12.4 Prove $\frac{1}{n^2+1} \to 0$ as $n \to \infty$.

Solution Given $\epsilon > 0$ take $N > \frac{1}{\sqrt{\epsilon}}$. If $n \geq N$ then $n^2 + 1 \geq N^2 + 1 \geq \frac{1}{\epsilon}$ so $\left|\frac{1}{n^2+1}\right| < \epsilon$.

12.6 Prove $\frac{n+2}{2n+3} \to \frac{1}{2}$ as $n \to \infty$.

Solution Given $\epsilon > 0$ let $N > \frac{1}{4\epsilon}$. If $n \geq N$ then

$$\left|\frac{n+2}{2n+3} - \frac{1}{2}\right| = \left|\frac{2n+4-2n-3}{4n+6}\right| = \left|\frac{1}{4n+6}\right| < \frac{1}{4n} < \frac{1}{4N} < \epsilon.$$

12.7 Negate the definition of $\lim_{n\to\infty} a_n = L$ to get a definition of $\lim_{n\to\infty} a_n \neq L$. Then write a definition of $\{a_n\}$ diverges.

Solution The definition of $\lim_{n\to\infty} a_n = L$ is $\forall\epsilon > 0 \exists\delta > 0, 0 < |x-a| < \delta \implies |f(x)-L| < \epsilon$. Negating step by step we invert the quantifiers (including the tacit $\forall x$ that goes with $0 < |x-a| < \delta$) and finally negate the $A \implies B$ as A and NOT B, to obtain $\exists\epsilon > 0 \forall\delta > 0 \exists x, 0 < |x-a| < \delta$ AND $|f(x)-L| \geq \epsilon$.

The defintion of $\{a_n\}$ converges is just $\exists L$, $\lim_{n\to\infty} a_n = L$, so its negation is $\forall L$, $\lim_{n\to\infty} a_n \neq L$. Combining it with the above we get $\forall L \exists\epsilon > 0 \forall\delta > 0 \exists x, 0 < |x-a| < \delta$ AND $|f(x)-L| \geq \epsilon$.

12.9 Show $\frac{n^5+2n}{n^2} \to \infty$ as $n \to \infty$.

Solution Given M take $N > M$. If $n \geq N$ then $\frac{n^5+2n}{n^2} > \frac{n^5}{n^2} = n^3 \geq n \geq N > M$.

12.10 Prove $\sum_{j=1}^{n} \frac{1}{j} < 2\sqrt{n}$ for every $n \in \mathbb{N}$.

Solution Use induction. It is true for $n = 1$ because the sum is 1 and $1 < 2$; this is our base case. If it is true for n then we have

$$\sum_{1}^{n+1} \frac{1}{j} = \frac{1}{n+1} + \sum_{1}^{n} \frac{1}{j} < \frac{1}{n+1} + 2\sqrt{n},$$

but if we square the right side and use $\frac{1}{(n+1)^2} \leq \frac{1}{n+1}$ and $\sqrt{n} \leq n$ then we get

$$\frac{1}{(n+1)^2} + \frac{4\sqrt{n}}{n+1} + 4n \leq \frac{1}{n+1} + \frac{4n}{n+1} + 4n = \frac{4n+1}{n+1} + 4n < 4 + 4n = 4(n+1),$$

so taking the square root give $\frac{1}{n+1} + 2\sqrt{n} < 2\sqrt{n+1}$, completing the inductive step and proving the result by induction.

12.11 Prove that if $s_n \to L$ then $s_{n^2} \to L$.

Solution Given $\epsilon > 0$ use $s_n \to L$ to obtain N_1 so $n \geq N_1$ implies $|s_n - L| < \epsilon$. Let $N > \sqrt{N_1}$. If $n \geq N$ then $n^2 \geq N^2 > N_1$ so $|s_{n^2} - L| < \epsilon$.

12.12 Prove that the series $\sum_1^\infty \frac{1}{(3k-2)(3k+1)}$ converges and determine its sum.

Solution One may compute the partial sums for $n = 1, 2, 3, 4$ to be $\frac{1}{4}$, $\frac{2}{7}$, $\frac{3}{10}$, $\frac{4}{13}$, which suggests the partial sum $s_n = \frac{n}{3n+1}$. We have a base case $n = 1$ for induction, and if the formula is valid for n then

$$\begin{aligned} s_{n+1} &= s_n + \frac{1}{(3(n+1)-2)(3(n+1)+1)} = \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} \\ &= \frac{n(3n+4)+1}{(3n+1)(3n+4)} = \frac{(n+1)(3n+1)}{(3n+1)(3(n+1)+1)} = \frac{n+1}{3(n+1)+1} \end{aligned}$$

which verifies the induction. Finally $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{3+(1/n)} = \frac{1}{3}$.

12.14 The terms a_n are given by $a_1 = \frac{1}{6}$ and $a_n = a_{n-1} - \frac{2}{n(n+1)(n+2)}$. Prove $\sum_1^\infty a_k$ converges and determine the sum.

Solution There is a difference between finding an argument that works and giving a proof. I remark that the fact $a_n - a_{n-1}$ is given means one can write $a_n - a_1 = \sum_{j=2}^n (a_j - a_{j-1})$ and thus compute a formula for a_j. Once you have it you can simply verify by induction that $a_k = \frac{1}{k+1} - \frac{1}{k+2}$, because for $k = 1$ this gives $a_1 = \frac{1}{6}$, so we have a base case, and if the formula is true for $k = n-1$ then

$$\begin{aligned} a_n &= a_{n-1} - \frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{1}{n+1} - \frac{2}{n(n+1)(n+2)} \\ &= \frac{(n+1)(n+2) - n(n+2) - 2}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \end{aligned}$$

which verifies the induction. Finally,

$$\sum_{k=1}^n a_k = \sum_{k=1}^n \frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{2} - \frac{1}{n+2} \to \frac{1}{2}$$

as $n \to \infty$.

12.15 Prove the series $\sum_1^\infty \frac{k+3}{(k+1)^2}$ diverges to ∞.

Solution In essence we show it is bigger than the harmonic series, but we do so for partial sums. Observe $\sum_1^n \frac{k+3}{(k+1)^2} > \sum_1^n \frac{1}{k+1} = \sum_2^{n+1} \frac{1}{k}$, where we just used the inequality for each term in the (finite sum). Now we proved in class that $\sum_1^\infty \frac{1}{k}$ diverges to ∞, so given M there is N so $n > N$ implies $\sum_1^n \frac{1}{k} > (M+1)$. Then $n \geq N$ also implies

$$\sum_1^n \frac{k+3}{(k+1)^2} > \sum_2^{n+1} \frac{1}{k} > \Big(\sum_1^{N+1} \frac{1}{k}\Big) - 1 > M + 1 - 1 = M$$

which shows the result.

12.18 Give an ϵ–δ proof that $\lim_{x\to 2} \frac{3}{2}x + 1 = 4$.

Solution Given $\epsilon > 0$ let $\delta = \frac{2}{3}\epsilon$. Then $0 < |x-2| < \delta$ implies $|\frac{3}{2}x + 1 - 4| = \frac{3}{2}|x-2| < \frac{3}{2}\delta = \epsilon$.

12.20 Give an ϵ–δ proof that $\lim_{x\to 2} 2x^2 - x - 5 = 1$.

Solution Given $\epsilon > 0$ let $\delta = \min\{1, \epsilon/9\}$. If $0 < |x-2| < \delta$ then $|x-2| < 1$ so $|2x+3| \le 2|x|+3 < 9$ and therefore $|2x^2-x-5-1| = |2x+3||x-2| < 9\delta \le \epsilon$.

12.26 For

$$f(x) = \begin{cases} 1 & x < 3 \\ 1.5 & x = 3 \\ 2 & x > 3 \end{cases}$$

determine (with proof) whether $\lim_{x\to 3} f(x)$ exists and whether $\lim_{x\to\pi} f(x)$ exists.

Solution Suppose $\lim_{x\to 3} f(x)$ exists and is equal to L. Then with $\epsilon = \frac{1}{2}$ we can take $\delta > 0$ so $0 < |x-3| < \delta$ implies $|f(x) - L| < \epsilon = \frac{1}{2}$. Now $3 \pm \frac{\delta}{2}$ are points in this interval and $f(3 \pm \delta/2) = 1.5 \pm .5$. But then both $|1-L| < \epsilon$ and $|2-L| < \epsilon$, producing $1 = 2 - 1 \le |2-L| + |1-L| < \epsilon + \epsilon = 1$. This contradiction ($1 < 1$) implies the limit $\lim_{x\to 3} f(x)$ does not exist.

We can also prove $\lim_{x\to\pi} f(x) = 2$, because if $\epsilon > 0$ is given we may take $\delta = \pi - 3 > 0$ and observe that $0 < |x-\pi| < \delta$ implies $x > \pi - \delta = 3$ so $f(x) = 2$ and thus $|f(x) - 2| = 0 < \epsilon$.