1.4.1 $u_t = u_{xx}$, $u(x,0) = x^2$.

Soln is $u(x,t) = 2t + x^2$

as one can check via $u_t = 2$, $u_{xx} = 2$, $u(x,0) = x^2$.

1.4.4 Rod has const heat source $f(x) = \begin{cases} 0 & 0 \le x \le \frac{L}{2} \\ H & \frac{L}{2} < x \le L \end{cases}$

2 Heat eqn is $u_t = u_{xx} + f(x)$

Ends held at temperature $= 0$ so $u(0,t) = u(L,t) = 0$ for all t.

a) Find steady state temperature.

So $u_t = 0$ and $u_{xx} = -f(x)$ 2

$$\Rightarrow u_x = c_1 - \int_0^x f(x) = c_1 - \begin{cases} 0 & x \le \frac{L}{2} \\ \int_{\frac{L}{2}}^{x} H & \frac{L}{2} < x \le L \end{cases}$$

$$= \begin{cases} c_1 & x \le \frac{L}{2} \\ c_1 - H(x - \frac{L}{2}) & \frac{L}{2} \le x \le L \end{cases}$$

2 $$\Rightarrow u = \begin{cases} c_1 x + c_2 & x \le \frac{L}{2} \\ (c_1 + \frac{HL}{2})(x - \frac{L}{2}) - \frac{H}{2}(x^2 - \frac{L^2}{4}) + c_3 & \frac{L}{2} < x \le L \end{cases}$$

But need

2 $u(0) = 0$ so $c_2 = 0$

$u(L) = 0$ so $(c_1 + \frac{HL}{2})(L - \frac{L}{2}) - \frac{H}{2}(L^2 - \frac{L^2}{4}) + c_3 = 0$

$$\Rightarrow (c_1 + \tfrac{HL}{2})(\tfrac{L}{2}) - \frac{3HL^2}{8} = 0 \qquad (*)$$

and also continuity at $\frac{L}{2}$: $c_1 \frac{L}{2} = c_3$ 1

So $*$ becomes $(2c_1 + \frac{HL}{2})(\frac{L}{2}) = \frac{3HL^2}{8}$ $\Rightarrow 2c_1 = \frac{3HL}{4} - \frac{HL}{2} = \frac{HL}{4}$

$\Rightarrow c_1 = \frac{HL}{8}$

1.6.2 For the eqn $(1+x)u_{xx} + 2xy\,u_{xy} - y^2 u_{yy} = 0$

we have coeff matrix for 2nd partials is $\begin{bmatrix} 1+x & xy \\ xy & -y^2 \end{bmatrix}$

with determinant $-(1+x)y^2 - x^2y^2$

$= -y^2(1+x+x^2)$

This is ~~positive~~ negative if $y \neq 0$ and $1+x+x^2 > 0$,

so $\frac{3}{4} + (x+\frac{1}{2})^2 > 0$ which is always true

~~$\Leftrightarrow (x-\frac{1}{2})^2 < \frac{5}{4}$~~

~~$\Leftrightarrow -\frac{\sqrt5}{2} < x - \frac{1}{2} < \frac{\sqrt5}{2}$~~

~~$\Leftrightarrow \frac{1-\sqrt5}{2} < x < \frac{\sqrt5+1}{2}$~~

~~negative if $y \neq 0$ and $1+x-x^2 < 0$,~~

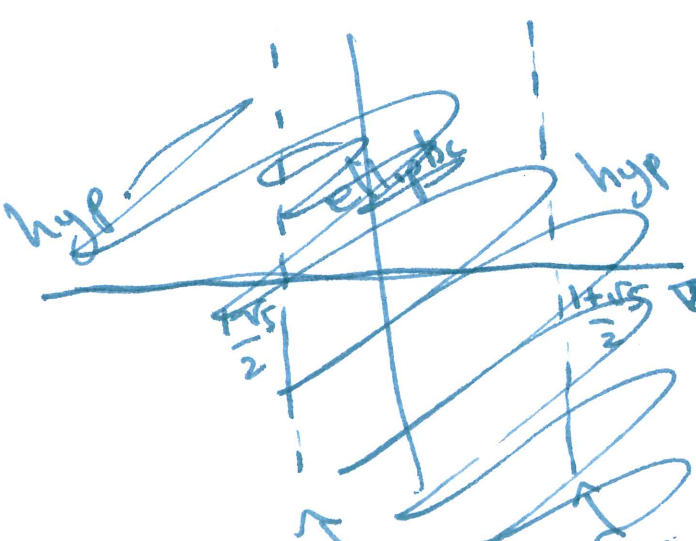
~~so $x < \frac{1-\sqrt5}{2}$ or $x > \frac{1+\sqrt5}{2}$~~

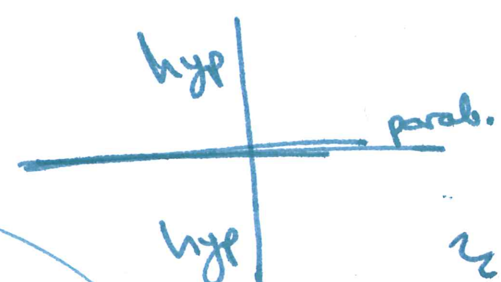
and zero if $y=0$ ~~or $x = \frac{1\pm\sqrt5}{2}$~~.

Hence the eqn is ~~elliptic if $y \neq 0$ and $\frac{1-\sqrt5}{2} < x < \frac{1+\sqrt5}{2}$~~

hyperbolic if $y \neq 0$ ~~and $x < \frac{1-\sqrt5}{2}$ or $x > \frac{1+\sqrt5}{2}$~~

parabolic if $y=0$ ~~or $x = \frac{1\pm\sqrt5}{2}$~~





parabolic on these lines and the x axis.

1.6.5 $$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0 \qquad (*)$$

Let $u = v e^{\alpha x + \beta y}$

then $$u_x = (v_x + \alpha v) e^{\alpha x + \beta y}$$

$$u_{xx} = (v_{xx} + \alpha v_x + \alpha v_x + \alpha^2 v) e^{\alpha x + \beta y}$$
$$= (v_{xx} + 2\alpha v_x + \alpha^2 v) e^{\alpha x + \beta y}$$

and similarly $$u_y = (v_y + \beta v) e^{\alpha x + \beta y}$$
$$u_{yy} = (v_{yy} + 2\beta v_y + \beta^2 v) e^{\alpha x + \beta y}$$

Multiplying $(*)$ by $e^{-(\alpha x + \beta y)}$ we find that

$$v_{xx} + 2\alpha v_x + \alpha^2 v + 3(v_{yy} + 2\beta v_y + \beta^2 v)$$
$$-2(v_x + \alpha v) + 24(v_y + \beta v) + 5v = 0$$

$$\Rightarrow 0 = v_{xx} + 3v_{yy} + (2\alpha - 2)v_x$$
$$+ (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v$$

$$= v_{xx} + 3v_{yy} - 44v$$

if $\alpha = 1$, $\beta = -4$

Now let $y' = \gamma y$ so $v_y = v_{y'} \gamma$
$v_{yy} = \gamma^2 v_{y'y'}$

and with $\gamma = \frac{1}{\sqrt{3}}$ we have

$$0 = v_{xx} + v_{y'y'} - 44v$$