^4}{4} + b\frac{x^3}{3} + c\frac{x^2}{2}\Big|_{-1}^{1}$$

$$= \frac{2}{3}b$$

So we must have $b=0$, $c = -\frac{a}{3}$. Thus $f = x^2 - \frac{1}{3}$.
(or any const multiple of it)

c) Find a cubic $\perp$ to quadratics.

At this point it is convenient to use Gram-Schmidt.

We have an orthogonal set $1, x, 3x^2-1$.

Take x^3 and remove projection on each:

$$x^3 - \frac{\langle x^3,1\rangle}{\langle 1,1\rangle}1 - \frac{\langle x^3,x\rangle}{\langle x,x\rangle}x - \frac{\langle x^3,3x^2-1\rangle}{\langle 3x^2-1,3x^2-1\rangle}(3x^2-1)$$

$$= x^3 - \left(\frac{\int_{-1}^{1} x^3}{\int_{-1}^{1} 1}\right) - \left(\frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx}\right)x - \left(\frac{\int_{-1}^{1}(3x^5-x^3)\,dx}{\int_{-1}^{1}(3x^2-1)^2\,dx}\right)(3x^2-1)$$

$$= x^3 - 0 - \frac{\left(\frac{2}{5}\right)}{(2/3)}x - 0$$

$$= x^3 - \frac{3}{5}x \text{, or any multiple of this.}$$

5.3.6 $\frac{d}{dx}$ has $\mathbb{C}$ evals when bdy condit is $X(0)=X(1)$.

Find them & determine if efns are $\perp$.

Soln $\frac{df}{dx} = \lambda f \Rightarrow f = e^{\lambda x}$, so all multiples $Ae^{\lambda x}$

With $f(0) = f(1)$ we have $e^{\lambda 0} = e^{\lambda}$

$\Rightarrow e^{\lambda} = 1$

$\Rightarrow \lambda = 2\pi i n \quad n \in \mathbb{Z}$.

The $\boxed{\begin{array}{l}\text{eigenvalues are } 2\pi i n, \ n \in \mathbb{Z} \\ \text{eigenfunctions are } e^{2\pi i n x}\end{array}}$

Orthogonality would be $\int_0^1 e^{2\pi i n x} e^{-2\pi i m x}$

$= 0$ if $n \neq m$.

Well, $\int_0^1 e^{2\pi i (n-m)x} dx = \frac{1}{2\pi i (n-m)} e^{2\pi i (n-m)x} \Big|_0^1$ if $n \neq m$

$= \frac{1}{2\pi i (n-m)} \left(e^{2\pi i (n-m)} - e^0 \right)$

$= 0$

and if $n = m$ get $\int_0^1 1 \, dx = 1$.

So in fact these efns are <u>orthonormal</u>.

S-3-9 Consider bdy condits
$$\left.\begin{aligned} X(b) &= \alpha X(a) + \beta X'(a) \\ X'(b) &= \gamma X(a) + \delta X'(a) \end{aligned}\right\} *$$

on $a \le x \le b$, for operator $\frac{d^2}{dx^2}$.

Show bdy condits symmetric iff $\alpha\delta - \beta\gamma = 1$.

Soln Sym bdy condits means that if X_1 and X_2 satisfy (*) then
$$-X_1' X_2 + X_2' X_1 \Big|_a^b = 0.$$

Here $-X_1'(b) X_2(b) + X_2'(b) X_1(b) + X_1'(a) X_2(a) - X_2'(a) X_1(a)$

$$\begin{aligned}
&= -\left(\gamma X_1(a) + \delta X_1'(a)\right)\left(\alpha X_2(a) + \beta X_2'(a)\right) \\
&\qquad + \left(\gamma X_2(a) + \delta X_2'(a)\right)\left(\alpha X_1(a) + \beta X_1'(a)\right) \\
&\qquad + X_1'(a) X_2(a) - X_2'(a) X_1(a)
\end{aligned}$$

$$\begin{aligned}
&= X_1(a) X_2(a) \left(-\alpha\gamma + \alpha\gamma\right) \\
&\quad + X_1'(a) X_2(a) \left(-\delta\alpha + \beta\gamma + 1\right) \\
&\quad + X_1(a) X_2'(a) \left(-\gamma\beta + \delta\alpha - 1\right) \\
&\quad + X_1'(a) X_2'(a) \left(-\delta\beta + \delta\beta\right)
\end{aligned}$$

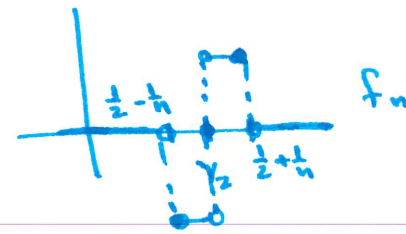
$$= \left(\alpha\delta - \beta\gamma - 1\right)\left(X_1(a) X_2'(a) - X_1'(a) X_2(a)\right)$$

This is zero for all choices of $X_1(a), X_1'(a), X_2(a), X_2'(a)$ if and only if $\alpha\delta - \beta\gamma - 1 = 0$

i.e. $\alpha\delta - \beta\gamma = 1$.

S.4.3 Let $\gamma_n \to \infty$ and f_n for $n \geq 2$ be

$$f_n = \begin{cases} 0 & \text{at } \frac{1}{2} \\ \gamma_n & \text{on } [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}) \\ -\gamma_n & \text{on } [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}] \\ 0 & \text{elsewhere} \end{cases}$$



Show:

a) $f_n \to 0$ pointwise.
Fix x. If $x = \frac{1}{2}$ then $f_n(\frac{1}{2}) = 0$ for all n, so cvges.
If $x \neq \frac{1}{2}$ then take N so large that $\frac{1}{N} < |x - \frac{1}{2}|$.
When $n \geq N$ then $f_n(x) = 0$ so $f_n(x) \to 0$ b/c is eventually constant.

b) the convergence is not uniform because for any n we have $\max_{0 \leq x \leq 1} |f_n(x) - 0| = \gamma_n \to \infty$.

c) $f_n \to 0$ in L^2 if $\gamma_n = n^{1/3}$.
We can compute
$$\|f_n - 0\|_{L^2}^2 = \int_0^1 |f_n - 0|^2 \, dx = \int_{\frac{1}{2}-\frac{1}{n}}^{1/2} \gamma_n^2 + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \gamma_n^2 = \gamma_n^2 \left(\frac{2}{n}\right)$$

So this goes to zero if $\gamma_n = n^{1/3}$ because it is $\frac{2n^{2/3}}{n} = 2n^{-1/3} \to 0$.

d) f_n does not converge in L^2 if $\gamma_n = n$.
Using the previous computation $\|f_n\|_{L^2} = \sqrt{\frac{2}{n}}\, \gamma_n = 2\sqrt{n}$ in this case, and $2\sqrt{n} \to \infty$ as $n \to \infty$.

5.4.6 Find sine series of $\cos x$ on $(0,\pi)$. What is sum of series on $[-\pi,\pi]$?

Well $\cos x = \sum_1^\infty A_n \sin(nx)$

$$\Rightarrow \quad A_n = \frac{2}{\pi}\int_0^\pi \cos x \sin(nx)\,dx \qquad n = 1,2,3,\dots$$

$$= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left(\sin((n+1)x) - \sin((n-1)x)\right)dx$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n+1} + \frac{(-1)^{n+1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\left((-1)^n + 1\right)\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left(1+(-1)^n\right)\left(\frac{-2}{n^2-1}\right)$$

$$= \begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{-4}{\pi(n^2-1)} & \text{if } n \text{ even} \end{cases}$$

So $$\cos x = \frac{-4}{\pi}\sum_{m=1}^\infty \frac{1}{4m^2-1}\sin(2mx) \qquad (*)$$

Since $\cos x$ is cts on $[0,\pi]$ with cts derivative, the Applying Thm 4 (i) pointwise sum of the series $(*)$ is $\cos x$ on $(0,\pi)$. By the fact that sin is odd it is $-\cos x$ on $(-\pi,0)$. At the remaining points $-\pi, 0$ and π we compute directly using $\sin 2\pi m = \sin(2\pi m) = \sin 0 = 0$ that the sum is 0. ∎

5.4.18 Consider soln of $u_{tt} = u_{xx}$ on $[0,l]$ with homog Dir or Neum bdy condits.

a) Show $E = \frac{1}{2}\int_0^l (u_t^2 + u_x^2)\,dx$ is const.

(This is similar to what did earlier in book)

$E = E(t)$, and $\frac{dE}{dt} = \frac{1}{2}\int_0^l (2u_t u_{tt} + 2u_x u_{xt})\,dx$

and integ by parts $= \frac{1}{2}\left(\int_0^l (2u_t u_{tt} - 2u_{xx}u_t)\,dx + 2u_x u_t\Big|_0^l\right)$

$= \frac{1}{2}\int_0^l 2u_t(u_{tt} - u_{xx})\,dx + u_x u_t\Big|_0^l$

$= u_x(l,t)u_t(l,t) - u_x(0,t)u_t(0,t)$.

Evidently, homog Neum condits $u_x(0,t) = u_x(l,t) = 0$ imply that $\frac{dE}{dt} = 0$

Also, homog Dir condits $u(0,t) = u(l,t) = 0$

$\Rightarrow u_t(0,t) = u_t(l,t) = 0$ just by differentiating wrt t

So also $\frac{dE}{dt} = 0$.

b) The nth harmonic is $X_n(x)T_n(t)$ when $u = \sum X_n T_n A_n$ is expansion in eigenfunctions. then

$u_t = \sum' A_n X_n(x) T_n'(t) \Rightarrow \int u_t^2 = \sum_{n,m} A_n T_n' T_m' A_m \int X_n X_m$

$= A_n^2 (T_n')^2 \int X_n^2$

because X_n, X_m orthog

and $u_x = \sum_1' A_n X_n'(x) T_n(t)$

$\Rightarrow \int u_x^2 = \sum A_n A_m T_n T_m \int X_n' X_m'$ but $\int_0^l X_n' X_m' = X_n' X_m\Big|_0^l - \int_0^l X_n'' X_m$

$= 0 - \int_0^l \lambda_n X_n X_m$

because $X_n'' = \lambda_n X_n$

So $\int u_x^2 = -\sum_1' A_n^2 (T_n)^2 \lambda_n \int X_n^2$.

but then $\frac{1}{2}\int_0^L u_t^2 + u_x^2 \, dx$

$$= \frac{1}{2}\sum_n{}' A_n^2 (T_n')^2 \int X_n^2 - A_n^2 T_n^2 \lambda_n \int X_n^2$$

$$= \frac{1}{2}\sum_n{}' A_n^2 \int_0^L \left((X_n T_n')^2 - X_n'' X_n T_n^2 \right)$$

$$= \frac{1}{2}\sum_n A_n^2 \left(\int_0^L \left((X_n T_n')^2 + (X_n' T_n)^2 \right) dx + \underbrace{\cancel{X_n' X_n \Big|_0^L}}_{=0} \right)$$

$$= \sum_n \frac{1}{2} \int_0^L (A_n X_n T_n')^2 + (A_n X_n' T_n)^2 \, dx$$

$$= \sum_n{}' E_n \, .$$

(I didn't do this very efficiently, but I hope result is clear).