6.2.8

$\Delta u = 0$ on $(0,\pi)\times(0,\pi)$

$u_y = 0$ on $y=0$ and $y=\pi$

$u = 0$ on $x=0$

$u = \frac{1}{2}(1+\cos 2y)$ on $x=\pi$.

We separate variables so $u = \sum X_n(x) Y_n(y)$

with $\frac{X_n''}{X_n} + \frac{Y_n''}{Y_n} = 0$ and $\frac{Y_n''}{Y_n} = -\lambda_n$ is basis with homog Neum condits,

so $Y_n = \cos\left(\frac{n\pi}{\pi} y\right) = \cos ny$, $\lambda_n = n^2$

$n = 0,1,2,\ldots$

Then $\frac{X_n''}{X_n} = \lambda_n = n^2$ has solns $X_n(x) = A_n \cosh nx + B_n \sinh nx$

for $n=1,2,3$ and $X_0(x) = A_0 + B_0 x$

So have $u(x,y) = A_0 + B_0 x + \sum_{n=1}^{\infty} (A_n \cosh nx + B_n \sinh nx)\cos ny$

But $u(0,y) = 0 \Rightarrow \sum_{n=0}^{\infty} A_n \cos ny = 0 \Rightarrow A_n = 0 \quad n=0,1,2,\ldots$

$u(\pi,y) = \frac{1}{2} + \frac{1}{2}\cos 2y$

$\Rightarrow B_0 \pi + \sum_{1}^{\infty} B_n \sinh(n\pi)\cos ny = \frac{1}{2} + \frac{1}{2}\cos 2y$

$\Rightarrow B_0 = \frac{1}{2\pi}$, $B_2 = \frac{1}{2\sinh 2\pi}$ and all other $B_n = 0$

$\Rightarrow u(x,y) = \frac{1}{2\pi} x + \frac{\sinh 2x \cos 2y}{2\sinh 2\pi}$

6-2.6 $\quad \Delta u = 0$ on $(0,1)^3$

Homog Neum on all bdys except

$u_z(x,y,1) = g(x,y)$ s.t. $\int_0^1\int_0^1 g(x,y)\,dx\,dy = 0.$

Expand in $X_n(x)$ and $Y_n(y)$ efns for homog Neum bdy, ie.

$$X_n(x) = \cos(n\pi x), \quad Y_m(y) = \cos(m\pi y)$$

$$u(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} X_n(x) Y_n(y) Z_{n,m}(z)$$

with $\frac{Z''_{n,m}}{Z} = -\frac{X_n''}{X_n} - \frac{Y_m''}{Y_m} = n^2+m^2$

$$\Rightarrow \quad Z_{n,m} = A_{n,m}\cosh(\sqrt{n^2+m^2}\,\pi z) + B_{n,m}\sinh(\sqrt{n^2+m^2}\,\pi z)$$

unless $n=m=0$ where $Z_{0,0} = A_{0,0} + B_{0,0} z$.

Given $\quad 0 = u_z(x,y,0) = \sum_{n=0}\sum_{m=0} X_n(x) Y_n(y) Z'_{n,m}(0)$

$$= \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sqrt{n^2+m^2}\,\pi\, B_{n,m} X_n Y_m\right) + B_{0,0}$$

$\Rightarrow$ All $B_{n,m} = 0$

And given

$$* \begin{cases} g(x,y) = u_z(x,y,1) \\ \quad = A_{0,1}(\pi\sinh\pi)\cos(m\pi y) + A_{1,0}(\pi\sinh\pi)\cos(n\pi x) \\ \quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m}\left(\sqrt{n^2+m^2}\,\pi\sinh\sqrt{n^2+m^2}\,\pi\right)\cos(n\pi x) \\ \quad \cdot\cos(m\pi y) \end{cases}$$

Thus $A_{n,m} = \frac{4}{(n^2+m^2)^{1/2}\pi\sinh\sqrt{n^2+m^2}\,\pi}\int_0^1\int_0^1 g(x,y)\cos(n\pi x)\cos(m\pi y)\,dx\,dy$

for $n,m \geq 0$ such that $n+m \geq 1$.

And we dont know $A_{0,0}$, b/c any const can be added to soln without changing that it is a soln. Note, too, that the condit $\int_0^1\int_0^1 g(x,y) = 0$ was needed in order that the expansion (*) is possible.

6.3.1

$\Delta u = 0$ on $\{r < 2\} = D$

$u = 3\sin 2\theta + 1$ on $\{r = 2\}. = \partial D$

i) By max principle, max of u on $\bar{D}$

$= $ max of u on ∂D

$= 4$ (achieved at $\theta = \frac{\pi}{4}$) b/c $|\sin 2\theta| \leq 1$

ii) The Poisson kernel at 0 is uniform density on circle with mass 1, so $u(0) = \frac{1}{2\pi}\int_0^{2\pi} (3\sin 2\theta + 1)\, d\theta = 1$

(b/c $\sin 2\theta$ is periodic & integrates to 0).