Math 2144 Midterm: Things you should definitely know.

Note: This is not an exhaustive list of topics we covered in class, but if you have a solid grasp of these topics and can do problems with them you should be fine on the midterm.

1. Surface integrals. Sections 12.1 through 12.4 introduce several concepts that are important for dealing with surfaces. From this, you need to be able to do the following:

   i) Parameterize a surface as $r(u, v) = (X(u, v), Y(u, v), Z(u, v))$ for $(u, v) \in T$ a region in $\mathbb{R}^2$. This might involve using cylindrical polar or spherical polar coordinates, or using the fact that the surface is a graph, or some basic geometry of surfaces (like planes, cones, cylinders, spheres). Bear in mind that $r(u, v)$ has to be a function.
   ii) Compute the fundamental vector product $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$.
   iii) Understand the geometric properties of this vector product, that it is normal to the surface and that its magnitude is the area scaling factor relating the area of a small parallelogram on the surface to that of a corresponding square in T.

2. Compute the area of a surface by parameterizing and integrating the magnitude of the fundamental vector product (equation (12.7) page 424). This could involve directly parameterizing a simple geometric object or a graph of a function. You may find it useful to know the formula (12.9), but if you understand (12.7) you can get (12.9) in a few lines.

3. Compute surface integrals $\iint f\, dS$ and flux integrals $\iint F \cdot n\, dS$. (Section 12.7)

4. All you really need to know from Theorems 12.1 and 12.2 (page 433-434) is that the value of a surface integral does not depend on how it is parameterized. Of course for a flux integral the sign can depend on the direction of the normal vector. (There are other things one can do with Theorem 12.1 but we did not cover them.)

5. Section 12.9. The important thing here is the wedge notation, e.g. $dx \wedge dy$. You should know that whenever it occurs there is a parametrization of the surface lurking in the background. Since these come from flux integrals and only the sign of the flux integral can change with the parametrization, all that matters is what direction (there are only two that are possible) the fundamental vector product of the parametrization has. With the correct parametrization $r(u, v) = (X(u, v), Y(u, v), Z(u, v))$, meaning the one with the correct fundamental vector product direction, one has

   $$dx \wedge dy = \frac{\partial(X, Y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

   and similarly for $dy \wedge dz$ (replace X with Y and Y with Z) etc. Note that $dx \wedge dy = -dy \wedge dx$. Knowing the above lets you convert any wedge expression into an integral you can compute.

6. Stokes theorem (page 438). Know and be able to apply Stokes theorem. I suppose this has to be memorized, and I recommend you learn the version $\iint_S (\nabla \times F) \cdot n \, dS = \int_C F \cdot d\alpha$. Here you must be careful about the orientation of the curve C which is the boundary of S. It should be related to the normal n in the surface integral by the right hand rule (equivalently the rule about walking around the boundary with head in normal direction and region on left side). Note that if you start out with a flux integral through a surface then this theorem is only usable if the vector field is a curl. Not all vector fields are curls (this occurs iff the field has zero divergence on a suitable domain) so you can't apply Stokes theorem to every problem of the form $\iint G \cdot n \, dS$.

7. A vector field F is a gradient on an open convex set iff $\nabla \times F = 0$ (Theorem 12.4 page 442). (Note that the set is important as well as the condition on F.)

8. Sections 12.12 and 12.14 have an introduction to the notation in which $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ is treated as if it were a vector. The highlights are that $\nabla \cdot (\nabla f) = \nabla^2 f$ $\nabla \cdot (\nabla \times F) = 0$ and the product rules $\nabla \cdot (fF) = \nabla f \cdot F + f \nabla \cdot F$ and $\nabla \times (fF) = \nabla f \times F + f(\nabla \times F)$.

9. From Section 12.16, the theorem that given F on a rectangular region there is G with $\nabla \times G = F$ if and only if $\nabla . F = 0$ in the region, and that one can get G by assuming $G = (0, M, N)$ and solving for M and N using partial integration. Then all solutions are of the form $G + \nabla \phi$ for scalar functions ϕ.

10. From Section 12.18, Stoke's theorem for surfaces with holes (the relevant bit is the orientation of the holes) and surfaces that cannot be given using a single one-to-one smooth parametrization but must be parametrized in pieces. The key to the latter is that the curves where the pieces join must be in opposite directions so the path integrals on the boundaries cancel.

11. The divergence theorem 12.6 page 457. $\iiint_V \nabla \cdot F \, dV = \iint_S F \cdot n \, dS$ where V is the region bounded by the surface S, and S is orientable with outward orientation. The latter means that there is a way to choose a normal direction to the surface consistently everywhere (this is the same as there being a consistent inside and outside of the region), and that we choose the outward-directed normal.

12. Definition of an eigenvector and eigenvalue (page 97)

13. A linear transformation can be represented by a diagonal matrix if and only if there is a basis of eigenvectors (Theorem 4.1 page 96).

14. An eigenvector has only one eigenvalue, but an eigenvalue can have many eigenvectors (eg, the identity map).

15. What eigenvalues a transformation has depends on the field you are using (a transformation might have no real eigenvalues but will always have some complex eigenvalues).

16. Eigenvectors corresponding to distinct eigenvalues form a linearly independent set. (Theorem 4.2 page 100). You should know how to prove this, at least for 2 or 3 eigenvalues. Instead of trying to learn the proof from the book, try to write your own for these cases.

17. Consequently a linear transformation on an n dimensional space has at most n distinct eigenvalues (it may have less) (Theorem 4.3 page 100). And if it does have n distinct eigenvalues then it can be represented by a diagonal matrix (by Theorem 4.1) and the diagonal entries are the eigenvalues.

18. How to find eigenvalues of a matrix by computing the characteristic polynomial (see page 103 for Definition and Theorem 4.5). You can be certain this will be tested.

19. A matrix always has n eigenvalues (counting multiplicity) in $\mathbb{C}$ (though it may have less or none in $\mathbb{R}$). (You should know why, and be able to give examples of each possible situation.)

20. How to find the eigenvectors of a matrix once you know the eigenvalues (examples pages 104-106).

21. Given a linear transformation which has a basis of eigenvectors, how to change to that basis so as to make the matrix diagonal. (Theorems 4.6 and 4.7 page 110.

22. Similar matrices represent the same linear transformation and have the same eigenvalues (Definition, Theorems 4.8, 4.9, page 111).

23. If the characteristic polynomial of a matrix has n distinct roots then the matrix is similar to the diagonal matrix with eigenvalues on the diagonal. (Theorem 4.10 page 112).

24. However, if the characteristic polynomial of a matrix does not have n distinct roots then it might be able to be diagonalized (e.g. the identity) or not (you should know an example that can't be diagonalized over the reals).