4.1.1 a) The wave solns for a string length l with fixed ends are linear combs of

$$\sin\left(\frac{n\pi}{l}ct\right)\sin\left(\frac{n\pi}{l}x\right)$$
$$\cos\left(\frac{n\pi}{l}ct\right)\sin\left(\frac{n\pi}{l}x\right)$$

If we clamp the string at its midpoint then it behaves like a string of half its length, so we replace l by $\frac{l}{2}$ and the frequencies $\frac{n\pi}{l}ct$ of the time oscillations (that produce the sound) become $\frac{2n\pi}{l}ct$. Doubling frequency is an octave increase in the sound.

(Equivalently one only sees the spatial vibrations $\sin\left(\frac{n\pi x}{l}\right)$ that are zero at the midpt, i.e. $\sin\left(\frac{n\pi}{l}\left(\frac{l}{2}\right)\right)=0$ so n must be even. Again the time frequencies we see are all double those for the original string).

b) Here the issue is that c changes. Recall that $c^2 = \frac{T}{\rho}$, the tension over the linear density. If the tension increases so does c^2 and thus c (i.e. speed of wave in string increases). Increasing the wave speed increases the ~~speed~~ frequency of the time oscillations (because they are sin or cos of $\frac{n\pi}{l}ct$) and thus the frequencies the ear hears - causing higher notes.

4.1.4 Write a series soln for

$$\begin{cases} u_{tt} = c^2 u_{xx} - r u_t & 0<x<l \\ u(0,t) = u(l,t) = 0 & \text{Dir bdy} \\ u(x,0) = \phi(x) & \\ u_t(x,0) = \psi(x) & \end{cases} \Big\} \; 0<x<l.$$

where $0 < r < \frac{2\pi c}{l}$.

Separating vars $u(x,t) = X(x)T(t)$ we get

$$X(x)T''(t) = c^2 X''(x)T(t) - rX(x)T'(t)$$

$$\Rightarrow \frac{T''(t) + rT'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Now $\frac{X''}{X} = -\lambda$ and $X(0) = X(l) = 0$ from Dir bdy condits

$\Rightarrow X(x) = \sin\left(\frac{n\pi}{l}x\right)$ $n = 1,2,3,\ldots$ and eval corresp " $\lambda_n = \left(\frac{n\pi}{l}\right)^2$.

So must solve $T_n''(t) + rT_n'(t) = -c^2\left(\frac{n\pi}{l}\right)^2 T_n(t)$

Solns are exponentials $e^{\alpha_n t}$ with $\alpha_n^2 + r\alpha_n + \left(\frac{cn\pi}{l}\right)^2 = 0$

so that $\alpha_n = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - 4\left(\frac{cn\pi}{l}\right)^2}$. However

$0 < r < \frac{2\pi c}{l} \Rightarrow r^2 - 4\left(\frac{cn\pi}{l}\right)^2 < \frac{4\pi^2 c^2}{l^2}(1-n^2) < 0$. So we

let $\beta_n = \frac{1}{2}\sqrt{4\left(\frac{nc\pi}{l}\right)^2 - r^2} = \sqrt{\left(\frac{n\pi c}{l}\right)^2 - \left(\frac{r}{2}\right)^2}$ and have $\alpha_n = -\frac{r}{2} \pm i\beta_n$.

This gives solns $T_n(t) = e^{-rt/2}\, A_n \cos(\beta_n t) + B_n \sin(\beta_n t)$,

and therefore soln to the original PDE is series

$$u(x,t) = \sum_{n=1}^{\infty} e^{-rt/2}\left(A_n \cos(\beta_n t) + B_n \sin(\beta_n t)\right) \sin\left(\frac{n\pi}{l}x\right)$$

with A_n and B_n given by init condits

$$\phi(x) = u(x,0) = \sum_{1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\psi(x) = u_t(x,0) = \sum_{n=1}^{\infty}\left(B_n\beta_n - \frac{r}{2}A_n\right)\sin\left(\frac{n\pi}{l}x\right).$$

4.16 Sep vars in $\begin{cases} t u_t = u_{xx} + 2u \\ u(0,t) = u(\pi,t) = 0 \end{cases}$

and show there are ∞ly many solns with $u(x,0) = 0$.

Let $u(x,t) = X(x)T(t)$, so

$$t\,T'(t)\,X(x) = X''(x)T(t) + 2X(x)T(t)$$

$$\Rightarrow \quad \frac{tT'}{T} - 2 = \frac{X''}{X} = -\lambda$$

and know $\frac{X''}{X} = -\lambda$, $X(0) = X(\pi) = 0 \Rightarrow X_n(x) = \sin nx$, $\lambda_n = n^2$

Then $\frac{tT'}{T} = 2-\lambda \quad \Rightarrow T(t) = C t^{2-\lambda}$

so $T_n(t) = C_n t^{2-n^2}$.

This gives separated solns $u_n(x,t) = t^{2-n^2} \sin(nx)$, $n = 1, 2, 3, \ldots$

Now we want to solve $u(x,0) = 0$,

but $u_n(x,0)$ is undefined at $t = 0$ when $n = 2, 3, 4, \ldots$

so only case is $n = 1$ where $X_1(x)T_1(t)$

$$= t \sin x$$

Then we see that any multiple of $t\sin x$ solves the initial value problem: $At\sin x$ is a soln for any $A \in \mathbb{R}$ and therefore solution to IVP is not unique.

4.2.1 Solve $\begin{cases} u_t = k u_{xx} & \text{in } 0 < x < l \\ u(0,t) = u_x(l,t) = 0 \end{cases}$

Set $u(x,t) = X(x)T(t)$ so eqn is $\frac{X''}{X} = -\lambda = \frac{T'}{T}$

We get $X(x) = C\cos(\sqrt{\lambda}\,x) + D\sin(\sqrt{\lambda}\,x)$

and $0 = X(0) = C$, $\quad 0 = X'(l) = D\sqrt{\lambda}\cos(\sqrt{\lambda}\,l)$

Since we dont want $D = C = 0$ (trivial soln) must have

either $\lambda = 0$ or $\cos(\sqrt{\lambda}\,l) = 0 \Rightarrow \sqrt{\lambda} = \frac{(n+\frac{1}{2})\pi}{l}$. $n = 0, 1, 2, \ldots$

We also note that $\lambda = 0$ gives $X(x) = D\sin(0) = 0$, so is trivial. Hence have evals $\frac{(n+\frac{1}{2})\pi}{l}$, efns $\sin\left(\frac{(n+\frac{1}{2})\pi}{l}x\right)$.

The solns are then $e^{-\left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 kt} \sin\left(\frac{(n+\frac{1}{2})\pi}{l}x\right)$

and our general soln is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 kt} \sin\left(\frac{(n+\frac{1}{2})\pi}{l}x\right).$$

4.2.2 $\quad u_{tt} = c^2 u_{xx} \quad 0 < x < l$

$u_x(0,t) = u(l,t) = 0$.

a) One can check the efns are $\cos\left((n+\frac{1}{2})\pi x/l\right)$ by direct substitution. They solve the eqn $X'' = \lambda_n X$ for $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2$ and the bdy conditns

b) Then wave soln is

$$u(x,t) = \sum \left[A_n \cos\left(\frac{(n+\frac{1}{2})\pi}{l}ct\right) + B_n \sin\left(\frac{(n+\frac{1}{2})\pi}{l}ct\right)\right]\cos\left(\frac{(n+\frac{1}{2})\pi}{l}x\right)$$

4.3.2 Consider $-X'' = \lambda X$

$$X'(0) - a_0 X(0) = 0$$
$$X'(l) + a_l X(l) = 0$$

a) For 0 to be an eval we must have $-X'' = 0$

so $X = A + Bt$

$$\Rightarrow 0 = X'(0) - a_0 X(0) = B - a_0 A$$

and $0 = X'(l) + a_l X(l) = B + a_l(A + Bl)$

$$= B(1 + la_l) + a_l$$

Thus bdy condits are $\begin{bmatrix} -a_0 & 1 \\ a_l & 1+la_l \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and has a nontrivial soln iff det $= 0$

i.e. $0 = -a_0(1 + la_l) - a_l$

$$= -(a_0 + a_l) - a_0 a_l l$$

$$\Leftrightarrow \quad a_0 + a_l = -a_0 a_l l \,. \quad (*)$$

This argument is reversible: if the condit $(*)$ holds then $X(t) = A + Bt$ satisfies bdy condits and has $X'' = 0$, so is an efn with eval 0.

b) The efns are ~~[illegible]~~ $X(x) = A + Bt$, ~~with~~

with $-a_0 A + a_l B = 0$ so $B = \frac{a_0}{a_l}$

so $X(x) = A\left(1 + \frac{a_0}{a_l} t\right)$