3.3.1 Solve $\left.\left.\begin{array}{l} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \end{array}\right\} x>0 \\ u(0,t) = 0 \right\} t>0$

Dir bdy $\Rightarrow$ can use odd reflection. So equivalent to solving

$$\left.\begin{array}{l} u_t - k u_{xx} = \begin{cases} f(x,t) & x>0 \\ -f(-x,t) & x<0 \end{cases} \\ u(x,0) = \begin{cases} \phi(x) & x>0 \\ -\phi(-x) & x<0 \end{cases} \end{array}\right\} \begin{array}{l} t>0 \\ x \in \mathbb{R} \end{array}$$

and restricting to $x>0$.

We know soln is

$$u(x,t) = \int_0^\infty S(x-y,t)\phi(y)\,dy + \int_{-\infty}^0 S(x-y,t)(-\phi(-y))\,dy$$
$$+ \int_0^t \left(\int_0^\infty S(x-y,t-s) f(y,s)\,dy + \int_{-\infty}^0 S(x-y,t-s)(-f(-y,s))\,dy \right) ds$$

$$= \int_0^\infty \left(S(x-y,t) - S(x+y,t)\right)\phi(y)\,dy$$
$$+ \int_0^t \int_0^\infty \left(S(x-y,t-s) - S(x+y,t-s)\right) f(y,s)\,dy\,ds$$

by making change of vars $y \leftrightarrow -y$ in $\int_{-\infty}^0$ integrals.

3.3.3

$$\left.\begin{array}{l} w_t - k w_{xx} = 0 \\ w(x,0) = \phi(x) \end{array}\right\} x>0 \quad \Big\} \; t>0$$
$$w_x(0,t) = h(t)$$

The book suggests setting $u(x,t) = w(x,t) - h(t)x$

so that
$$\left\{\begin{array}{l} u_t - k u_{xx} = w_t - k w_{xx} - h'(t)x = -h'(t)x \\ u(x,0) = w(x,0) - h(0)x = \phi(x) - h(0)x \\ u_x(0,t) = h(t) - h(t) = 0 \end{array}\right\} x>0, \; t>0$$

We can do this by even reflection, so solving on $\mathbb{R}$

$$u_t - k u_{xx} = \begin{cases} -h'(t)x \\ h'(t)x \end{cases}$$

$$u(x,0) = \begin{cases} \phi(x) - h(0)x \\ \phi(-x) + h(0)x \end{cases}$$

which has solution

$$u(x,t) = \int_0^\infty (S(x-y,t) + S(x+y,t))\phi(y)\,dy$$
$$+ \int_0^\infty S(x-y,t)(-h(0)y)\,dy + \int_{-\infty}^0 S(x-y,t)h(0)y\,dy$$
$$+ \int_0^t \left(\int_0^\infty S(x-y,s)(-h'(s)y)\,dy\,ds + \int_{-\infty}^0 S(x-y,s)(h'(s)y)\,dy\,ds \right)$$

Now it is obviously useful to compute

$$-\int_0^\infty S(x-y,s)\,y\,dy + \int_{-\infty}^0 S(x-y,s)\,y\,dy$$

~~[illegible]~~ Putting $z = x-y$ so $y = x-z$ we can rewrite this as

$$= -\int_{-\infty}^x S(z,s)(x-z)\,dz + \int_x^\infty S(z,s)(x-z)\,dz$$

$$= x\left(\int_x^\infty S(z,s)\,dz - \int_{-\infty}^x S(z,s)\,dz\right) + \int_{-\infty}^x z S(z,s)\,dz - \int_x^\infty z S(z,s)\,dz$$

Now $$\int_x^\infty S(z,s)\,dz - \int_{-\infty}^{x} S(z,s)\,dz = 2\int_x^\infty S(z,s)\,dz - \int_{-\infty}^{\infty} S(z,s)\,dz$$

and putting $p = \frac{z}{\sqrt{4ks}}$ $$= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4ks}}}^{\infty} e^{-p^2}\,dp - 1$$

$$= -\mathrm{Erf}\left(\frac{x}{\sqrt{4ks}}\right)$$

And $$\int_{-\infty}^{x} z\,S(z,s)\,dz - \int_x^\infty z\,S(z,s)\,dz$$ can be transformed putting

$q = \frac{z^2}{4ks}$, so $dq = \frac{z\,dz}{2ks}$

and x endpt is

just $q = \frac{x^2}{4ks}$

$$= \int_{\infty}^{\frac{x^2}{4kt}} \frac{2kt}{\sqrt{4ks\pi}} e^{-q}\,dq - \int_{\frac{x^2}{4kt}}^{\infty} \frac{2kt}{\sqrt{4ks\pi}} e^{-q}\,dq$$

$$= \frac{4kt}{\sqrt{4ks\pi}} \int_{\frac{x^2}{4ks}}^{\infty} e^{-q}\,dq$$

$$= \sqrt{\frac{4ks}{\pi}}\left[-e^{-q}\right]_{\frac{x^2}{4ks}}^{\infty} = \sqrt{\frac{4ks}{\pi}}\, e^{-\frac{x^2}{4ks}}$$

Thus, substituting into soln for $u(x,t)$ we arrive at

$$w(x,t) = u(x,t) + x h(t) = \int_0^\infty \left(S(x-y,t) + S(x+y,t)\right)\phi(y)\,dy$$
$$+ h(0)\left(x\left(\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right) - 1\right)\right) + h(0)\sqrt{\frac{4kt}{\pi}}\, e^{-\frac{x^2}{4kt}}$$
$$+ \int_0^t \left(h'(s)\,x\left(\mathrm{Erf}\left(\frac{x}{\sqrt{4ks}}\right) - 1\right) + h'(s)\sqrt{\frac{4ks}{\pi}}\, e^{-\frac{x^2}{4ks}}\right) ds + x h(t)$$

We could, perhaps, integ by parts in the second integral to simplify further, but I am not sure it would achieve much.

3·4·3 $\quad u_{tt} = c^2 u_{xx} + \cos x$

$u(x,0) = \sin x$

$u_t(x,0) = 1 + x$

Inhomog wave eqn on $\mathbb{R}$. Soln is

$$u(x,t) = \frac{1}{2}\big(u(x+ct,0) + u(x-ct,0)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(y,0)\,dy$$

$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos y \,dy\,ds$$

$$= \frac{1}{2}\big(\sin(x+ct) + \sin(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct}(1+y)\,dy$$

$$+ \frac{1}{2c}\int_0^t \Big[\sin(x+c(t-s)) - \sin(x-c(t-s))\Big]\,ds$$

$$= \frac{1}{2}\big(\sin(x+ct) + \sin(x-ct)\big) + \frac{1}{2c}\Big(x+ct + \tfrac{1}{2}(x+ct)^2 - (x-ct) - \tfrac{1}{2}(x-ct)^2\Big)$$

$$+ \frac{1}{2c}\int_{x+ct}^{x} \sin p \,\frac{dp}{-c} - \frac{1}{2c}\int_{x-ct}^{x} \sin q\,\frac{dq}{c}$$

(by putting $p = x + c(t-s)$, $dp = -c\,ds$
$q = x - c(t-s)$, $dq = c\,ds$)

$$= \frac{1}{2}\big(\sin(x+ct) + \sin(x-ct)\big) + t + xt$$

$$+ \frac{1}{2c^2}\Big(\cos x - \cos(x+ct) + \cos x - \cos(x-ct)\Big]$$

One may simplify using $\sin(a+b) + \sin(a-b) = 2\sin a\cos b$

and $\cos(a+b) + \cos(a-b) = 2\cos a \cos b$

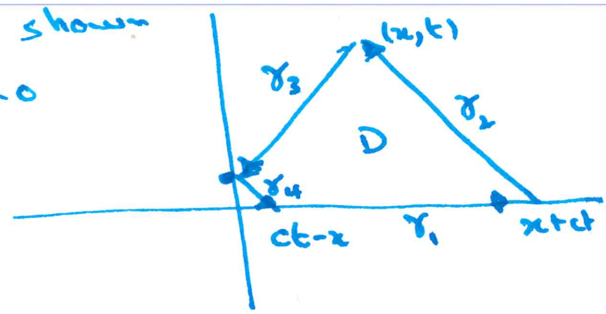
to get $= \sin x \cos ct + t + xt + \frac{1}{c^2}(\cos x)(1 - \cos ct)$

but this step is not essential.

3.4.10 Problem is
$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \\ u(0,t) = 0 \end{cases} \quad x>0 \quad \text{for } t>0$$

Asked to verify that $u(x,t) = \frac{1}{2c}\iint_D f(y,s)\,dy\,ds$ satisfies this, where

D is region shown when $x-ct<0$ and $t>0$



(The labels $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are for later; they are directed curves)

Evidently D is empty if $t=0$, so that $u(x,0)=0$. Also the area of D grows as t^2 for t near 0 so $u_t(x,0)=0$. More precisely, $\left|\frac{1}{2c}\iint_D f\right| \le \frac{M\,\text{Area}(D)}{2c}$ if $|f| \le M$

and $\text{Area}(D) \le \frac{1}{2}t\,(2ct)$ because it is contained in a triangle of base 2ct and height t

so $\frac{M\,\text{Area}(D)}{2c} \le \frac{1}{2}t^2$, whereupon
$$\left|\frac{u(x,t) - u(x,0)}{t}\right| = \left|\frac{u(x,t)}{t}\right| \le \frac{1}{2}t \to 0 \text{ as } t \to 0$$

$$\Rightarrow u_t(x,0) = 0.$$

When $x=0$ the triangle D is empty, so again $\frac{1}{2c}\iint_D f = 0$ in this case, showing $u(0,t)=0$.

Finally we must show that for $u(x,t) = \frac{1}{2c}\iint_D f$ we have $u_{tt} - c^2u_{xx} = f$.

One option is to follow the book in noting that by Green's

thus $$\iint_D u_{tt} - c^2u_{xx} = -\int_{\partial D} u_t\,dx + c^2u_x\,dt = \sum_{j=1}^{4}\int_{\gamma_j} -(u_t\,dx + c^2u_x\,dt) \quad (*)$$

Now recall (from multivar calc) that what this means is that we should param each curve as $(x(s), t(s))$ at which point $\int u_t\,dx + c^2 u_x\,dt$ means $\int \left(u_t \frac{dx}{ds} + c^2 u_x \frac{dt}{ds}\right) ds$.

We param γ_1 as $(s, 0)$ for s from $ct-x$ to $x+ct$

so $\frac{dx}{ds} = 1$, $\frac{dt}{ds} = 0$ and

$$-\int_{\gamma_1} u_t\,dx + c^2 u_x\,dt = -\int_{ct-x}^{ct+x} u_t(s,0)\,ds$$

Param γ_2 as $(x + c(t-s), s)$ from $s=0$ to $s=t$

So $\frac{dx}{ds} = -c$, $\frac{dt}{ds} = 1$ and

$$-\int_{\gamma_2} u_t\,dx + c^2 u_x\,dt = -\int_0^t (-c u_t + c^2 u_x)\,ds$$

Now we recognize this as in the form $\left(\frac{dx}{ds}, \frac{dt}{ds}\right) \cdot \nabla u$ so we can apply the fund thm on line integrals. This also applies to the integrals on γ_3 and γ_4. Thus

$$-\int_{\gamma_2} u_t\,dx + c^2 u_x\,dt = c\int_0^t (c,-1) \cdot \nabla u\,ds$$

$$= c\left(u(x,t) - u(x+ct, 0)\right)$$

Similarly param γ_3 as $(x - c(t-s), s)$ from $s=t$ to $s = t - \frac{x}{c}$

so $\frac{dx}{ds} = c$, $\frac{dt}{ds} = 1$ and

$$-\int_{\gamma_3} u_t\,dx + c^2 u_x\,dt = -\int_t^{t-\frac{x}{c}} (c u_t + c^2 u_x)\,ds = -c\int_t^{t-\frac{x}{c}} (c,1) \cdot \nabla u\,ds$$

$$= -c\left(u(0, t - \tfrac{x}{c}) - u(x,t)\right) = c\left(u(x,t) - u(0, t - \tfrac{x}{c})\right)$$

And param γ_4 as $(c(t-s) - x, s)$ from $s = t - \frac{x}{c}$ to $s = 0$

so $\frac{dx}{ds} = -c$, $\frac{dt}{ds} = 1$ and

$$-\int_{\gamma_4} u_t\,dx + c^2 u_x\,dt = \int_{t-\frac{x}{c}}^0 (-c u_t + c^2 u_x)\,ds = c\int_{t-\frac{x}{c}}^0 (-c,1) \cdot \nabla u\,ds$$

$$= c\left(u(ct-x, 0) - u(0, t - \tfrac{x}{c})\right)$$

Putting these together we have from (*)

$$\frac{1}{2c}\iint_D u_{tt} - c^2 u_{xx} = \frac{-1}{2c}\int_{ct-x}^{ct+x} u_t(y,0)\,dy + \frac{c}{2c}\Big(u(x,t) - u(x+ct,0)\Big)$$

$$+ \frac{c}{2c}\Big(u(x,t) - u(0,t-\tfrac{x}{c})\Big) + \frac{c}{2c}\Big(u(ct-x,0) - u(0,t-\tfrac{x}{c})\Big)$$

$$= u(x,t) - \frac{1}{2}\Big(u(x+ct,0) - u(ct-x,0)\Big) - u(0,t-\tfrac{x}{c}) - \frac{1}{2c}\int_{ct-x}^{ct+x} u_t(y,0)\,dy$$

So that

$$u(x,t) = \frac{1}{2}\Big(u(x+ct,0) - u(ct-x,0)\Big) + u(0,t-\tfrac{x}{c}) + \frac{1}{2c}\int_{ct-x}^{ct+x} u_t(y,0)\,dy$$

$$+ \frac{1}{2c}\iint_D u_{tt} - c^2 u_{xx}$$

but we ~~[illegible]~~ were given $u(x+ct,0) = u(ct-x,0) = 0$

and $u_t(y,0) = 0$

and $u(0,t-\frac{x}{c}) = 0$

so $u(x,t) = \frac{1}{2c}\iint_D u_{tt} - c^2 u_{xx}$ and thus

$$u(x,t) = \frac{1}{2c}\iint_D f \iff u_{tt} - c^2 u_{xx} = f.$$

We showed this for D as in diagram under the assumption $x - ct < 0$ i.e. $ct > x$. If $x < ct$ then the domain is the usual light cone and the computation is in the book.

3.4.12

Find general soln for $t>0$ for system
$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t) \\ v(x,0) = \phi(x) \\ v_t(x,0) = \psi(x) \\ v(0,t) = h(t) \end{cases} \quad x>0$$

by integrating over domain of dependence using Green's thm.

This is actually what we did in 3.4.10 where we found that
$$v(x,t) = \frac{1}{2}\left(v(x+ct) - v(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{ct+x} v_t(y,0)\,dy + \frac{1}{2c}\iint_D v_{tt} - c^2 v_{xx}$$
$$+ v\left(0, t-\frac{x}{c}\right).$$

Substituting our boundary data we get (for $ct>x$)
$$v(x,t) = \frac{1}{2}\left(\varphi(x+ct) - \varphi(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\iint_D f(y,s)\,dy\,ds$$
$$+ h\left(t-\frac{x}{c}\right).$$

3.4.13 Solve
$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = x \\ u_t(x,0) = 0 \\ u(0,t) = t^2 \end{cases} \quad x>0$$

By previous problem soln for $ct>x$ is
$$u(x,t) = \frac{1}{2}\left(x+ct - (ct-x)\right) + \left(t-\frac{x}{c}\right)^2$$
$$= x + \left(t-\frac{x}{c}\right)^2$$

while for $ct \le x$ it is the usual wave soln on $\mathbb{R}$
$$u(x,t) = \frac{1}{2}\left((x+ct) + (x-ct)\right) = x$$

So
$$u(x,t) = \begin{cases} x & \text{if } x \le ct \\ x + \left(t-\frac{x}{c}\right)^2 & \text{if } x > ct. \end{cases}$$