3.1.2

$$u_t = k u_{xx} \quad \text{on } x>0$$
with $u(x,0)=0$
$u(0,t)=1. \quad t>0$

We can set $v = u-1$, so $v_t = k v_{xx}$ for $x>0$
$v(x,0) = -1$ for $x>0$
$v(0,t)=0 \quad t>0$

this is solved in the book as eqn (6) on pg 59:

$$u = v+1 = 1 + \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left(e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\right)(-1)\,dy$$

$$\boxed{= 1 - \mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right)}$$

3.1.3

$w_t - k w_{xx} = 0 \quad x>0 \quad t>0$
$w_x(0,t) = 0 \quad t>0$
$w(x,0) = \phi(x) \quad x>0$

We extend the initial data evenly to $\mathbb{R}$, letting $u(x,0) = \begin{cases} \phi(x) & x>0 \\ \phi(-x) & x<0 \end{cases}$
and look for soln of $u_t - k u_{xx} = 0 \quad x \in \mathbb{R},\ t>0$
with $u(x,0)$ as above.

The soln is
$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, u(y,0)\,dy$$
$$= \int_0^\infty S(x-y,t)\phi(y)\,dy + \int_{-\infty}^0 S(x-y,t)\phi(-y)\,dy$$
$$= \int_0^\infty S(x-y,t)\phi(y) + S(x+y,t)\phi(y)\,dy$$
$$\boxed{= \int_0^\infty \phi(y)\left(e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right)\frac{dy}{\sqrt{4kt\pi}}}$$

Setting $w(x,t) = u(x,t)$ for $x>0$ we have $w_t = k w_{xx}$ because it is true for u, and $w(x,0) = \phi(x)$ for $x>0$ because this is also true for u (in the sense $\lim_{t\to 0} u(x,t) = \phi(x)$ if $x>0$).

Also, $u(-x,t) = \int_0^\infty \phi(y)\left(S(-x-y,t) + S(-x+y,t)\right)dy$
$= \int_0^\infty \phi(y)\left(S(x+y,t) + S(x-y,t)\right)dy = u(x,t)$ because

$S(-v,t) = S(v,t)$ (this is clear from $S(v,t) = \frac{e^{-v^2/4kt}}{\sqrt{4\pi kt}}$.)

~~Therefore~~ And $u(-x,t) = u(x,t)$

$\Rightarrow$ $\frac{\partial}{\partial x} u(x,t) = 0$ when $x = 0$, so $u_x(0,t) = 0$ for $t > 0$.

3.2.2 Given $u_{tt} = c^2 u_{xx}$ $x > 0$

$u_x(0,t) = 0$ (so Neumann bdy condits)

with $u(x,0) = 0$

$$u_t(x,0) = \begin{cases} V & \text{if } a < x < 2a \\ 0 & \text{otherwise.} \end{cases}$$

Asked for plots of u for $t = 0, \frac{a}{c}, \frac{3a}{2c}, \frac{2a}{c}, \frac{3a}{c}$.

Observe that for $t=0$ and $t = \frac{a}{c}$ the wave (travelling at speed $\leq c$ and initially supported in $[a,2a]$) has not reached zero, so the picture is just as it would be on $\mathbb{R}$ instead of $[0,\infty)$ and is given by

$$u(x,t) = \frac{V}{2c} \int_{x-ct}^{x+ct} u_t(y,0)\, dy$$

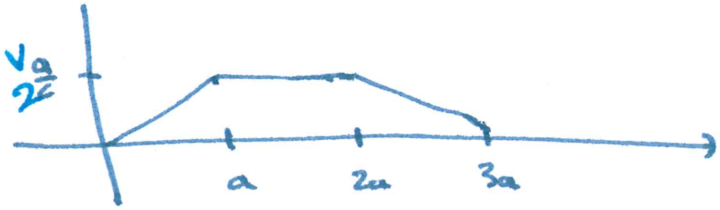
$$= \frac{V}{2c} \cdot \text{length of} \left([x-ct, x+ct] \cap [a, 2a] \right)$$

Obviously for $t=0$ this is zero (which matches $u(x,0)=0$)

For $t = \frac{a}{c}$ we have an interval $[x-a, x+a] \cap [a,2a]$, which has length 0 if $x > 3a$ or $x = 0$ ($x < 0$ is not in the domain). ~~It has length a precisely when x =~~ Its maximum length is a, which occurs when $x \in [a, 2a]$. For $x \in [0,a]$ or $[2a, 3a]$ the length changes linearly.

Accordingly we have graphs:

$t=0$
(undisturbed)



$t = \frac{a}{c}$

For the ~~x~~ later times I think it is easier to ~~use the~~ directly use the fact that even reflections will provide Neumann solution than to first derive the formulas for Neumann solutions.

The even reflection of boundary data gives $u(x,0)=0$ on $\mathbb{R}$

and $u_t(x,0) = \begin{cases} V & \text{if } a<x<2a \text{ or } -2a<x<-a \\ 0 & \text{otherwise} \end{cases}$ on $\mathbb{R}$

and we know the solution of this ~~wave~~ is

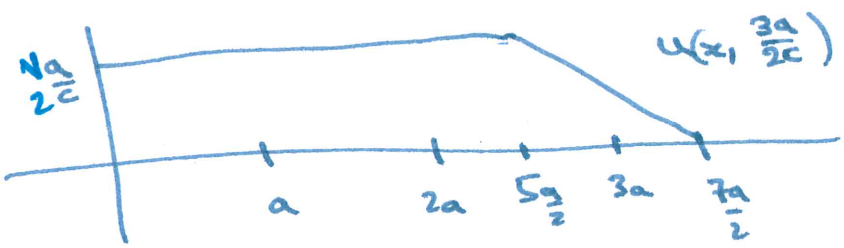
$$u(x,t) = \frac{V}{2c}\int_{x-ct}^{x+ct} u_t(y,0)\,dy$$

$$= \frac{V}{2c} \text{ length of}\Big([x-ct,x+ct] \cap \big([-2a,-a] \cup [a,2a]\big)\Big)$$

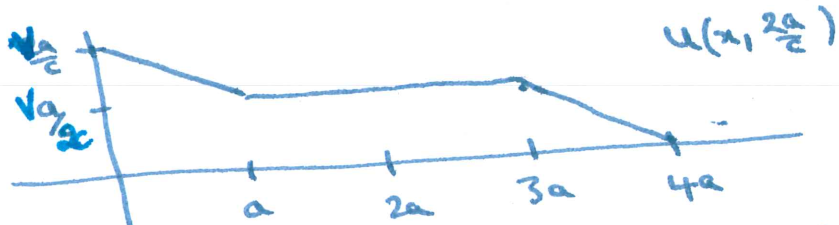
which satisfies the wave equation and is symmetric, hence has Neumann condition at $x=0$. Its ~~initial data~~ restriction to $x>0$ also has our desired initial data, so it is the solution we want.

We are now asked for the cases $t = \frac{3a}{2c}$, $t = \frac{2a}{c}$, $t = \frac{3a}{c}$.

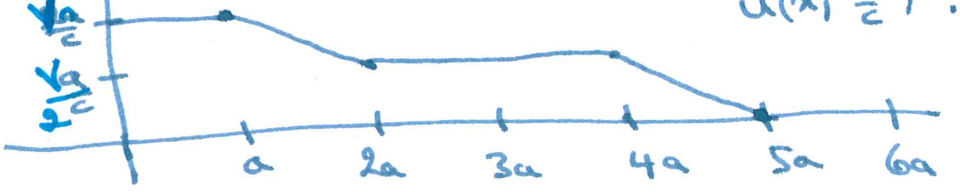
In the case $t = \frac{3a}{2c}$ the interval $[x - \frac{3a}{2}, x + \frac{3a}{2}]$ has length $3a$ and center at x. Evidently when $x = 0$ it overlaps $[-2a, -a]$ on length $\frac{a}{2}$ and $[a, 2a]$ also by length $\frac{a}{2}$. As x increases, the amount of length in $[-2a,-a] \cap [x - \frac{3a}{2}, x + \frac{3a}{2}]$ plus that in $[a, 2a] \cap [x - \frac{3a}{2}, x + \frac{3a}{2}]$ is constant equal to a, until $x = \frac{a}{2}$ where the first intersection becomes empty but the second is all of $[a, 2a]$. This persists until $x = \frac{5a}{2}$ at which point the intersection drops linearly to 0 at $x = \frac{7a}{2}$. The graph is



Similarly, when $t = \frac{2a}{c}$ the intersection of interest is $[x-2a, x+2a] \cap ([-2a,-a] \cup [a, 2a])$. It has length $2a$ when $x = 0$, decreasing linearly to length a at $x = a$, staying at that level until $x = 3a$ and dropping linearly to 0 at $x = 4a$



And when $t = \frac{3a}{c}$ the same analysis yields



3.2.5

$$u_{tt} = 4u_{xx} \qquad 0 < x < \infty$$

$$u(0,t) = 0$$

$$\left.\begin{aligned} u(x,0) &\equiv 1 \\ u_t(x,0) &\equiv 0 \end{aligned}\right] \quad x > 0$$

As in the text, we consider the odd extension of our initial data to $\mathbb{R}$ and solve the wave eqn on $\mathbb{R}$ for this data. The result is, by (2) and (3) on page 62. Using that $u_t(x,0) \equiv 0$ for all x we can ignore the integral term

$$u(x,t) = \begin{cases} \frac{1}{2}\left(u(x+2t,0) - u(2t-x,0)\right) & 0 < x < 2|t| \\ \frac{1}{2}\left(u(x+2t,0) + u(x-2t,0)\right) & x > 2|t| \end{cases}.$$

Now $u(y,0) = \begin{cases} 1 & y > 0 \\ -1 & y < 0 \end{cases}$ because it is the odd extension of the initial data.

It is convenient to observe from this that

$$\frac{1}{2}\left(u(y_1,0) - u(y_2,0)\right) = \begin{cases} 0 & \text{if } y_1 \text{ and } y_2 \text{ have the same sign} \\ 1 & \text{if } y_1 > 0 \text{ and } y_2 < 0 \\ -1 & \text{if } y_1 < 0 \text{ and } y_2 > 0 \end{cases}$$

$$\frac{1}{2}\left(u(y_1,0) + u(y_2,0)\right) = \begin{cases} 0 & \text{if } y_1 \text{ and } y_2 \text{ have opposite signs} \\ 1 & \text{if } y_1 \text{ and } y_2 > 0 \\ -1 & \text{if } y_1 \text{ and } y_2 < 0. \end{cases}$$

We may then compute for the case $x > 2|t|$ that either $t > 0$ and $x > 2t$ whence $x+2t, x-2t$ are both > 0 and $u(x,t) = 1$ or $t < 0$ and $x > -2t$ whence $x+2t > 0$ and $x-2t > 0$ so $u(x,t) = 1$.

For the case $x < 2|t|$, if $t > 0$ this says $x < 2t$ so $x+2t > 0$ and $2t - x > 0$, giving $u(x,t) = 0$. If $t < 0$, $x < -2t$ so $x+2t < 0$ and $2t-x < 0$ so also $u(x,t) = 0$

This gives us our solution

$$u(x,t) = \begin{cases} 0 & 0 < x < 2|t| \\ 1 & x > 2|t| \end{cases}$$

Evidently this is discontinuous (singular) along both characteristic lines $x = \pm 2t$.

3.2.6

$$u_{tt} = c^2 u_{xx} \qquad x > 0$$

$$u(x,0) \equiv 0$$

$$u_t(x,0) \equiv V$$

$$u_t(0,t) + a u_x(0,t) = 0$$

with V, a, c all positive and $a > c$.

There are several ways to do this. I think one of the easiest is as follows:

Consider a location $x > 0$.
If $c|t| < x$ then since the speed of the wave is $\le c$, the location x is only affected by the initial data on the interval $[x-ct, x+ct]$, which does not contain the origin. So at x you don't know you are on a half line until $c|t| = x$. Thus for $c|t| < x$ we have just the full line formula for a solution, which here is:

$$u(x,t) = \frac{1}{2c}\int_{x-c|t|}^{x+c|t|} V\,dy = \frac{2c|t|V}{2c} = |t|V$$

If $c|t| > x$ then the interval $[x-ct, x+ct]$ contains 0, and the wave will consist of two pieces, ~~one of the form~~
For convenience first take $t > 0$. One piece has the form

$$\frac{1}{2c}\int_0^{x+ct} V\,dy = \left(\frac{x+ct}{2c}\right)V$$

The other is from the piece $[0,|x-ct|] = [0, ct-x]$ but has been reflected at the origin. Since the reflection condition $u_t + a u_x = 0$ does not depend on the ~~origin~~ location of initial data we expect the reflected wave to be a multiple of $\frac{1}{2c}\int_0^{ct-x} V\,dy = \frac{V}{2c}(ct-x)$.

Call the multiple α, so our wave is

$$u(x,t) = \frac{V}{2c}\left(x+ct + \alpha(ct-x)\right)$$

$$= \frac{V}{2c}\left((1+\alpha)ct + (1-\alpha)x\right)$$

where α is such that $u_t + a u_x = 0$ at $x=0$,

i.e. $(1+\alpha)c + a(1-\alpha) = 0$

$$\Rightarrow \quad \alpha(c-a) + a + c = 0$$

$$\Rightarrow \quad \alpha = \frac{a+c}{a-c}$$

Thus $1+\alpha = \frac{2a}{a-c}$, $1-\alpha = \frac{-2c}{a-c}$ and

$$u(x,t) = \frac{V}{2c}\left(\frac{2a}{a-c}ct - \frac{2c}{a-c}x\right) = \frac{(at-x)V}{a-c}.$$

One can repeat this computation for $t<0$ to get

$$u(x,t) = \frac{V(x-at)}{a+c}.$$

Summarizing, we have:

For $t>0$

$$u(x,t) = \begin{cases} \dfrac{V}{a-c}(at-x) & 0<x<ct \\ Vt & x>ct \end{cases}$$

For $t<0$

$$u(x,t) = \begin{cases} \dfrac{V}{a+c}(x-at) & 0<x<-ct \\ V|t| & x>-ct \end{cases}$$