6.1.2 $\Delta u = k^2 u$ in $\mathbb{R}^3$

⇒ Find solutions depending only on r.

Using polar form of Δ we have

$$u_{rr} + \frac{2}{r} u_r = k^2 u$$

Try the hint: $u = \frac{v}{r} \Rightarrow u_r = \frac{v_r}{r} - \frac{v}{r^2}$

$$u_{rr} = \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3}$$

Then $u_{rr} + \frac{2}{r} u_r = k^2 u$

$$\Rightarrow \frac{v_{rr}}{r} - \cancel{\frac{2v_r}{r^2}} + \cancel{\frac{2v}{r^3}} + \cancel{\frac{2v_r}{r^2}} - \cancel{\frac{2v}{r^3}} = k^2 \frac{v}{r}$$

$$\Rightarrow v_{rr} = k^2 v$$

$$\Rightarrow v(r) = A\cosh(k^2 r) + B\sinh(k^2 r)$$

$$\Rightarrow u(r) = \left(A\cosh(k^2 r) + B\sinh(k^2 r)\right)/r.$$

Equivalently: $u(r) = \left(Ce^{k^2 r} + De^{-k^2 r}\right)/r.$

6.1.5 $u_{xx} + u_{yy} = 1$ on $r < a$.

$u = 0$ on $r = a$.

$$\Rightarrow u_{rr} + \frac{1}{r} u_r = 1$$

$$\Rightarrow \frac{\partial}{\partial r}(r u_r) = r$$

$$\Rightarrow u_r = \frac{r}{2} + \frac{c_1}{r}$$

$$\Rightarrow u = \frac{r^2}{4} + c_1 \log r + c_2$$

Continuity at $r = 0 \Rightarrow c_1 = 0$.

At $r = a$, $0 = \frac{a^2}{4} + c_2 \Rightarrow u = \frac{r^2 - a^2}{4}.$

6.1.8

$\Delta u = 1$ on $a < r < b$ in $\mathbb{R}^3$

$u = 0$ on $r = a$

$u_r = 0$ on $r = b$

Radial sym $\Rightarrow$ $u = u(r)$ whence PDE is

$$u_{rr} + \frac{2}{r} u_r = 1$$

$$\Rightarrow \frac{\partial}{\partial r}(r^2 u_r) = r^2$$

$$\Rightarrow u_r = \frac{r}{3} + \frac{c_1}{r^2} \quad \text{and } u_r(b) = 0$$

$$\Rightarrow c_1 = -\frac{b^3}{3}$$

$$\Rightarrow u_r = \frac{r}{3} - \frac{b^3}{3r^2}$$

$$\Rightarrow u = \frac{r^2}{6} + \frac{b^3}{3r} + c_2$$

and $u(a) = 0 \Rightarrow c_2 = -\frac{a^2}{6} - \frac{b^3}{3a}$

Hence $u(r) = \frac{r^2 - a^2}{6} + \frac{b^3}{3}\left(\frac{1}{r} - \frac{1}{a}\right)$.

When $a \to 0$ we see that $u \to -\infty$ at all $r > 0$. There are various physical interpretations. Mathematically one might simply say that any function with $\Delta u = 1$ and $u_r = 0$ on $r = b$ has a singularity at 0, so there is no way to force $u = 0$ at $r = 0$.