4-3-6 Robin problem on $[0, l]$ with $a_0 = a_l = 1$.

a) If $a \geq 0$ show there are no -ve evals
If $-\frac{2}{l} < a < 0$ show there is one -ve eval
If $a < -\frac{2}{l}$ show there are two -ve evals.

b) Show 0 is an eval iff $a = 0$ or $a = -\frac{2}{l}$.

Soln a) Using the analysis in the book there is a -ve eval if and only if there is $\beta > 0$ such that

$$\tanh(\beta l) = \frac{-(a_0 + a_l)\beta}{\beta^2 + a_0 a_l} = \frac{-2a\beta}{\beta^2 + a^2} \qquad (*)$$

and each -ve eval is then $-\beta^2$, where β is the root of this eqn.

It is easy to check that $\tanh(\beta l)$ is increasing, concave, and has limit 1 as $\beta \to \infty$. Also $\tanh(0) = 0$, so $\tanh(\beta l) > 0$ for $\beta \in (0, \infty)$. The rational function $\frac{-2a\beta}{\beta^2 + a^2}$ has ~~[illegible]~~ negative values if $a > 0$, so in this case there are no -ve evals b/c $(*)$ is never true.

If $a < 0$ then $\frac{-2a\beta}{\beta^2 + a^2}$ is positive on $(0, \infty)$, increasing on $(0, -a)$ to a max of 1 at $\beta = -a$ and decreasing on $(-a, \infty)$ with limit 0. It follows immediately that $(*)$ has exactly one root on $(-a, \infty)$, so there is at least one -ve eval. If, in addition, we have $a < -\frac{2}{l}$ then the slope l of $\tanh(\beta l)$ at 0 is less than the slope $-\frac{2}{a}$ of $\frac{-2a\beta}{\beta^2 + a^2}$ so that $\frac{-2a\beta}{\beta^2 + a^2} < \tanh(\beta l)$ when β is close to 0, but $\tanh(\beta l) < 1 = \frac{-2a(-a)}{a^2 + (-a)^2}$, so there is a root on $(0, -a)$. Thus there are at least two -ve evals if $a < -\frac{2}{l}$.

This is not actually a proof that there is at most one root in $(0, -a)$, nor that it is impossible to have more than one if $-\frac{2}{l} < a < 0$, but it is a sufficient answer to this problem for this course. (You could have also drawn graphs and reasoned from them)

b) The case of eigenvalue 0 requires a different analysis.

In this situation $X'' = 0 \Rightarrow X = A + Bx$

and $X'(0) - a_0 X(0) = 0 \Rightarrow B - a_0 A = 0$

$X'(l) + a_l X(l) = 0 \Rightarrow B + a_l(A + Bl) = 0$

So that
$$\begin{bmatrix} -a_0 & 1 \\ a_l & 1 + l a_l \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and ~~[illegible]~~ a non-trivial soln exists iff

$$-a_0(1 + l a_l) - a_l = 0$$

$$\Leftrightarrow \quad a_0 + a_l = -a_0 a_l l$$

(you could use a previous HW problem to get this)

Since $a_0 = a_l$ this says $2a = -a^2 l$

so $a(2 + al) = 0$

$\Leftrightarrow \quad a = 0$ or $a = -\frac{2}{l}$.

4.3.10 Solve the wave eqn with Robin bdy such that $a_0 < 0$, $0 < a_0 + a_l < -a_0 a_l l$.

From the analysis in the book there is then one -ve eval $\lambda_0 < 0$. It has efn $\cosh\sqrt{|\lambda_0|}\,x + \frac{a_0}{\sqrt{|\lambda_0|}}\sinh(\sqrt{|\lambda_0|}\,x)$. The time dependence is $T_0(t) = A_0\cosh(\sqrt{|\lambda_0|}\,ct) + B_0\sinh(\sqrt{|\lambda_0|}\,ct)$. Otherwise solns correspond to +ve evals λ_n, $n = 1, 2, 3, \ldots$ with $\left(\frac{n\pi}{l}\right)^2 \le \lambda_n \le \left(\frac{(n+1)\pi}{l}\right)^2$ and $\lim_n \lambda_n - \left(\frac{n\pi}{l}\right)^2 = 0$. Their efcts and time dependence can be copied from book to obtain wave soln

$$u(x,t) = \left(A_0\cosh\sqrt{|\lambda_0|}\,ct + B_0\sinh\sqrt{|\lambda_0|}\,ct\right)\left(\cosh\sqrt{|\lambda_0|}\,x + \frac{a_0}{\sqrt{|\lambda_0|}}\sinh\sqrt{|\lambda_0|}\,x\right)$$

$$+ \sum_{n=1}^{\infty}\left(A_n\cos\sqrt{\lambda_n}\,ct + B_n\sin\sqrt{\lambda_n}\,ct\right)\left(\cos\sqrt{\lambda_n}\,x + \frac{a_0}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\,x\right)$$

where $\tan(l\sqrt{\lambda_n}) = \frac{(a_0 + a_l)\sqrt{\lambda_n}}{\lambda_n - a_0 a_l}$ $n = 1, 2, 3, \ldots$

4.3.13 A string is fixed at one end and has mass m at the other.

a) Show waves in string satisfy
$$\left.\begin{aligned} u_{tt} &= c^2 u_{xx} \quad 0<x<l \\ u(0,t) &= 0 \\ \ddot{u}(l,t) &= -k\, u_x(l,t) \end{aligned}\right\} \quad (*)$$

b) Find eval problem

c) Find the evals & efns.

a) The Dir condit at 0 is clear.
At $x=l$ the tension force on the string is a vector with transverse cpt $\frac{-Tu_x}{\sqrt{1+u_x^2}} \approx -Tu_x$ (book, ch 1 § 4)

so the eqn of motion is $m u_{tt}(l,t) = -Tu_x(l,t)$

$\Rightarrow$ $*$ with $k = \frac{T}{m} > 0$.

b) If $u(x,t) = X(x)T(t)$ then $\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$ as usual.

Clearly $X(0)=0$. At l we have $T''(t)X(l) = -kX'(l)T(t)$

so that $-k\frac{X'(l)}{X(l)} = \frac{T''(t)}{T(t)} = -c^2\lambda$

and get $kX'(l) = c^2\lambda X(l)$

or $X'(l) - \frac{c^2\lambda}{k} X(l) = 0$, a Robin condit, of sorts, with a variable ~~rate~~ endpt interaction depending on λ.

c) Positive evals give $X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$
and $X(0) \Rightarrow C=0$ so $X(x) = D\sin\sqrt{\lambda}\,x$.
The $x=l$ condit gives $\sqrt{\lambda}\cos(\sqrt{\lambda}l) - \frac{c^2\lambda}{k}\sin(\sqrt{\lambda}l) = 0$
Clearly $\lambda \neq 0$ and $\cos(\sqrt{\lambda}l) \neq 0$ else soln is trivial, so efns are $\sin\sqrt{\lambda_n}x$ with evals λ_n being solns of

$$\tan(\sqrt{\lambda}l) = \frac{k}{c^2\sqrt{\lambda}}$$

4·3·14 For $x^2u'' + 3xu' + \lambda u = 0 \qquad 1 < x < e$

$u(1) = u(e) = 0$

Assume $\lambda > 1$ and solve the eval problem.

Soln Take $u(x) = x^m$ so the eqn is

$$(m(m-1) + 3m + \lambda)x^m = 0$$

$$\Rightarrow \quad m^2 + 2m + \lambda = 0$$

$$\Rightarrow \quad m = -1 \pm \sqrt{1-\lambda}$$

Fit soln $u(x) = A x^{-1+\sqrt{1-\lambda}} + B x^{-1-\sqrt{1-\lambda}}$

to bdy data $u(1) = u(e) = 0$ and get

$$A + B = 0$$

$$Ae^{\sqrt{1-\lambda}} + Be^{-\sqrt{1-\lambda}} = 0$$

so $\begin{bmatrix} 1 & 1 \\ e^{\sqrt{1-\lambda}} & e^{-\sqrt{1-\lambda}} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which has non trivial

soln only if $e^{-\sqrt{1-\lambda}} - e^{\sqrt{1-\lambda}} = 0$

$$\Leftrightarrow \quad e^{2\sqrt{1-\lambda}} = 1$$

$$\Leftrightarrow \quad \sqrt{1-\lambda} = n\pi i \qquad n \in \mathbb{Z}$$

$$\Leftrightarrow \quad \lambda = 1 + n^2\pi^2$$

Since we assumed $\lambda > 0$, and $n^2 = (-n)^2$ we can say

$\lambda = 1 + n^2\pi^2, \; n = 1, 2, 3, \ldots$ (of course it is also clear $n=0$ leads to $A = B = 0$ so not an eval)

Thus evals $1 + n^2\pi^2, \; n = 1, 2, 3, \ldots$

with corresp efns $u_n(x) = x^{-1}\left(x^{in\pi} - x^{-in\pi}\right)$

$$= x^{-1}\left(e^{in\pi\log x} - e^{-in\pi\log x}\right)$$

$$= 2i\, x^{-1} \sin(n\pi \log x).$$

Series soln to ODE is

$$u(x) = \sum_1^{\infty} A_n x^{-1} \sin(n\pi\log x).$$

5.1.5 In Example 3 pg 109 we find

$$x = \sum_1^\infty \frac{(-1)^{n+1} 2l}{n\pi} \sin\left(\frac{n\pi}{l} x\right) \quad \text{on } (0,l)$$

Assuming this can be integrated term by term we get

$$\frac{1}{2}x^2 = \sum_1^\infty \frac{(-1)^{n+2} 2l^2}{(n\pi)^2} \cos\left(\frac{n\pi}{l}x\right) + \frac{1}{2}A_0$$

and since $\int_0^l \frac{1}{2}x^2\,dx = \frac{l^3}{6} = \int_0^l \frac{1}{2}A_0\,dx = \frac{1}{2}A_0 l$ by the fact that $\int_0^l \cos\left(\frac{n\pi}{l}x\right)dx = 0$ for all $n = 1,2,3,\dots$ we conclude $\frac{1}{2}A_0 = \frac{l^2}{6}$.

Now substituting $x=0$ (and assuming the sum cvges at 0, which is not so obvious given we originally assumed it only on $(0,l)$) we get

$$\frac{l^2}{6} = \frac{2l^2}{\pi^2} \sum_1^\infty \frac{(-1)^{n+1}}{n^2}$$

$$\Rightarrow \sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

5.1.8 Rod has $u_t = u_{xx} \quad 0 < x < 1$

$u(0,t) = 0$

$u(1,t) = 1$

$$u(x,0) = \phi(x) = \begin{cases} \frac{5x}{2} & 0 < x < \frac{2}{3} \\ 3-2x & \frac{2}{3} < x < 1 \end{cases}$$

Find $u(x,t)$.

Following suggestion in book we take $U(x)$ to be the equilibrium soln, i.e. with no t dependence, so $U'' = 0$ and $U(x) = A + Bx$.

Then $U(0) = 0$, $U(1) = 1 \Rightarrow U(x) = x$.

Now let $v(x,t) = u(x,t) - U(x) = u(x,t) - x$

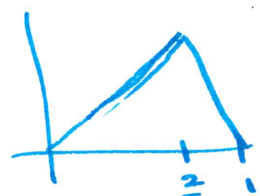
$$v_t = u_t = u_{xx} = v_{xx}$$

$$v(0,t) = u(0,t) - 0 = 0$$

$$v(1,t) = u(0,t) - 1 = 0$$

$$v(x,0) = \phi(x) - U(x) = \phi(x) - x$$

$$= \begin{cases} \frac{3}{2}x & 0 < x < \frac{2}{3} \\ 3-3x & \frac{2}{3} < x < 1 \end{cases}$$



We try solving for $v(x,t)$ using Fourier sine series, so

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

with $v(x,0) = \sum_1^{\infty} A_n \sin(n\pi x) = \phi(x) - x$ as above.

Then $$A_n = 2\int_0^1 \sin(n\pi x)\,(\phi(x) - x)\,dx$$

$$= 2\int_0^{2/3} \frac{3}{2} x \sin(n\pi x)\,dx + 2\int_{2/3}^1 (3-3x)\sin(n\pi x)\,dx$$

$$= 3\left[\frac{\sin n\pi x}{(n\pi)^2} - \frac{x\cos(n\pi x)}{n\pi}\right]_0^{2/3} + 6\left[-\frac{\cos(n\pi x)}{n\pi}\right]_{2/3}^1$$

$$-6\left[\frac{\sin n\pi x}{(n\pi)^2} - \frac{x\cos(n\pi x)}{n\pi}\right]_{2/3}^{1}$$

$$= 3\left(\frac{\sin\left(\frac{2n\pi}{3}\right)}{(n\pi)^2} - \frac{2\cos\left(\frac{2n\pi}{3}\right)}{3n\pi}\right) + 6\left(-\frac{\cos n\pi}{n\pi} + \frac{\cos\left(\frac{2n\pi}{3}\right)}{n\pi}\right)$$

$$-6\left(\frac{\sin n\pi}{(n\pi)^2} - \frac{\cos n\pi}{n\pi} - \frac{\sin\left(\frac{2n\pi}{3}\right)}{(n\pi)^2} + \frac{2\cos\left(\frac{2n\pi}{3}\right)}{3n\pi}\right)$$

but $\cos n\pi = (-1)^n$, $\sin(n\pi) = 0$, $\sin\frac{2n\pi}{3} = \begin{cases} 0 & \text{if } n = 3m \\ \frac{\sqrt{3}}{2} & \text{if } n = 3m+1 \\ -\frac{\sqrt{3}}{2} & \text{if } n = 3m+2 \end{cases}$

$$\cos\left(\frac{2n\pi}{3}\right) = \begin{cases} 1 & \text{if } n = 3m \\ -\frac{1}{2} & \text{if } n = 3m+1 \text{ or } 3m+2 \end{cases}.$$

So $$A_n = \left(\frac{\sin\frac{2\pi n}{3}}{(n\pi)^2}\right)(3+6) + \left(\frac{\cos\frac{2\pi n}{3}}{n\pi}\right)(-2+6-4) + \frac{\cos n\pi}{n\pi}(-6+6)$$

$$= \frac{9\sin\frac{2\pi n}{3}}{(n\pi)^2} \quad \text{or} \quad \begin{cases} \frac{9\sqrt{3}}{2(n\pi)^2} & \text{if } n = 3m+1 \\ -\frac{9\sqrt{3}}{2(n\pi)^2} & \text{if } n = 3m+2 \\ 0 & \text{if } n = 3m \end{cases} \quad (\text{for } m = 0,1,2,\ldots)$$

So $$v(x,t) = \sum_{1}^{\infty} e^{-n^2\pi^2 t} \sin\left(\frac{2\pi n}{3}\right)\sin(n\pi x)\left(\frac{9}{(n\pi)^2}\right)$$

$$u(x,t) = x + \frac{9}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2 t}\sin\left(\frac{2\pi n}{3}\right)\sin(n\pi x)$$

OR if prefer

$$u(x,t) = x + \frac{9\sqrt{3}}{2\pi^2}\sum_{m=0}^{\infty}\left[\frac{1}{(3m+1)^2}e^{-(3m+1)^2\pi^2 t^2}\sin((3m+1)\pi x) - \frac{1}{(3m+2)^2}e^{-(3m+2)^2\pi^2 t}\sin((3m+2)\pi x)\right]$$

5.2.1

a) $\sin(ax)$, $a>0$. Periodic, period $\frac{2\pi}{a}$, odd.

b) e^{ax}. Not periodic, even or odd.

c) x^m $m \in \mathbb{Z}$ not periodic. Even if $m \in 2\mathbb{Z}$, odd if $m \in 2\mathbb{Z}+1$

d) $\tan x^2$. Not periodic, even.

e) $|\sin(\frac{x}{b})|$, $b>0$. Periodic, period $b\pi$. Even.

f) $x\cos ax$, $a>0$ .. Not periodic. Odd.