9.9.3

\( f: \mathbb{R} \to \mathbb{R} \) is uniformly continuous and \((x_n)^\infty_{n=1} < x \) is Cauchy.

Show \((f(x_n))^\infty_{n=1}\) is Cauchy.

\( \forall \varepsilon > 0 \exists S > 0 \; \text{s.t.} \; \forall x, y \in S \implies |f(x) - f(y)| < \varepsilon. \)

(i.e. Given \( \varepsilon > 0 \), we can take \( S \) small enough
\( x \) intervals (\(< S \) in size) such that \( f \) varies by
less than \( \varepsilon \) on these intervals).

Now \((x_n)^\infty_{n=1}\) Cauchy means that given
some positive distance \( S > 0 \), the sequence is
eventually \( S \)-stable, i.e.

\[ \exists N \in \mathbb{N} \; \forall k > N \implies |x_k - x_N| < S. \]

I used \( S \) here deliberately, because if \( |x_j - x_k| < S \)
than \( |f(x_j) - f(x_k)| < \varepsilon \), meaning that

Given \( \varepsilon > 0 \) \( \exists S > 0 \; \text{s.t.} \; \forall k > N \)

\[ \implies |f(x_j) - f(x_k)| < \varepsilon \]

\[ \therefore (f(x_j))^\infty_{j=1} \text{ is Cauchy}. \]
9.9.4 \( f: X \rightarrow \mathbb{R} \) unif. cts.
An adherence pt of \( X \)
Show \( \lim_{x \to x_0} f(x) \) exists.

\[ f(x) \] 

**Solu** Start by following the hint to use result of 9.9.3.
So know \( (x_n) \) Cauchy \( \Rightarrow (f(x_n)) \) Cauchy.

We could use this in the following way:
As \( x_0 \) is adh pt of \( X \), \( \exists (x_n), \) c\( \mathbb{R} \) s.t.
\[ \lim_{n \to \infty} x_n = x_0 \] by 9.1.14, \( (x_n) \) c\( \mathbb{R} \) s.t. (to \( x_0 \))
so \( (x_n) \) is Cauchy, so \( (f(x_n)) \) is Cauchy by
Ex 9.9.3, so \( \lim_{n \to \infty} f(x_n) = L \in \mathbb{R} \) if limit exists.

Ex 9.9.3, so \( \lim_{n \to \infty} f(x_n) = L \in \mathbb{R} \) (quadratically).

However we are not done, because the fact
that a seq \( (x_n) \) c\( \mathbb{R} \) has \( (f(x_n)) \) c\( \mathbb{R} \)
doesn't imply \( \lim_{x \to x_0} f(x) \) exists. To overcome this,
we use Prop 9.3.9, which was our way of reducing
limits of functions to limits of sequences. From it we see
that we have to prove:
If \( (y_n) \) is any seq with \( \lim_{n \to \infty} y_n = x_0 \) then
\( \lim_{n \to \infty} f(y_n) = L \). We know this is true for the seq.
(\( x_n \)), \( \Rightarrow \) but must prove it for any \( (y_n) \), c\( \mathbb{R} \)
with \( \lim_{n \to \infty} y_n = x_0 \). However, if \( \lim_{n \to \infty} y_n = x_0 \) then
by 9.9.7, \( (y_n) \), c\( \mathbb{R} \), and \( (y_n) \), are equivalent, so
by 9.9.8, \( (f(y_n)) \), c\( \mathbb{R} \), and \( (f(y_n)) \), are equivalent
Applying 9.9.7 again to \( (f(x_n)) \), and \( f(y_n) \),
we find \( \lim_{n \to \infty} f(x_n) - f(y_n) = 0 \).

And since \( \lim_{n \to \infty} f(x_n) = L \), the limit laws say
\[
\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} f(x_n) - \lim_{n \to \infty} (f(x_n) - f(y_n)) = L.
\]

This shows that any \( (y_n) \), \( y_n \to Y \) with \( \lim_{n \to \infty} f(y_n) = L \) has
\( \lim_{n \to \infty} f(y_n) = L \) so by 9.3.9, \( \lim_{x \to \infty} f(x) = L \).
10.1.2

Let \( x \in \mathbb{R} \), \( x_0 \) a limit pt of \( X \), \( f: X \to \mathbb{R} \) function, and \( L \in \mathbb{R} \).

Show \((a)\) \( f \) diff at \( x_0 \) on \( X \) with \( f'(x_0) = L \)

\((b)\) \( \forall \varepsilon > 0 \exists \delta > 0 \) s.t. \( f(x) - f(x_0) + L(x-x_0) \leq \varepsilon |x-x_0| \)

whenever \( x \in X \) and \( |x-x_0| \leq \delta. \)

Solu \( \)

Sp. \((a)\) is true, so

\[ \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = L. \]

Then given \( \varepsilon > 0 \) \( \exists \delta > 0 \) s.t.

\[ \left| \frac{f(x)-f(x_0)}{x-x_0} - L \right| \leq \varepsilon \quad \text{whenever} \quad x \in X \setminus \{x_0\}, \quad |x-x_0| \leq \delta. \]

So \( \left| f(x) - f(x_0) + L(x-x_0) \right| \leq \varepsilon \quad \text{whenever} \quad x \in X \setminus \{x_0\}, \quad |x-x_0| \leq \delta. \)

But if \( x = x_0 \) then

\[ \left| f(x) - f(x_0) - L(x-x_0) \right| = 0 \leq \varepsilon. \]

So \( \left| f(x) - f(x_0) + L(x-x_0) \right| \leq \varepsilon \quad \text{whenever} \quad x \in X \]

and \( |x-x_0| \leq \delta. \)
(g) Suppose that \( \forall \varepsilon > 0 \exists \delta > 0 \) s.t.

\[
|f(x) - f(x_0) + L(x-x_0)| < \varepsilon |x-x_0| \quad \text{ whenever} \\
\quad \forall x \in \mathbb{R} \quad \text{and} \quad |x-x_0| < \delta.
\]

Then if \( x \neq x_0 \) we have for \( x \in \mathbb{R}\setminus \{x_0\} \), \( |x-x_0| < \delta \),

that

\[
\left| \frac{f(x) - f(x_0)}{x-x_0} - L \right| \leq \varepsilon \quad \text{(\( \# \))}
\]

Since \( \forall \varepsilon > 0 \exists \delta > 0 \) s.t. (\( \# \)) is true, we conclude that

\[
\lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x-x_0} \right) = L
\]

\( \forall x \in \mathbb{R}\setminus \{x_0\} \)
As suggested in the hint, we use the Newton approx from the previous exercise. It helps to record what we know and what we want to determine.

Known by assumption:

(i) \( \forall \epsilon > 0 \ \exists \delta > 0 \text{ st. whenever } y \in Y, |y - y_0| < \delta, \text{ we have } |g(y) - (g(y_0) + g'(y_0)(y - y_0))| \leq \epsilon_1 |y - y_0| \).

(ii) \( \forall \epsilon_2 > 0 \ \exists \delta_2 > 0 \text{ st. when } x \in X, |x - x_0| < \delta_2 \) we have \( |f(x) - (f(x_0) + f'(x_0)(x - x_0))| \leq \epsilon_2 |x - x_0| \).

Want to show that \( g f'(x_0) = g'(y_0) f'(x_0) \), which in terms of Newton approx is that \( \forall \epsilon > 0 \ \exists \delta > 0 \text{ st. } \) whenever \( x \in X, |x - x_0| < \delta \).

\[ |g(f(x)) - (g(f(x_0)) + g'(y_0)f'(x_0)(x - x_0))| \leq \epsilon x \text{ whenever } x \in X, |x - x_0| < \delta \].

Now the question is how to get from (i) and (ii) to (iii). The answer is that the \( f(x_0)(x - x_0) \) on the left of (iii) is close to \( f(x) - f(x_0) \) by (ii), so writing \( y = f(x) \) and using \( y_0 = f(x_0) \) we have

\[ |g(f(x)) - (g(f(x_0)) + g'(y_0)f'(x_0)(x - x_0))| \]

\[ = |g(f(x)) - (g(f(x_0)) + g'(y_0)f'(x_0)(x - x_0))| \]

\[ = |(g(f(x)) - g(y_0)) + g'(y_0)(y - y_0)) + (g(y_0) - f(x_0)(x - x_0))| \]

\[ \leq |g(f(x)) - g(y_0) + g'(y_0)(y - y_0)| + |g'(y_0)| |f(x_0) - f(x_0) + f'(x_0)(x - x_0)| \]

\[ \leq \epsilon_1 |y - y_0| + \epsilon_2 |x - x_0| \text{ whenever } x \in X, y \in Y, |x - x_0| < \delta_2 \text{ and } |y - y_0| < \delta_1. \]
So now we need to know how to use the assumption 
\[ |x-x_0| \leq \delta \] for (iii) to conclude that
\[ \varepsilon_1 |y-y_0| + \varepsilon_2 |x-x_0| |g'(x_0)| |x-x_0| \leq \varepsilon. \]
and that \(|y-y_0| \leq \delta_1\), so that (i) is applicable.

Both things are about making \(|y-y_0|\) small when \(|x-x_0|\) is small. This is possible by (ii) and the triangle inequality, which implies
\[ |y-y_0| \leq |f'(x_0)||x-x_0| + \varepsilon_2 |x-x_0| \]
whenever \(x \in Y\) and \(|x-x_0| \leq \delta_2\).

in particular we find that \(|x-x_0| \leq \delta_2\)
\[ \Rightarrow |y-y_0| \leq (|f'(x_0)| + \varepsilon_2) \delta_2. \]

So if \((|f'(x_0)| + \varepsilon_2) \delta_2 \leq \delta_1\), then \(|y-y_0| \leq \delta_1\), and (i) can be used. (iv)

and we find that when (i) and (iii) are valid
then from the working on the previous page,
\[ |g(\delta_1) - g(\delta_1)| \leq |\delta_1 f'(x_0)| + \varepsilon_2 |x-x_0| |g'(x_0)| |x-x_0| \]
\[ \leq \varepsilon_1 |f'(x_0)| |x-x_0| + \varepsilon_2 |x-x_0| + \varepsilon_2 |g'(x_0)| |x-x_0|. \]

Now suppose we are given \(\varepsilon > 0\).

- Take \(\varepsilon_1 > 0\) such that \(\varepsilon_1 |f'(x_0)| < \frac{\varepsilon}{3}\).
- Take \(\delta_1\) such that (i) is valid for \(|y-y_0| < \delta_1\)
- Take \(\varepsilon_2\) such that \(\varepsilon_1 \leq \frac{\delta_1}{2}\) and \(\varepsilon_2 |g'(x_0)| < \frac{\varepsilon}{3}\)
Then take \( S_2 \) such that (ii) is valid.

Lastly, take \( S = \min \left( S_2, \frac{S_1}{|f'(x_0)| + \varepsilon_2} \right) \)

So that by (i) \( |x-x_0| < S \Rightarrow |f(x) - f(x_0)| < \varepsilon_1 \), i.e. \( |y-y_0| < \varepsilon_1 \).

So (i) is valid when \( |x-x_0| < S_2 \).

Combining these we see that
\[ \forall \varepsilon_1 \text{ s.t. } \varepsilon_1 > 0 \text{ and } \forall \varepsilon_2 \text{ s.t. } \varepsilon_2 > 0 \text{ and } \]
\[ |x-x_0| < S \Rightarrow (y-y_0) \in Y \text{ and } |y-y_0| < \varepsilon_1 \text{ and } |y-y_0| < \varepsilon_2, \]

and in combination they gave
\[
\begin{align*}
|g(f(x)) - (g(f(x_0)) + g'(y_0) f'(x_0) (x-x_0))| \\
\leq \varepsilon_1 |f'(x_0)| |x-x_0| + \varepsilon_2 |x-x_0| + \varepsilon_2 |g'(x_0)| |x-x_0| \\
\leq \left( \frac{\varepsilon_1}{3} + \frac{\varepsilon_2}{3} + \frac{\varepsilon_2}{3} \right) |x-x_0| = \varepsilon_1 |x-x_0|
\end{align*}
\]

by choice of \( \varepsilon_1 \) and \( \varepsilon_2 \).

This completes the proof.
10.2.1 Let \( a < b \) and \( f : (a, b) \to \mathbb{R} \) be
such that
\[ x_0 \in (a, b), \quad f'(x_0) \text{ exists and } f \text{ has a local max or a local min at } x_0. \]
Show \( f'(x_0) = 0 \).

**Solution**

By def\( \frac{f(x) - f(x_0)}{x-x_0} \)

Suppose \( f \) has a local max at \( x_0 \).

If \( x \in (x_0, b) \) then \( f(x) \leq f(x_0) \)
\[ \Rightarrow \frac{f(x) - f(x_0)}{x-x_0} \leq 0 \]
\[ \Rightarrow f'(x_0) \leq 0. \]

If \( x \in (a, x_0) \) then \( f(x) \leq f(x_0) \)
\[ \Rightarrow \frac{f(x) - f(x_0)}{x-x_0} \geq 0 \]
\[ \Rightarrow f'(x_0) \geq 0. \]

Hence \( f'(x_0) = 0 \)

10.2.2 Let \( f(x) = 1 - |x| \) on \((-1, 1)\). Then \( f \) is \( C^1 \) (example in book)
and \( f \) not diff at 0 (example in book)
but \( f \) has a global max at 0 (else \( f(0) = 1 \) and \( f(x) \leq f(0) \) for \( x \neq 0 \) \( \Rightarrow f'(x) < 0 \) for \( x \neq 0 \) \( \Rightarrow f'(0) \text{ exists and in this case the assumption is false as } f'(x) \text{ exists} \))
10.2.6  Let \( m > 0 \). \( f : [a,b] \to \mathbb{R} \) be on \([a,b]\)
diff on \((a,b)\)
has \( |f'(x)| \leq M \) on \((a,b)\).

Show if \( x \leq y \) have \( |f(x) - f(y)| \leq M |x-y| \).

**Soln**

Take \( x \leq y \).

No loss of generality in assuming \( x < y \).

If \( x = y \) then \( |f(x) - f(y)| = 0 = M |x-y| \).

Else \( x < y \). Applying MVT to \( f \) on \([x,y]\) have that \exists z \in (x,y) \text{ s.t.} 

\[
\frac{f(x) - f(y)}{x-y} = f'(z),
\]

\[
\Rightarrow |f(x) - f(y)| = |f'(z)| \leq M |x-y|
\]

\[
\Rightarrow |f(x) - f(y)| \leq M |x-y|.
\]

10.3.5 Consider the following set

\[
X = (0,1) \cup (2,3) \subset \mathbb{R}
\]

and \( f(x) = \begin{cases} 
    x & x \in (0,1) \\
    x-2 & x \in (2,3)
\end{cases}.
\]

Then by 10.1.13 (a), (b) and (c), \( f(x) = \begin{cases} 
    x & x \in (0,1) \\
    1 & x \in (2,3)
\end{cases} \)

so \( f'(x) > 0 \) on \( X \). But \( f \) is not strictly incr b/c

\[
f\left(\frac{5}{2}\right) = \frac{5}{2} \neq f\left(\frac{5}{2}\right).
\]