9.1.3  
(a) \( \overline{\mathbb{N}} = \mathbb{N} \), and (b) \( \overline{\mathbb{Z}} = \mathbb{Z} \)

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Any set is contained in its closure, so \( \mathbb{N} \subseteq \overline{\mathbb{N}} \) and \( \mathbb{Z} \subseteq \overline{\mathbb{Z}} \). (see 9.1.1)

To prove that \( \overline{\mathbb{N}} = \mathbb{N} \) and \( \overline{\mathbb{Z}} = \mathbb{Z} \), we use the contrapositive, so we prove

\[ x \notin \mathbb{N} \implies x \notin \overline{\mathbb{N}} \]
\[ x \notin \mathbb{Z} \implies x \notin \overline{\mathbb{Z}} \]

In both cases the idea is that if \( x \notin \mathbb{N} \) (or \( \mathbb{Z} \)) then \( x \) has some positive distance from \( \mathbb{N} \), and setting \( \varepsilon = \frac{1}{2} \) of this distance we get that \( x \) is not \( \varepsilon \)-adherent to \( \mathbb{N} \) (respectively \( \mathbb{Z} \)). The main issue is to determine the distance from \( x \) to \( \mathbb{N} \) (or \( \mathbb{Z} \)).

One easy way is to use Ex 5.4.3., which says that if \( x \notin \mathbb{Z} \) then there is \( y \in \mathbb{Z} \) s.t. \( y < x < y + 1 \).

Then if \( m \in \mathbb{Z} \) we have either

\[ m - y + 1 \leq x \leq m - y \]  
or \( m \leq y \) so \( x - m \leq n - y \) these are positive.

hence, \( |x - m| \geq \min (|x - y|, |y + 1 - x|) \) \( \forall x \in \mathbb{Z} \).

Intuitively this expresses that the distance from \( x \) to \( \mathbb{Z} \) is \( \varepsilon = \frac{1}{2} \min (|x - y|, |y + 1 - x|) \).

Following the above idea we set \( \varepsilon = \frac{1}{2} \min (|x - y|, |y + 1 - x|) \) so

\[ |m - x| > \varepsilon \quad \forall m \in \mathbb{Z} \]

\[ \implies x \text{ is not } \varepsilon \text{-adherent to } \mathbb{Z} \]
\[ \Rightarrow x \notin \overline{\mathbb{Z}} \]

Hence \( \overline{\mathbb{Z}} \leq \mathbb{Z} \), and so \( \overline{\mathbb{Z}} = \overline{\mathbb{Z}} \).

We could prove \( \overline{\mathbb{N}} \leq \mathbb{N} \) in essentially the same way, but at this point there is an alternative, namely, that \( \overline{\mathbb{N}} \leq \mathbb{Z} \). Knowing this, we need only check that if \( m \in \mathbb{Z} \), \( m < 0 \) then \( m \notin \overline{\mathbb{N}} \), but if \( m \in \mathbb{Z} \), \( m < 0 \) and \( \varepsilon = \frac{1}{2} \) then for any \( n \in \mathbb{N} \)

\[ n - m > n + 1 > 1 > \frac{1}{2} \]

because \( m \leq -1 \)

\[ \Rightarrow |n-m| > \frac{1}{2} \quad \forall \ n \in \mathbb{N} \]

\[ \Rightarrow m \text{ is not } \frac{1}{2} \text{-adherent to } \mathbb{N} \]

\[ \Rightarrow m \notin \overline{\mathbb{N}} \]

So \( \overline{\mathbb{N}} \leq \mathbb{Z} \) and does not contain any negative values, so \( \overline{\mathbb{N}} \leq \mathbb{N} \)

and thus \( \overline{\mathbb{N}} = \mathbb{N} \).

(like the picture on a cereal packet, this proof was enlarged to show detail - specifically the reasoning behind the choice of \( \varepsilon \) etc. You do not need to include this detail in your own proofs.)
(c) \( \overline{A} = \mathbb{R} \).

By our def of adherent pt, any adherent pt is in \( \mathbb{R} \), so the closure of any set is in \( \mathbb{R} \), hence \( \overline{A} \subseteq \mathbb{R} \).

Now we wish to show \( x \in \mathbb{R} \Rightarrow x \in \overline{A} \), so \( \mathbb{R} \subseteq \overline{A} \).

If \( x \in \mathbb{R} \) and \( \varepsilon > 0 \) then the interval \( (x-\varepsilon, x+\varepsilon) \) contains some \( y \in \overline{A} \) by 5.4.14. Thus \( x \) is \( \varepsilon \)-adherent to \( \overline{A} \).

Since this is true for any \( \varepsilon > 0 \) (with \( y \) depending on \( \varepsilon \)) we have \( x \in \overline{A} \).

(d) \( \overline{\mathbb{R}} = \mathbb{R} \).

As in (c), the closure of any set is in \( \mathbb{R} \) so \( \overline{\mathbb{R}} \subseteq \mathbb{R} \).

Then by 9.1.11, \( \mathbb{R} \subseteq \overline{\mathbb{R}} \), so \( \overline{\mathbb{R}} = \mathbb{R} \).

(e) \( \overline{\emptyset} = \emptyset \).

Proof: Suppose \( x \in \overline{\emptyset} \). Then for any \( \varepsilon > 0 \), \( x \) is \( \varepsilon \)-adherent to \( \emptyset \), so there is \( y \in \emptyset \) with \( |x-y| \leq \varepsilon \). This is a contradiction because there is no \( y \in \emptyset \) at all. Hence there is no \( x \in \overline{\emptyset} \), so \( \overline{\emptyset} = \emptyset \).
The union of a finite collection of bounded sets is bounded.

When confronted with a problem like this it is often a good idea to think first of a simple case and see how the argument works there. One simplification we can do here is to look at just a small and finite number.

E.g.: If the collection contains no sets, then its union is empty and φ is odd.

If the collection contains one odd set, then its union is the same set, and this set is odd.

(So far very easy)

If the collection is two odd sets A₁ and A₂

with \( |x| \leq M₁ \) \& \( x \in A₁ \)
\( |x| \leq M₂ \) \& \( x \in A₂ \)

then \( x \in A₁ \cup A₂ \) \( \Rightarrow \) \( x \in A₁ \), or \( x \in A₂ \)
\( \Rightarrow \) \( |x| \leq M₁ \) or \( |x| \leq M₂ \)
\( \Rightarrow \) \( |x| \leq \max (M₁, M₂) \).

At this point it looks like we have an induction available, with the inductive argument being a simple modification of the two set case.

Statement \( P(n) \) is “A union of \( n \) bounded sets is bounded”.

We have shown above that \( P(0) \), \( P(1) \) and \( P(2) \) are true, so can take \( P(0) \) as base case.

For the inductive step, suppose \( P(n) \) is true. Consider a union \( \bigcup_{j=1}^{n+1} A_j \) of \( (n+1) \) bounded sets.
Then \[ A = \bigcup_{j=1}^{n} A_j \cup A_{M_1} = B \cup A_{M_1} \text{ with } B = \bigcup_{j=1}^{n} A_j \]

Since \( P(n) \) is true, \( B \) is odd, so \( \exists M_1 \),

\[ 1x1 \leq M_1, \text{ if } x \in B \]

also \( A_{M_1} \) is odd so \( \exists M_2 \),

\[ 1x1 \leq M_2, \text{ if } x \in A_{M_1} \].

Now if \( x \in A \) then \( x \in B \) or \( x \in A_{n+1} \)

\[ \Rightarrow 1x1 \leq M_1 \text{ or } 1x1 \leq M_2 \]

\[ \Rightarrow 1x1 \leq \max(M_1, M_2) \]

So \( \bigcup_{j=1}^{n} A_j \) is odd by \( \max(M_1, M_2) \)

\[ \Rightarrow P(n+1) \text{ is true.} \]

and so the result follows by induction.

It is not true that an infinite collection of odd sets has odd union. For example let \( A_j = [-j, j] \),

so \( A_j \) is odd by \( j \), but \( \bigcup_{j=1}^{\infty} A_j = \mathbb{R} \) which is not odd.
\[ 9.1.13 \]

\[ X \text{ is closed if and only if } \forall (a_n)_0 \subset X \exists \text{ subseq } (a_{n_j})_{j=0}^{\infty} \text{ s.t. } \]
\[ x = \lim_{j \to \infty} a_{n_j} \in X. \]

**Proof:**

If \( X \) is closed and \( (a_n)_0 \subset X \), then since \( X \) is closed, \( X \) contains all convergent subsequences of \( (a_n)_0 \), which converges to some \( x = \lim_{j \to \infty} a_{n_j} \) by 6.6.8. Then since \( X \) is closed, by 9.1.17 \( x \in X \). This proves \( \Rightarrow \).

We prove the converse by contrapositive. So we show:

- If it is not the case that \( X \) is closed and \( (a_n)_0 \subset X \),

then there is a seq \( (a_n)_0 \subset X \) which does not have any subseq \( (a_{n_j})_{j=0}^{\infty} \) such that \( a_{n_j} \to x \in X \).

This is a little tricky, but we have to negate both of the compound statements:

- \( (i) \) \( X \) is closed and \( (a_n)_0 \subset X \)

**Negation:** Either \( X \) is not closed or \( (a_n)_0 \subset X \) is not \( X \).

and \( (ii) \) \( \forall (a_n)_0 \subset X \exists \text{ subseq s.t. } a_{n_j} \to x \in X \)

**Negation:** \( \exists (a_n)_0 \subset X \) \( \forall \text{ subseq } (a_{n_j})_{j=0}^{\infty} \)

either \( a_{n_j} \) doesn't converge or \( \lim_{j \to \infty} a_{n_j} \notin X \).
Conveniently, the fact that we know $\text{closures}$ is connected to the existence of sets $\text{that cage}$, and closed is connected to the limits being in $X$, gives some idea of how to build the desired $(a_n)_{n=0}^\infty \subset X$. If $X$ is not closed we should make a seq that doesn't cage, while if $X$ is not closed we should make a seq that cages to a pt in $X$ that is not in $X$.

Following this intuition:

Sp. $X$ not closed. Then $\forall n \in \mathbb{N} \exists a_n \in X$ with 
\[ |a_n| > n \quad (\text{else it would be a cd for } X) \]

So taking such $a_n$ for each $n$ gives a seq $(a_n)_{n=0}^\infty \subset X$.

If $(a_{n_j})_{j=0}^\infty$ is any subseq of $(a_n)_{n=0}^\infty$ then $|a_{n_j}| > n_j \quad \forall j$, so $(a_{n_j})_{j=0}^\infty$ is not $\text{closed}$. Since $\text{caged seqs}$ are closed, $(a_{n_j})_{j=0}^\infty$ is not $\text{caged}$.

Thus we have exhibited a seq with no $\text{caged}$ subseq.

Sp. $X$ not closed. Let $x \in X \setminus X$.

Since $x \in X$ there is a seq $(a_n)_{n=0}^\infty \subset X$ with 
\[ \lim_{n \to \infty} a_n = x. \quad (9.1.17) \]
But every subseq \((a_n)_{j=0}^\infty\) converges to \(x\) as well (by 6.6.5).

So we have exhibited a seq \((a_n)_{j=0}^\infty \subset X\) for which there is no subseq converging to a pt in \(X\). \(\{x \notin X\}\).

We have thus shown that

\[
\neg (X \text{ closed } \& \text{ bdd}) \Rightarrow \neg (\forall (a_n)_{j=0}^\infty \subset X \exists \text{ subseq } (a_n)_{j=0}^\infty \text{ s.t. } a_n \to x \in X).
\]

and thus \(X\) closed \& bdd

\[
\Rightarrow (\forall (a_n)_{j=0}^\infty \subset X \exists \text{ subseq } (a_n)_{j=0}^\infty \text{ s.t. } a_n \to x \in X).
\]
(a) \((f+g)oh = foh + goh\)

Observe that the domain of \((f+g)oh\) and of \(foh\) and \(goh\) is \(\mathbb{R}\).

Also, for \(x \in \mathbb{R}\),
\[
(f+g)oh(x) = (f+g)(h(x)) = f(h(x)) + g(h(x)) = foh(x) + goh(x).
\]

So \((f+g)oh\) and \(foh + goh\) agree on their domains, so are equal.

(b) It is not generally the case that
\[
fo(g+g) = fog + foh.
\]

The reason is that \(f\) sums \(\sum f\) (values) in general - this only happens for linear functions. Accordingly we can take the simplest non-linear function we know, \(f(x) = x\), and put in more or less anything. E.g. \(g(x) = 0\), \(h(x) = 0\) (These are all constant functions).

Then
\[
\begin{align*}
fo(g+h)(x) &= f(g(x) + h(x)) = f(0) = 1 \\
foh(x) &= f(0) = 1
\end{align*}
\]

\(\Rightarrow fog(x) + fo(h(x) = 2 \forall x\)

But
\[
fo(g+h)(x) = f'(g(x) + h(x)) = f'(0) = f(0) = 1 \forall x
\]

So \(fo(g+h) \neq fog + foh\)
(c) \((f+g) \cdot h = f \cdot h + g \cdot h\)

\[
(f+g) \cdot h(x) = (f(x)+g(x))h(x) = f(x)h(x) + g(x)h(x) = f \cdot h(x) + g \cdot h(x)
\]

(a) \(f \circ (g \cdot h) = f \circ g + f \circ h\)

\[
f \circ (g \cdot h)(x) = f(g(x)) \cdot (g(x) + h(x)) = f(g(x)) \cdot g(x) + f(g(x)) \cdot h(x) = f \circ g(x) + f \circ h(x)
\]

9.3.1

Let \(f : X \rightarrow R\) be a function

\(E \subseteq X, x_0 \in E, L \in \mathbb{R}\).

Show \(\lim_{x \to x_0} f(x) = L \iff \forall (a_n)_{n \in \mathbb{N}} \subseteq E \text{ with } n \to \infty, a_n \to x_0 \implies \lim_{n \to \infty} f(a_n) = L\).

It is useful to use the fact that

\(\lim_{x \to x_0} f(x) = L\) if \(\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \forall x \in E, 0 < |x-x_0| < \delta \implies |f(x) - L| < \varepsilon\).

Suppose first that \(\lim_{x \to x_0} f(x) = L\), and that
\((a_n)_{n=0}^{\infty} \subseteq E \text{ is a seq with } \lim a_n = \xi_0.\)

Then \(N \in \mathbb{N} \exists \delta > 0 \Rightarrow (|x-x_0| < \delta \Rightarrow (f(x) - L) < \varepsilon \).

And given \(\varepsilon > 0 \exists N \in \mathbb{N}, \ n > N \Rightarrow |a_n - \xi_0| < \delta, \)

\(\therefore \text{ from which } \ |f(a_n) - L| < \varepsilon, \)

Hence \(N \in \mathbb{N} \exists \delta > 0 \Rightarrow (|x-x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon \).

which verifies that \(\lim f(a_n) = L.\)

For the converse it is helpful to use the contrapositive.

Suppose \(\lim_{x \to x_0} f(x) \) does not exist or is not \(L, \) \(\forall x \in E \)

then \(\forall \varepsilon > 0 \exists \delta > 0 \forall x \in E \text{ with } |x-x_0| < \delta \text{ but } |f(x) - L| \geq \varepsilon. \)

In particular this is true for each \(\varepsilon = \frac{1}{n}, \) so for each \(n \) there is \(x_n \) with \(|x_n-x_0| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon. \)

Taking such \(x_n \) for each \(n \in \mathbb{N} \) gives a seq \(\{x_n\}_{n=0}^{\infty} \subseteq E \)

Since \(|x_n-x_0| < \frac{1}{n} \text{ the squeeze test } \Rightarrow \lim_{n \to \infty} x_n = \xi_0. \)
However \( (f(a_n)) \) is never \( \varepsilon \) close to \( L \) because

\[
|f(a_n) - L| \geq \varepsilon \Rightarrow \varepsilon > \frac{\varepsilon}{2} \quad \forall \ n.
\]

So it is not true that \( \lim_{n \to \infty} f(a_n) = L \)

(it could be that the limit exists \( \neq L \) or that it does not exist - we cannot determine this from the construction).

We have shown the implication is true, so the proof is complete.