5.5.4

a) $u_t = k u_{xx} \quad x \in (0,l)$

$u(x,0) = \phi(x)$

$u(0,t) = u(l,t) = \dfrac{u(l,t) - u(0,t)}{l}$.

Separating vars we get eval prob $-X'' = \lambda X$ with bdy condit,

So if $\lambda > 0$ then

$$X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

$$\frac{X(l) - X(0)}{l} = \frac{C(\cos\sqrt{\lambda}l - 1) + D\sin\sqrt{\lambda}l}{l}$$

and $X'(0) = D\sqrt{\lambda}$, $X'(l) = (-C\sin(\sqrt{\lambda}l) + D\cos\sqrt{\lambda}l)\sqrt{\lambda}$

Bdy condits are then

$$\left.\begin{aligned} C(\cos\sqrt{\lambda}l - 1) + D(\sin\sqrt{\lambda}l - \sqrt{\lambda}l) &= 0 \\ \sqrt{\lambda}\left(-C\sin\sqrt{\lambda}l + D(\cos\sqrt{\lambda}l - 1)\right) &= 0 \end{aligned}\right] \quad (*)$$

Since assumed $\lambda > 0$ we can cancel $\sqrt{\lambda}$ from 2^{nd} eqn and reduce existence of nontrivial soln for C, D to

$$(\cos\sqrt{\lambda}l - 1)^2 + \sin\sqrt{\lambda}l\,(\sin\sqrt{\lambda}l - \sqrt{\lambda}l) = 0$$

It is suggested in 4.3.12 to set $\gamma = \frac{1}{2}\sqrt{\lambda}l$ so $(\cos 2\gamma - 1)^2 = (-2\sin^2\gamma)^2 = 4\sin^4\gamma$, $\sin 2\gamma = 2\sin\gamma\cos\gamma$ and we obtain

$$4\sin^4\gamma + 4\sin\gamma\cos\gamma(\sin\gamma\cos\gamma - \gamma) = 0$$

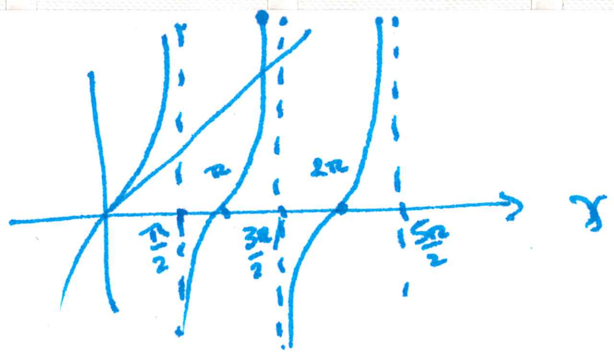
$$\Rightarrow 4\sin^2\gamma(\sin^2\gamma + \cos^2\gamma) - 4\gamma\sin\gamma\cos\gamma = 0$$

$$\Rightarrow 4\sin\gamma(\sin\gamma - \gamma\cos\gamma) = 0$$

So either $\sin\gamma = 0$ whence $\gamma = n\pi \quad n \in \mathbb{Z}$

$$\Rightarrow \lambda_n = \frac{4n^2\pi^2}{l^2}, \quad \sqrt{\lambda} = \frac{2n\pi}{l}$$

or $\sin\gamma = \gamma\cos\gamma$ which has solns $\gamma = 0$ (incompatible with our assumption $\lambda > 0$) or $\tan\gamma = \gamma$, which has infinitely many solns, one in each period of $\tan\gamma$.



We conclude there are γ_n with $n\pi \le \gamma_n \le (n+1)\pi$ for $n = 1, 2, 3, \dots$ and $\lim(\gamma_n - (n+\frac{1}{2})\pi) = 0$ as $n \to \infty$. The corresp λ_n is $\left(\frac{2\gamma_n}{L}\right)^2$, and the corresp efn is, by solving (*)

$$X_n(x) = (\cos\sqrt{\lambda_n}L - 1)\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}L)\sin(\sqrt{\lambda_n}x)$$

$$= \cos(\sqrt{\lambda_n}(L-x)) - \cos(\sqrt{\lambda_n}x)$$

The same is true for the efns with $\lambda_n = \frac{4n^2\pi^2}{L^2}$.

To summarize, for $\lambda > 0$ we have two sets of evals and efns.

- Evals $\frac{4n^2\pi^2}{L^2}$, efns $\cos\left(\frac{2n\pi}{L}(L-x)\right) - \cos\left(\frac{2\pi n}{L}x\right) = Y_n(x)$
- Evals solns of $\tan\frac{1}{2}\sqrt{\lambda_n}L = \frac{1}{2}\sqrt{\lambda_n}L$
  efns $\cos\sqrt{\lambda_n}(L-x) - \cos(\sqrt{\lambda_n}x) = X_n(x)$

We also have, for $\lambda = 0$, the efns $Ax + B$ with bdy condits given by $X'(0) = A = X'(L) = \frac{X(L) - X(0)}{L} = \frac{AL + B - B}{L}$, so A and B are unconstrained. This says that 0 is an eval of multiplicity 2 with efns 1 and x.

Further observe that if X_1 and X_2 satisfy the bdy condits

then $\displaystyle -X_1'X_2 + X_1X_2' \Big|_0^l$

$$= -X_1'(l)X_2(l) + X_1'(0)X_2(0) + X_1(l)X_2'(l) - X_1(0)X_2'(0)$$

$$= X_1'(0)\big(X_2(0) - X_2(l)\big) + X_2'(0)\big(X_1(l) - X_1(0)\big)$$

$$= X_1'(0)\big(-X_2'(0)\,l\big) + X_2'(0)\big(X_1'(0)\,l\big) = 0$$

So the bdy condits are symmetric and efns corresp to distinct evals are orthogonal. Hence all efns except 1 and x are orthogonal, and we can orthogonalize $1, x$ using Gram-Schmidt to get $\displaystyle x - \frac{\int_0^l 1\,x\,dx}{\int_0^l 1\,dx} = x - \frac{l^2}{2l} = x - \frac{l}{2}$.

Thus we can solve the eqn as a series

$$(**) \quad u(x,t) = \sum_{n=1}^{\infty}\left(A_n X_n(x)\, e^{-k\lambda_n t} + B_n Y_n(x)\, e^{-k\left(\frac{4n^2\pi^2}{l^2}\right)t}\right) + A_0 + B_0\left(x - \frac{l}{2}\right)$$

where $\displaystyle \phi(x) = u(x,0) = \sum_{n=1}^{\infty}\left(A_nX_n + B_nY_n\right) + A_0 + B_0\left(x - \frac{l}{2}\right)$

and thus (by orthogonality)

$$A_n = \frac{\int_0^l \phi(x)X_n(x)\,dx}{\int_0^l (X_n(x))^2\,dx}, \qquad B_n = \frac{\int_0^l \phi(x)Y_n(x)\,dx}{\int (Y_n(x))^2\,dx}$$

$$A_0 = \frac{1}{l}\int_0^l \phi(x)\,dx \quad \text{and} \quad B_0 = \frac{\int_0^l \phi(x)\left(x - \frac{l}{2}\right)}{\int_0^l \left(x - \frac{l}{2}\right)^2 dx} = \frac{\int_0^l \phi(x)\left(x - \frac{l}{2}\right)}{\left(l^3/12\right)}$$

b) It is easy to see in (**) that term-by-term limits have all terms except $A_0 + B_0\left(x - \frac{l}{2}\right)$ go to 0.

c) We have for $-X'''' = \lambda X$

$$\lambda \int_0^l X^2\,dx = -\int_0^l X''''X\,dx = \int_0^l (X'')^2\,dx - XX'\Big|_0^l$$

but $XX'\Big|_0^l = X'(0)\left(X(l)-X(0)\right) = \frac{\left(X(l)-X(0)\right)^2}{l} \leq \int_0^l (X')^2$ by

result of exercise 3,

So $\lambda \int_0^l X^2\,dx \geq 0$ and there are no -ve evals.

d) We ~~[illegible]~~ found $A_0 = \frac{1}{l}\int_0^l \phi(x)\,dx$

$$B_0 = \frac{12}{l^3}\int_0^l \phi(x)\left(x-\frac{l}{2}\right)dx$$

But clearly from our solution $B = B_0$

and $A = A_0 - \frac{l}{2}B_0$

$$= \frac{1}{l}\int_0^l \phi - \frac{6}{l^2}\int_0^l \phi\left(x-\frac{l}{2}\right)$$

$$= -\frac{2}{l}\int_0^l \phi(x) - \frac{6}{l^2}\int_0^l x\phi(x)$$

5.5.7 Given $\int_{-\pi}^{\pi} |f|^2 + |g|^2 < \infty$,

with $g(x) = \frac{f(x)}{e^{ix}-1}$, c_n the coeffs of f in Full Fourier series.

Compute $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)(e^{ix}-1)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} g(x)\, e^{-i(n-1)x} - \int_{-\pi}^{\pi} g(x)\, e^{-inx}\, dx\right)$$

So if the fourier series for g is $\sum_{-\infty}^{\infty} a_n e^{inx}$ (which exists & is cvgnt b/c $\int_{-\pi}^{\pi} |g|^2 < \infty$, then

$$c_n = a_{n-1} - a_n$$

So then $\sum_{n=-N}^{N} c_n = \sum_{n=-N}^{N} (a_{n-1} - a_n)$

$$= \sum_{n=-(N+1)}^{N-1} a_n - \sum_{n=-N}^{N} a_n$$

$$= a_{-(N+1)} - a_N$$

but $\sum_{-\infty}^{\infty} |a_n|^2$ converges because $\int |g|^2 < \infty$, so both $a_{-(N+1)}$ and a_N converge to 0, and therefore so does $\sum_{-N}^{N} c_n$.

5.6.1

a) Solve as a series $u_t = u_{xx}$ in $(0,1)$

$u_x(0,t) = 0$

$u(1,t) = 1$

$u(x,0) = x^2$

Compute the first two terms explicitly.

One Soln: We are given a (homog) Neumann condit at 0 and an inhomog Dir condit at 1. We take a Fourier basis that is Neum @ 0 and Dir at 1, namely we solve

$$-X'' = \lambda X$$
$$X'(0) = 0, \quad X(1) = 0$$

which gives $X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$

and $X'(0) = 0 \Rightarrow D = 0$

$X(1) = 0 \Rightarrow C\cos\sqrt{\lambda} = 0$

$\Rightarrow \sqrt{\lambda} = (n+\frac{1}{2})\pi \quad n = 0,1,2,\ldots$

Thus we take a series expansion with respect to basis $\cos((n+\frac{1}{2})\pi x)$, for u, u_t and u_{xx}:

$$u(x,t) = \sum_0^\infty a_n(t)\cos((n+\tfrac{1}{2})\pi x)$$

$$u_t(x,t) = \sum_0^\infty b_n(t)\cos((n+\tfrac{1}{2})\pi x)$$

$$u_{xx}(x,t) = \sum_0^\infty c_n(t)\cos((n+\tfrac{1}{2})\pi x)$$

We have $a_n(t) = 2\int_0^1 u(x,t)\cos((n+\frac{1}{2})\pi x)\,dx$

$$b_n(t) = 2\int_0^1 u_t(x,t)\cos((n+\tfrac{1}{2})\pi x)\,dx$$
$$= a_n'(t)$$

and

$$c_n(t) = 2\int_0^1 u_{xx}(x,t)\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)dx$$

$$= 2\left[u_x(x,t)\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)\Big|_0^1 - \int_0^1 u_x(x,t)\left(n+\tfrac{1}{2}\right)\pi\,\sin\left(\left(n+\tfrac{1}{2}\right)\pi x\right)dx\right]$$

$$= 2\left[-u(x,t)\left(n+\tfrac{1}{2}\right)\pi\,\sin\left(\left(n+\tfrac{1}{2}\right)\pi x\right)\Big|_0^1 - \int_0^1 u(x,t)\left(n+\tfrac{1}{2}\right)^2\pi^2\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)dx\right]$$

$$= 2\left[\left(n+\tfrac{1}{2}\right)\pi\left((-1)^{n+1}u(1,t)+0\right) - \left(n+\tfrac{1}{2}\right)^2\pi^2 a_n(t)\right)$$

$$= (2n+1)\pi(-1)^{n+1} - \left(n+\tfrac{1}{2}\right)^2\pi^2 a_n(t).$$

However the eqn says $u_t = u_{xx}$ from which, using our expansions,

$$a_n'(t) = b_n(t) = c_n(t)$$
$$= (2n+1)\pi(-1)^{n+1} - \left(n+\tfrac{1}{2}\right)^2\pi^2 a_n(t)$$

Thus

$$\frac{d}{dt}\left(e^{\left(n+\frac{1}{2}\right)^2\pi^2 t}a_n(t)\right) = (-1)^{n+1}\pi(2n+1)e^{\left(n+\frac{1}{2}\right)^2\pi^2 t}$$

and so

$$a_n(t) = a_n(0)e^{-\left(n+\frac{1}{2}\right)^2\pi^2 t} + (-1)^{n+1}\pi(2n+1)e^{-\left(n+\frac{1}{2}\right)^2\pi^2 t}\int_0^t e^{\left(n+\frac{1}{2}\right)^2\pi^2 s}ds$$

$$= a_n(0)e^{-\left(n+\frac{1}{2}\right)^2\pi^2 t} + (-1)^{n+1}\frac{2}{\left(n+\frac{1}{2}\right)\pi}\left(1-e^{-\left(n+\frac{1}{2}\right)^2\pi^2 t}\right)$$

$$= \frac{(-1)^{n+1}2}{\left(n+\frac{1}{2}\right)\pi} + \left(a_n(0) + \frac{(-1)^n 2}{\left(n+\frac{1}{2}\right)\pi}\right)e^{-\left(n+\frac{1}{2}\right)^2\pi^2 t}$$

and $$u(x,t) = \sum_0^\infty a_n(t)\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)$$

We are also given

$$x^2 = u(x,0) = \sum_0^\infty a_n(0) \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)$$

So that by orthogonality

$$a_n(0) = 2\int_0^1 x^2 \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right) dx$$

$$= 2\left[x^2 \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\left(n+\frac{1}{2}\right)\pi}\Bigg|_0^1 - \int_0^1 2x \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\left(n+\frac{1}{2}\right)\pi} dx \right]$$

$$= 2\left[\frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi} + 2x \frac{\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\left(n+\frac{1}{2}\right)^2\pi^2}\Bigg|_0^1 - \int_0^1 \frac{2\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\left(n+\frac{1}{2}\right)^2\pi^2} dx \right]$$

$$= 2\left[\frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi} + 0 - \frac{2\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\left(n+\frac{1}{2}\right)^3\pi^3}\Bigg|_0^1 \right]$$

$$= \frac{2(-1)^n}{\left(n+\frac{1}{2}\right)\pi}\left[1 - \frac{2}{\left(n+\frac{1}{2}\right)^2\pi^2}\right]$$

and thus $\displaystyle a_n(0) + \frac{(-1)^n 2}{\left(n+\frac{1}{2}\right)\pi} = \frac{4(-1)^n}{\left(n+\frac{1}{2}\right)\pi}\left[1 - \frac{1}{\left(n+\frac{1}{2}\right)^2\pi^2}\right]$

and soln is

$$u(x,t) = \sum_0^\infty \left(\frac{2(-1)^{n+1}}{\left(n+\frac{1}{2}\right)\pi} + \frac{4(-1)^n}{\left(n+\frac{1}{2}\right)^3\pi^3}\left[\left(n+\tfrac{1}{2}\right)^2\pi^2 - 1\right] e^{-\left(n+\frac{1}{2}\right)^2\pi^2 t} \right)$$

We are asked for the first two terms, with $n=0$ and $n=1$,

which are $\displaystyle -\frac{4}{\pi} + \frac{32}{\pi^3}\left(\frac{\pi^2}{4} - 1\right) e^{-\frac{\pi^2 t}{4}}$ $(n=0)$

and $\displaystyle \frac{4}{3\pi} - \frac{32}{27\pi^3}\left(\frac{9\pi^2}{4} - 1\right) e^{-\frac{9\pi^2 t}{4}}$ $(n=1)$

b) The equilibrium state (limit as $t \to \infty$), computing limit term-by-term, is

$$\frac{2}{\pi} \sum_0^\infty \frac{(-1)^{n+1}}{(n+\frac{1}{2})}$$

(which is actually $= 1$ as it is the Fourier series for the const funct 1 at 0)

Alternative Soln

You might have chosen to subtract something to make the problem easier. One option is to let $v(x,t) = u(x,t) - 1$ so

$$v_t = v_{xx} \quad \text{in } (0,1)$$

$$v_x(0,t) = u_x(0,t) = 0 \qquad \text{Homog Neum at } 0$$

$$v(1,t) = u(1,t) - 1 = 0 \qquad \text{Homog Dir at } 1$$

$$v(x,0) = u(x,0) - 1 = x^2 - 1.$$

Since we have homog bdy condits we know that setting $v(x,t) = \sum_0^\infty a_n(t) X_n(x)$ for $X_n(x) = \cos((n+\frac{1}{2})\pi x)$ (the Fourier basis with these bdy condits) reduces the PDE to $a_n'(t) = -\lambda_n a_n(t) \Rightarrow a_n(t) = a_n(0)\, e^{-\lambda_n t}$.

Then using $x^2 - 1 = v(x,0) = \sum_0^\infty a_n(0) X_n(x)$ we can compute

$$a_n(0) = \frac{2}{1}\int_0^1 (x^2-1)\cos((n+\tfrac{1}{2})\pi x)\, dx$$ (using fact bdy condits symmetric so have X_n orthog)

$$= 2\frac{(-1)^n}{(n+\frac{1}{2})\pi}\left[1 - \frac{2}{(n+\frac{1}{2})^2\pi^2}\right] - \frac{2(-1)^n}{(n+\frac{1}{2})\pi}$$ (I reused the computation in previous solution)

$$= \frac{4(-1)^{n+1}}{\pi^3 (n+\frac{1}{2})^3}.$$

Thus $u(x,t) = 1 + v(x,t)$

$$= 1 + \sum_0^\infty \frac{4(-1)^{n+1}}{\pi^3(n+\frac{1}{2})^3} e^{-\left((n+\frac{1}{2})^2\pi^2\right)t} \cos((n+\tfrac{1}{2})\pi x)$$

And in the term by term limit as $t \to \infty$ we get

$$\lim_{t\to\infty} u(x,t) = 1.$$

5.6.4 Solve

$$u_{tt} = c^2 u_{xx} + k \qquad 0 < x < l$$
$$u(0,t) = 0$$
$$u_x(l,t) = 0$$
$$u(x,0) = 0$$
$$u_t(x,0) = V$$

Take Fourier basis $X_n = \sin\left(n+\frac{1}{2}\right)\frac{\pi}{l}x$, $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2$, $n = 0,1,2,\dots$ which has Dir condit at 0 and Neum at l. Since the bdy condits hold,

$$u = \sum_0^\infty a_n(t) X_n(x)$$
$$u_{tt} = \sum_0^\infty a_n''(t) X_n(x)$$
$$u_{xx} = \sum_0^\infty a_n(t)(-\lambda_n) X_n(x)$$

Also $k = k \sum_0^\infty d_n X_n(x)$

Thus PDE is $a_n''(t) = -c^2\lambda_n a_n(t) + k d_n \qquad n = 0,1,2,\dots$

which is an inhom ODE of easily solved type. One soln is $a_n(t) = \frac{k d_n}{c^2 \lambda_n}$, and the homog soln may be added to get

$$a_n(t) = \frac{k d_n}{c^2\lambda_n} + A_n \cos(c\sqrt{\lambda_n}\, t) + B_n \sin(c\sqrt{\lambda_n}\, t)$$

But $0 = u(x,0) = \sum_0^\infty a_n(0) X_n(x) \Rightarrow a_n(0) = 0 \Rightarrow A_n = \frac{-k d_n}{c^2\lambda_n}$

and $V = u_t(x,0) = \sum_0^\infty a_n'(0) X_n(x)$ and $V = V\sum_0^\infty d_n X_n(x)$ (b/c d_n are coeffs of expanding 1)

$$\Rightarrow a_n'(0) = V d_n \Rightarrow B_n c\sqrt{\lambda_n} = V d_n$$
$$\Rightarrow B_n = \frac{V d_n}{c\sqrt{\lambda_n}}$$

Then $u(x,t) = \sum_0^\infty a_n(t) X_n(x)$

$$= \sum_0^\infty \left(\frac{k d_n}{c^2\lambda_n}\left(1 - \cos c\sqrt{\lambda_n}\, t\right) + \frac{V d_n}{c\sqrt{\lambda_n}} \sin(c\sqrt{\lambda_n}\, t)\right) X_n(x).$$

but $d_n = \frac{2}{l}\int_0^l \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{l}x\right)dx = \left[\frac{-2}{l\sqrt{\lambda_n}}\cos\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)\right]_0^l = \frac{2}{\pi(n+\frac{1}{2})}$

and plug in λ_n to get

$$u(x,t) = \frac{2l^2}{c^2\pi^3}\sum^\infty \left[\frac{k}{(n+\frac{1}{2})^3}\left(1 - \cos\left(\frac{(n+\frac{1}{2})\pi c t}{l}\right)\right) + \frac{Vc\pi}{(n+\frac{1}{2})^2 l}\sin\left(\frac{(n+\frac{1}{2})\pi c t}{l}\right)\right] X_n(x)$$

Alternatively one could set $w(x,t) = u(x,t) - \frac{1}{2}kt^2$

So that
$$\begin{cases} w_{tt} = u_{tt} - k = c^2 u_{xx} = c^2 w_{xx} \\ w(0,t) = u(0,t) - \frac{1}{2}kt^2 = -\frac{1}{2}kt^2 \\ w_x(l,t) = u_x(l,t) = 0 \\ w(x,0) = u(x,0) = 0 \\ w_t(x,0) = u_t(x,0) = V \end{cases}$$

Again one can expand in $X_n(x) = \sin\left((n+\frac{1}{2})\frac{\pi}{l}x\right)$. This time one must compute using integration by parts:

$$w(x,t) = \sum_{0}^{\infty} \alpha_n(t) X_n(x)$$
$$w_{tt}(x,t) = \sum_{0}^{\infty} \beta_n(t) X_n(x)$$
$$w_{xx}(x,t) = \sum_{0}^{\infty} \gamma_n(t) X_n(x)$$

We have $\beta_n(t) = \alpha_n''(t)$ as usual, but

$$\gamma_n(t) = \frac{2}{l}\int_0^l w_{xx} X_n(x)\,dx = \frac{2}{l}\left[w_x X_n\Big|_0^l - w X_n'\Big|_0^l + \int_0^l w X_n''\,dx \right]$$
$$= \frac{2}{l}\left[w_x X_n - w X_n'\right]_0^l - \lambda_n \alpha_n(t)$$

Now using $X_n(0) = 0$, $X_n'(0) = \sqrt{\lambda_n}$, $X_n(l) = (-1)^n$, $X_n'(l) = 0$ and the data for $w(0,t), w_x(l,t)$ we get

$$\gamma_n(t) = -\lambda_n \alpha_n(t) + \frac{2}{l}\left(-\frac{1}{2}kt^2\sqrt{\lambda_n}\right)$$

From the PDE, $\alpha_n''(t) = \beta_n(t) = c^2\gamma_n(t) = -\lambda_n c^2 \alpha_n(t) - \frac{k\sqrt{\lambda_n}}{l}c^2 t^2$

which is an ODE. Its initial data are from

$$0 = w(x,0) = \sum_{0}^{\infty} \alpha_n(0) X_n(x) \Rightarrow \alpha_n(0) = 0$$
$$V = w_t(x,0) = \sum_{0}^{\infty} \alpha_n'(0) X_n(x) \Rightarrow \alpha_n'(0) = V d_n \text{ as in previous soln}$$
$$= \frac{2V}{l\sqrt{\lambda_n}}.$$

The ODE is inhomog of the form $\alpha_n'' + \mu_n \alpha_n = \nu_n t^2$, which has a particular soln $\frac{\nu_n}{\mu_n}t^2 - \frac{2\nu_n}{\mu_n^2} = -\frac{k\sqrt{\lambda_n}c^2}{l\lambda_n c^2}t^2 + \frac{2k\sqrt{\lambda_n}c^2}{l\lambda_n^2 c^4} = \frac{-k}{l\sqrt{\lambda_n}}t^2 + \frac{2k}{lc^2\lambda_n^{3/2}}$

The general soln is then $\frac{-kt^2}{l\sqrt{\lambda_n}} + \frac{2k}{lc^2\lambda_n^{3/2}} + \eta_n\cos(\sqrt{\lambda_n}ct) + \zeta_n\sin(\sqrt{\lambda_n}ct)$.

From $\alpha_n(0) = 0$ we deduce $\eta_n = \dfrac{-2k}{lc^2\lambda_n^{3/2}}$

and from $\alpha_n'(0) = \dfrac{2V}{l\sqrt{\lambda_n}}$ we get $\zeta_n\sqrt{\lambda_n}\,c = \dfrac{2V}{l\sqrt{\lambda_n}}$

$$\Rightarrow \zeta_n = \frac{2V}{lc\lambda_n}.$$

Thus $\alpha_n(t) = \dfrac{-kt^2}{l\sqrt{\lambda_n}} + \dfrac{2k}{lc^2\lambda_n^{3/2}}\left(1-\cos(\sqrt{\lambda_n}ct)\right) + \dfrac{2V}{lc\lambda_n}\sin(\sqrt{\lambda_n}ct).$

Substituting λ_n we get at last

$$\begin{aligned} u(x,t) &= kt^2 + w(x,t) \\ &= kt^2 + \sum_0^\infty \alpha_n(t)\, X_n(x) \\ &= kt^2 + \sum_0^\infty \left[\frac{-kt^2}{(n+\frac{1}{2})\pi} + \frac{2kl^2}{\pi^3c^2(n+\frac{1}{2})^3}\left(1-\cos\left(\frac{(n+\frac{1}{2})\pi c}{l}t\right)\right)\right. \\ &\quad \left. + \frac{2Vl}{\pi^2c(n+\frac{1}{2})^2}\sin\left(\frac{(n+\frac{1}{2})\pi c}{l}t\right)\right]\sin\left((n+\tfrac{1}{2})\frac{\pi}{l}x\right) \end{aligned}$$

I think the first version easier, but either way is fine.

5.6.8 Solve $u_t = k u_{xx}$ in $(0,l)$

$u(0,t) = 0$

$u(l,t) = At \qquad A$ const.

$u(x,0) = 0$

We have homog Dir at 0 & inhomog Dir at l, so try expanding in basis $\sin\left(\frac{n\pi}{l}x\right) = X_n(x)$.

One has
$$u(x,t) = \sum_1 a_n(t) X_n(x)$$
$$u_t(x,t) = \sum b_n(t) X_n(x)$$
$$u_{xx}(x,t) = \sum c_n(t) X_n(x).$$

As usual $b_n(t) = a_n'(t)$. For $c_n(t)$ we have

$$c_n(t) = \frac{2}{l}\int_0^l u_{xx}(x,t)\sin\left(\frac{n\pi}{l}x\right)dx$$
$$= \frac{2}{l}\left[\cancel{u_x(x,t)\sin\left(\frac{n\pi}{l}x\right)\Big|_0^l}_{=0} - \int_0^l u_x(x,t)\left(\frac{n\pi}{l}\right)\cos\left(\frac{n\pi}{l}x\right)dx\right]$$
$$= \frac{2}{l}\left[-u(x,t)\left(\frac{n\pi}{l}\right)\cos\left(\frac{n\pi}{l}x\right)\Big|_0^l - \int_0^l u(x,t)\left(\frac{n\pi}{l}\right)^2\sin\left(\frac{n\pi}{l}x\right)dx\right]$$
$$= \frac{2}{l}\left(\frac{n\pi}{l}\right)\left[(-1)^{n+1}At\right] - \left(\frac{n\pi}{l}\right)^2 a_n(t)$$

So from eqn $\quad a_n'(t) = b_n(t) = k c_n(t)$
$$= k\left[\frac{2}{l}\left(\frac{n\pi}{l}\right)(-1)^{n+1}At - \left(\frac{n\pi}{l}\right)^2 a_n(t)\right]$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{\left(\frac{n\pi}{l}\right)^2 kt} a_n(t)\right) = k\frac{2n\pi}{l^2}(-1)^{n+1}At\, e^{\left(\frac{n\pi}{l}\right)^2 kt}$$

$$\Rightarrow \quad e^{\left(\frac{n\pi}{l}\right)^2 kt} a_n(t) - a_n(0) = \frac{2n\pi k}{l^2}(-1)^{n+1}A\int_0^t t\, e^{\left(\frac{n\pi}{l}\right)^2 kt}\,dt$$
$$= \frac{2n\pi k}{l^2}(-1)^{n+1}A\left[\frac{t\,e^{\left(\frac{n\pi}{l}\right)^2 kt}}{\left(\frac{n\pi}{l}\right)^2 k} - \frac{e^{\left(\frac{n\pi}{l}\right)^2 kt}}{\left(\frac{n\pi}{l}\right)^4 k^2} + \frac{1}{\left(\frac{n\pi}{l}\right)^4 k^2}\right]$$

$$\Rightarrow a_n(t) = a_n(0)e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} + \frac{2(-1)^{n+1}At}{n\pi} - \frac{2A\ell^2(-1)^{n+1}}{(n\pi)^3 k} + \frac{2A(-1)^{n+1}\ell^2}{(n\pi)^3 k}e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

$$= \frac{2A(-1)^{n+1}}{(n\pi)^3 k}\left[(n\pi)^2 kt - \ell^2\right] + \left(a_n(0) + \frac{2A\ell^2(-1)^{n+1}}{(n\pi)^3 k}\right)e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}.$$

But $0 = u(x,0) = \sum_{1}^{\infty} a_n(0) X_n(x) \Rightarrow a_n(0) = 0.$

So soln is

$$\cancel{u(x,t) = \sum_{n=1}^{\infty} \frac{2A(-1)^{n+1}}{(n\pi)^3 k}\left[(n\pi)^2 kt - \ell^2\right] + 2A\ell^2(-1)}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2A(-1)^{n+1}}{(n\pi)^3 k}\left[(n\pi)^2 kt - \ell^2 + \ell^2 e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}\right]\sin\left(\frac{n\pi}{\ell}x\right)$$

5.6.10 Have conical rod with area $A(x) = b(1-\frac{x}{l})^2$ $0 \le x \le l$. (b is const.) Have Neum bdy condit on curved bdy, Dir on flat bdy ($x=0$), temp $u(x,t)$ indep of y and z coords and init temp $\phi(x)$. Find $u(x,t)$.

We start by saying that for cross section at x the heat can only escape into cross-sections at $x+dx$ (flow is along x axis) b/c of Neum condit on curved bdy. Then amt of heat in cross sect $[x, x+\delta x]$ is integral of area times temp

$$\int_x^{x+\delta x} u(y,t)(1-\tfrac{y}{l})^2 b\,dy$$

and time change is $\int_x^{x+\delta x} u_t(y,t)(1-\frac{y}{l})^2 b\,dy$. (*)

But only way get change is through surfaces at x and $x+\delta x$, i.e. flux out of cross sect is $\left(\frac{\partial}{\partial x} u(x,t)\right)(1-\frac{x}{l})^2 h \Big|_x^{x+\delta x}$ where h is a const of permeability

$$= \int_x^{x+\delta x} \frac{\partial}{\partial y}\left(u_x(y,t)(1-\tfrac{y}{l})^2 h\right) dy$$

Provided the permeability h is const, equating this to (*) for all δx gives

$$u_t(x,t)(1-\tfrac{x}{l})^2 b = h\frac{\partial}{\partial x}\left(u_x(x,t)(1-\tfrac{x}{l})^2\right)$$

so $u_t(x,t)(1-\frac{x}{l})^2 = k\frac{\partial}{\partial x}\left((1-\frac{x}{l})^2 u_x\right)$ as suggested in book.

Suppose $u(x,t) = X(x)T(t)$ ~~must~~

Following suggestion in book put $X(x) = \frac{v(x)}{1-\frac{x}{l}}$

so that $$u(x,t) = \frac{v(x)\,T(t)}{1-\frac{x}{l}}$$

$$\Rightarrow \quad u_t(x,t) = \frac{v(x)\,T'(t)}{1-\frac{x}{l}}$$

and $$u_x(x,t) = \frac{v'(x)\,T(t)}{1-\frac{x}{l}} + \frac{v(x)\,T(t)}{(1-\frac{x}{l})^2}\left(\frac{1}{l}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\left(1-\frac{x}{l}\right)^2 u_x\right) = \frac{\partial}{\partial x}\left(\left(1-\frac{x}{l}\right) v'(x)T(t) + \frac{v(x)T(t)}{l}\right)$$

$$= -\cancel{\frac{v'(x)T(t)}{l}} + \cancel{\frac{v'(x)T(t)}{l}} + \left(1-\frac{x}{l}\right) v''(x)T(t)$$

And so eqn $u_t(x,t)\left(1-\frac{x}{l}\right)^2 = k\frac{\partial}{\partial x}\left(\left(1-\frac{x}{l}\right)^2 u_x(x,t)\right)$

becomes $$v(x)\,T'(t)\left(1-\frac{x}{l}\right) = k\left(1-\frac{x}{l}\right) v''(x)\,T(t)$$

$$\Rightarrow \quad \frac{v''(x)}{v(x)} = \frac{T'(t)}{T(t)} \quad \text{on } x \in (0,l).$$

We know the soln to this heat eqn, provided we have bdy data. In this case $v''(x) = -\lambda v(x)$

and we have $v(x) = \left(1-\frac{x}{l}\right)X(x)$

so $v(0) = X(0) = 0$ (Dir bdy condit on flat surface)

$v(l) = 0\,X(l) = 0$

So v should be expanded in a Fourier sine series,

giving $v_n(x) = \sin\left(\frac{n\pi}{l}x\right)$

$T_n(x) = e^{-(\frac{n\pi}{l})^2 t}$

and therefore $$u(x,t) = \frac{1}{(1-\frac{x}{l})}\sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 t}\sin\left(\frac{n\pi}{l}x\right).$$

Our initial heat distribution is $\phi(x) = u(x,0)$

$$\Rightarrow \quad \phi(x)\left(1-\frac{x}{l}\right) = \sum_{1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right)$$

and therefore $A_n = \dfrac{2}{l}\displaystyle\int_0^l \phi(x)\left(1-\frac{x}{l}\right) \sin\left(\frac{n\pi}{l} x\right) dx$.