2.1.1

$$u_{tt} = c^2 u_{xx} \qquad u(x,0) = e^x$$
$$u_t(x,0) = \sin x$$

Soln is

$$u(x,t) = \frac{1}{2}\left(e^{x-ct} + e^{x+ct}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s \, ds$$

$$= \frac{1}{2}\left(e^{x-ct} + e^{x+ct}\right) + \frac{1}{2c}(-\cos s)\Big|_{x-ct}^{x+ct}$$

$$= \frac{1}{2}\left(e^{(x-ct)} + e^{(x+ct)}\right) + \frac{1}{2c}\left(\cos(x-ct) - \cos(x+ct)\right)$$

One could rewrite this using hyperbolic functs and trig fmla as

$$= \frac{1}{2}e^x\left(e^{ct} + e^{-ct}\right) + \frac{1}{2c}\left(\cos x \cos ct + \sin x \sin ct - \cos x \cos ct + \sin x \sin ct\right)$$

$$= e^x \cosh ct + \frac{1}{c}\sin x \sin ct.$$

2.1.2

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = \log(1+x^2)$$

$$u_t(x,0) = 4+x$$

Soln is

$$u(x,t) = \frac{1}{2}\left(\log(1+(x-ct)^2) + \log(1+(x+ct)^2)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}(4+s)\,ds$$

$$= \frac{1}{2}\left(\log\left(1+(x-ct)^2\right)\left(1+(x+ct)^2\right)\right) + \frac{1}{2c}\left(4s+\frac{s^2}{2}\right)\Big|_{x-ct}^{x+ct}$$

$$= \frac{1}{2}\log\left(1+(x-ct)^2+(x+ct)^2+(x-ct)^2(x+ct)^2\right) + \frac{1}{2c}\left(4(x+ct-x+ct)+\frac{1}{2}\left((x+ct)^2-(x-ct)^2\right)\right)$$

$$= \frac{1}{2}\log\left(1+2x^2+2(ct)^2+(x^2-(ct)^2)^2\right) + \frac{1}{2c}\left(8ct+\frac{1}{2}(4ctx)\right)$$

$$= \frac{1}{2}\log\left(1+2x^2+2c^2t^2+x^4+c^4t^4-2x^2c^2t^2\right) + 4t + tx$$

2.1.8

Given $u_{tt} = c^2(u_{rr} + \frac{2}{r}u_r)$

a) Let $v = ru$ so $v_t = ru_t$, $v_{tt} = ru_{tt}$

$v_r = u + ru_r$, $v_{rr} = 2u_r + ru_{rr}$

Then $v_{tt} = c^2 r\left(u_{rr} + \frac{2}{r}u_r\right) = c^2(ru_{rr} + 2u_r) = c^2 v_{rr}$.

b) Since $v(r,t) = f(r+ct) + g(r-ct)$

then $u = \frac{v}{r} = \frac{1}{r}\left(f(r+ct) + g(r-ct)\right)$

c) Alternatively $v(r,t) = \frac{1}{2}\left(v(r+ct,0) + v(r-ct,0)\right)$

$+ \frac{1}{2c}\int_{x-ct}^{x+ct} v_t(s,0)\, ds$

$\Rightarrow u(r,t) = \frac{1}{2r}\left(v(r+ct,0) + v(r-ct,0)\right)$

$+ \frac{1}{2cr}\int_{r-ct}^{r+ct} v_t(s,0)\, ds$

$= \frac{1}{2r}\left((r+ct)u(r+ct,0) + (r-ct)u(r-ct,0)\right)$

$+ \frac{1}{2cr}\int_{r-ct}^{r+ct} s\, u_t(s,0)\, ds$

$= \frac{1}{2r}\left((r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right)$

$+ \frac{1}{2cr}\int_{r-ct}^{r+ct} s\,\psi(s)\, ds$

(The significance of ~~the conditions~~ assuming $\phi(r) = \phi(-r)$ and $\psi(r) = \psi(-r)$ is that then these functions, which are supposed to be radial so only defined for $r > 0$, make sense and are consistent for $r < 0$.