2.2.5 $u_{tt} - c^2 u_{xx} + r u_t = 0 \qquad r > 0.$

Let $E = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2$

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \rho u_t u_{tt} + T u_x u_{xt} = \int_{-\infty}^{\infty} \rho u_t u_{tt} - T u_t u_{xx}$$

$$= \int_{-\infty}^{\infty} \rho u_t (u_{tt} - c^2 u_{xx})$$

$$= -\int_{-\infty}^{\infty} \rho u_t r u_t \le 0$$

b/c ρ, r, u_t^2 all ≥ 0.

So E is non incr.

2.3.2 $u_t = u_{xx} \qquad 0 \le x \le l,\ 0 \le t < \infty$

Let $M(T)$ max on $[0,l] \times [0,T]$.

$m(T)$ min on $[0,l] \times [0,T]$.

Max principle says $M(T)$ = max of u on $x=0$, or $x=l$, $t \in [0,T]$ and $t=0$, $x \in [0,l]$.

$m(T)$ = min similarly.

Then, if you know the bdy values at $x=0$, $x=l$ are constant for all time, or if they are btwn the min & max of $u(x,0)$ for $0 \le x \le l$ then $M(T)$ and $m(T)$ are achieved on $t=0$ and are therefore indep of T.

Otherwise you can't say anything.

2.3.4 Have $u_t = u_{xx}$ $0<x<1$, $0<t<\infty$

with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

a) Notice that extrema of $4x(1-x)$ on $(0,1)$

are at endpts where it is 0 and crit pt where $4(1-2x)=0$

$\Rightarrow x = \frac{1}{2}$

$\Rightarrow 4x(1-x) = 1$.

So max & min of u on bdy $\left\{\begin{array}{l}(0,t) : t>0 \\ (1,t) : t>0 \\ (x,0) : x\in(0,1)\end{array}\right\}$

are 1 and 0.

Thus weak max princ. says $0 \leq u(x,t) \leq 1$ for $x\in(0,1)$, $t>0$.

Strong max princ. says $0 < u < 1$ on the set.

b) Let $v(x,t) = u(1-x,t)$

Then $v(0,t) = v(1,t) = 0$ and $v(x,0) = 4x(1-x)$

and $v_x = -u_x$, $v_{xx} = u_{xx} = u_t$

So by uniqueness of soln of heat eqn, $u = v$

which says $u(1-x,t) = u(x,t)$

for $x\in(0,1)$ and $t>0$.

c) $\frac{d}{dt}\int_0^1 u^2\,dx = \int_0^1 2uu_t\,dx = 2\int_0^1 uu_{xx}\,dx = -2\int_0^1 u_x^2\,dx$

using that $u=0$ at 0 and 1 in the integ by parts.

Now strong max principle in (a) shows u is not constant and therefore can't have $u(x,t)=0$ for all $x\in(0,1)$ at any t and thus $\int_0^1 u_x^2 > 0$ for any t. This shows that $\int_0^1 u^2\,dx$ is strictly decreasing with time.

2.4.1 Solve $\begin{cases} u_t = k u_{xx} \\ u(x,0) = \begin{cases} 1 & |x| < l \\ 0 & |x| > l \end{cases} \end{cases}$

By our formula $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, u(y,0)\, dy$

$$= \int_{-l}^{l} S(x-y,t)\, dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}}\, dy \qquad p = \frac{x-y}{\sqrt{4kt}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{\sqrt{4tk}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-p^2}\, dp$$

$$= \mathcal{E}rf\left(\frac{x+l}{\sqrt{4kt}}\right) - \mathcal{E}rf\left(\frac{x-l}{\sqrt{4kt}}\right)$$

2.4.4 Solve $u_t = k u_{xx}$

$u(x,0) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$

We follow example 2 in the text, completing the square in

$$\frac{(x-y)^2}{4kt} + y = \frac{x^2+y^2-2xy+4kty}{4kt} = \frac{1}{4kt}\left((y+2kt-x)^2 - kt + x\right)$$

to transform $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, u(y,0)\, dy = \int_0^{\infty} S(x-y,t)\, e^{-y}\, dy$

$$= \int_0^{\infty} \frac{e^{\left(-(x-y)^2/4kt - y\right)}}{\sqrt{4kt\pi}}\, dy$$

$$= \int_0^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}}\, e^{kt-x} \frac{dy}{\sqrt{4\pi kt}} \qquad p = \frac{y+2kt-x}{\sqrt{4kt}}$$

$$= \left(\int_{\frac{2kt-x}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp\right) \frac{e^{kt-x}}{\sqrt{\pi}}$$

$$= e^{kt-x}\left(\frac{1}{2} - \frac{1}{2}\mathcal{E}rf\left(\frac{2kt-x}{\sqrt{4kt}}\right)\right) \qquad \begin{matrix} t > 0 \\ x \in \mathbb{R} \end{matrix}$$

2.4.9 $u_t = k u_{xx}$ with $u(x,0) = x^2$

Reason via: u_{xxx} also solves heat eqn and $u_{xxx}(x,0) = 0$

$\Rightarrow u_{xxx}(x,t) = 0$ for all $t > 0$ by uniqueness

$\Rightarrow u(x,t) = A(t)x^2 + B(t)x + C(t)$.

And $u(x,0) = A(0)x^2 + B(0)x + C(0) = x^2$

$\Rightarrow A(0) = 1, \; B(0) = 0, \; C(0) = 0.$

Also $u_t = k u_{xx}$

$\Rightarrow A'(t)x^2 + B'(t)x + C'(t) = k\,2A(t)$

True for all x

$\Rightarrow A'(t) = B'(t) = C'(t) - 2kA(t) = 0$

But then $A(t) = A(0) = 1$

$B(t) = B(0) = 1$

$C(t) = 2kt + C(0) = 2kt$

and so $u(x,0) = x^2 + 2kt$.

2.4.17 $u_t - k u_{xx} + bt^2 u = 0$

But $u_t - but^2 = e^{bt^3/3} \frac{\partial}{\partial t}\left(e^{-bt^3/3} u\right)$

$$\Rightarrow \frac{\partial}{\partial t}\left(e^{-bt^3/3} u\right) - k e^{-bt^3/3} u_{xx} = 0$$

So $v = e^{-bt^3/3} u$ satisfies $v_t = k v_{xx}$ and has soln

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, v(y,0)\, dy$$

and $v(y,0) = u(y,0) = \phi(y)$ so

$$u(x,t) = e^{bt^3/3} \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\, dy.$$