Math 2144 Final: Things you should definitely know.

Note: This is not an exhaustive list of topics we covered in class, but if you have a solid grasp of these topics and can do problems with them you should be fine on the final exam.

1. Surface integrals. Sections 12.1 through 12.4 introduce several concepts that are important for dealing with surfaces. From this, you need to be able to do the following:

   i) Parameterize a surface as $r(u, v) = (X(u, v), Y(u, v), Z(u, v))$ for $(u, v) \in T$ a region in $\mathbb{R}^2$. This might involve using cylindrical polar or spherical polar coordinates, or using the fact that the surface is a graph, or some basic geometry of surfaces (like planes, cones, cylinders, spheres). Bear in mind that $r(u, v)$ has to be a function.
   ii) Compute the fundamental vector product $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$.
   iii) Understand the geometric properties of this vector product, that it is normal to the surface and that its magnitude is the area scaling factor relating the area of a small parallelogram on the surface to that of a corresponding square in T.

2. Compute the area of a surface by parameterizing and integrating the magnitude of the fundamental vector product (equation (12.7) page 424). This could involve directly parameterizing a simple geometric object or a graph of a function. You may find it useful to know the formula (12.9), but if you understand (12.7) you can get (12.9) in a few lines.

3. Compute surface integrals $\iint f \, dS$ and flux integrals $\iint F \cdot n \, dS$. (Section 12.7)

4. All you really need to know from Theorems 12.1 and 12.2 (page 433-434) is that the value of a surface integral does not depend on how it is parameterized. Of course for a flux integral the sign can depend on the direction of the normal vector. (There are other things one can do with Theorem 12.1 but we did not cover them.)

5. Section 12.9. The important thing here is the wedge notation, e.g. $dx \wedge dy$. You should know that whenever it occurs there is a parametrization of the surface lurking in the background. Since these come from flux integrals and only the sign of the flux integral can change with the parametrization, all that matters is what direction (there are only two that are possible) the fundamental vector product of the parametrization has. With the correct parametrization $r(u, v) = (X(u, v), Y(u, v), Z(u, v))$, meaning the one with the correct fundamental vector product direction, one has

$$dx \wedge dy = \frac{\partial(X, Y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

   and similarly for $dy \wedge dz$ (replace X with Y and Y with Z) etc. Note that $dx \wedge dy = -dy \wedge dx$. Knowing the above lets you convert any wedge expression into an integral you can compute.

6. Stokes theorem (page 438). Know and be able to apply Stokes theorem. I suppose this has to be memorized, and I recommend you learn the version $\iint_S (\nabla \times F) \cdot n\, dS = \int_C F \cdot d\alpha$. Here you must be careful about the orientation of the curve C which is the boundary of S. It should be related to the normal n in the surface integral by the right hand rule (equivalently the rule about walking around the boundary with head in normal direction and region on left side). Note that if you start out with a flux integral through a surface then this theorem is only usable if the vector field is a curl. Not all vector fields are curls (this occurs iff the field has zero divergence on a suitable domain) so you can't apply Stokes theorem to every problem of the form $\iint G \cdot n\, dS$.

7. A vector field F is a gradient on an open convex set iff $\nabla \times F = 0$ (Theorem 12.4 page 442). (Note that the set is important as well as the condition on F.)

8. Sections 12.12 and 12.14 have an introduction to the notation in which $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ is treated as if it were a vector. The highlights are that $\nabla \cdot (\nabla f) = \nabla^2 f$ $\nabla \cdot (\nabla \times F) = 0$ and the product rules $\nabla \cdot (fF) = \nabla f \cdot F + f \nabla \cdot F$ and $\nabla \times (fF) = \nabla f \times F + f(\nabla \times F)$.

9. From Section 12.16, the theorem that given F on a rectangular region there is G with $\nabla \times G = F$ if and only if $\nabla . F = 0$ in the region, and that one can get G by assuming $G = (0, M, N)$ and solving for M and N using partial integration. Then all solutions are of the form $G + \nabla \phi$ for scalar functions ϕ.

10. From Section 12.18, Stoke's theorem for surfaces with holes (the relevant bit is the orientation of the holes) and surfaces that cannot be given using a single one-to-one smooth parametrization but must be parametrized in pieces. The key to the latter is that the curves where the pieces join must be in opposite directions so the path integrals on the boundaries cancel.

11. The divergence theorem 12.6 page 457. $\iiint_V \nabla \cdot F\, dV = \iint_S F \cdot n\, dS$ where V is the region bounded by the surface S, and S is orientable with outward orientation. The latter means that there is a way to choose a normal direction to the surface consistently everywhere (this is the same as there being a consistent inside and outside of the region), and that we choose the outward-directed normal.

12. Definition of an eigenvector and eigenvalue (page 97)

13. A linear transformation can be represented by a diagonal matrix if and only if there is a basis of eigenvectors (Theorem 4.1 page 96).

14. An eigenvector has only one eigenvalue, but an eigenvalue can have many eigenvectors (eg, the identity map).

15. What eigenvalues a transformation has depends on the field you are using (a transformation might have no real eigenvalues but will always have some complex eigenvalues).

16. Eigenvectors corresponding to distinct eigenvalues form a linearly independent set. (Theorem 4.2 page 100). You should know how to prove this, at least for 2 or 3 eigenvalues. Instead of trying to learn the proof from the book, try to write your own for these cases.

17. Consequently a linear transformation on an n dimensional space has at most n distinct eigenvalues (it may have less) (Theorem 4.3 page 100). And if it does have n distinct eigenvalues then it can be represented by a diagonal matrix (by Theorem 4.1) and the diagonal entries are the eigenvalues.

18. How to find eigenvalues of a matrix by computing the characteristic polynomial (see page 103 for Definition and Theorem 4.5). You can be certain this will be tested.

19. A matrix always has n eigenvalues (counting multiplicity) in $\mathbb{C}$ (though it may have less or none in $\mathbb{R}$). (You should know why, and be able to give examples of each possible situation.)

20. How to find the eigenvectors of a matrix once you know the eigenvalues (examples pages 104-106).

21. Given a linear transformation which has a basis of eigenvectors, how to change to that basis so as to make the matrix diagonal. (Theorems 4.6 and 4.7 page 110.

22. Similar matrices represent the same linear transformation and have the same eigenvalues (Definition, Theorems 4.8, 4.9, page 111).

23. If the characteristic polynomial of a matrix has n distinct roots then the matrix is similar to the diagonal matrix with eigenvalues on the diagonal. (Theorem 4.10 page 112).

24. However, if the characteristic polynomial of a matrix does not have n distinct roots then it might be able to be diagonalized (e.g. the identity) or not (you should know an example that can't be diagonalized over the reals).

25. Complex vector spaces and how the inner product is different than it is in the real case (page 114). You should know the formula for the standard dot product on $\mathbb{C}^n$ as well as the Hermitean symmetry property.

26. How to compute an eigenvalue from the eigenvector using the inner product (Theorem 5.1 page 114).

27. The notions of Hermitean (symmetric), skew-Hermitean (skew-symmetric) for linear transformations (this property is defined using inner products (Definition page 115).

28. Hermitean transformations have real eigenvalues, skew-Hermitean have imaginary eigenvalues.

29. For a Hermitean (or skew-Hermitean) transformation, eigenvectors corresponding to distinct eigenvalues are orthogonal. This is a major theorem (Theorem 5.3 page 117). You should know how to prove it.

30. A Hermitean (or skew-Hermitean) transformation on a finite dimensional space has a basis of orthonormal eigenvectors. This is another major result (Theorem 5.4 page 120).

31. To check if a transformation is Hermitean or skew-Hermitean it is enough to check on a basis (Theorem 5.5 page 121).

32. The notions of Hermitean (symmetric), skew-Hermitean (skew-symmetric) for matrices, and the adjoint of a matrix (Definitions page 122). It is crucial to understand how these are connected to the same notions for transformations. In particular, the notions for a transformation do not depend on the basis (at all), but a Hermitean transformation has a Hermitean matrix when written with respect to an orthonormal basis (Theorem 5.6 page 121). This need not be the case if the basis is not orthonormal. It is also not the case that having a Hermitean matrix means the transformation is Hermitean, as you know from the fact that some matrices which are not Hermitean can still be diagonalized (using a basis that is not orthonormal).

33. Since a Hermitean (or skew Hermitean) matrix has an orthonormal basis of eigenvectors it is similar to the diagonal matrix with the eigenvalues on the diagonal (by Theorem 4.1 page 96). The change of basis matrix C from the standard basis to an orthonormal basis is unitary (see Theorem 5.7 page 122, and Definition page 123).

34. The quadratic form Q associated to a real symmetric transformation T is the function $Q(x) = \langle Tx, x\rangle$. In a basis in which $x = (x_i)$, and T has matrix $A = (a_{ij})$ the function is $Q(x) = \sum_{i,j} a_{ij}x_ix_j = xAx^t$. (Theorem 5.8 and 5.9 page 127). This sum is called the quadratic form associated to A.

35. The function xAx^t makes sense even if A is not symmetric, but it is the same as xBx^t where B is the symmetric matrix $B = \frac{1}{2}(A + A^t)$, so we can always assume that a symmetric matrix is used. (Theorem 5.10 page 128).

36. More usefully, if A is real symmetric then it is similar to a diagonal matrix via an orthogonal matrix C. So $D = C^{-1}AC = C^tAC$ is diagonal with entries d_{ii} on the diagonal. This change of basis converts the quadratic form into a simpler expression $\sum_{i=1}^n d_{ii}y_i^2$, where $y = XC$. (Theorem 5.11 page 129).

37. The quadratic form gives an easy way of finding some eigenvalues of a symmetric matrix. If we look at the values of $Q(x)$ on the set $\{x : \|x\| = 1\}$, then all real eigenvalues occur in this collection of values. Also any maximum and minimum values of Q are eigenvalues and they occur when x is the corresponding eigenvector (this gives us at least two eigenvalues). (This summarizes Theorems 5.12,13,14.)

38. A transformation is called unitary if it preserves inner products: $\langle T(x), T(y)\rangle = \langle x, y\rangle$.

39. A unitary transformation preserves lengths (and distances) and angles. It is invertible and its inverse is unitary. (Theorem 5.15 page 138).

40. Eigenvalues of a unitary transformation have absolute value 1 (in $\mathbb{C}$). Eigenspaces for distinct eigenvalues are orthogonal and there is an orthonormal basis of eigenvectors. (Theorem 5.16 page 139).

41. To determine whether a transformation is unitary it suffices to check on a basis (Theorem 5.17 page 140) hence a unitary transformation has a unitary matrix when written with respect to an orthonormal basis (Theorem 5.18 page 140).

42. Chapter 9. Second order Taylor. You should know how to approximate a scalar function of several variables by the linear transformation given by the gradient with a second order term involving the Hessian, as in Theorem 9.4 (page 308).

43. Second derivative test for max and min (Theorem 9.6 page 311). This is based on Theorem 9.5 page 310 which says that a quadratic form is strictly positive if its eigenvalues are all positive and strictly negative if all eigenvalues are negative, and on Theorem 9.4 page 308. It tells you that if a scalar field f has a stationary point at $x = a$, (so $\nabla f(a) = 0$) then you may be able to tell the type of stationary point (max or min or saddle) using the eigenvalues of the Hessian. You should know how to do problems using this, including all cases that can occur.

44. There is a version of Theorem 9.6 given in Theorem 9.7 page 312 that you presumably know, and should be able to use.

45. Chapter 6. In this chapter a theory for using linear algebra to work with differential equations is built up step by step. Rather than repeat the steps here, I want to summarize the structure. To do any of it, you need both to be able to recognize a differential equation, and to identify whether a differential equation is linear or non-linear, whether it has constant or variable coefficients, and what the order of the differential equation is. In particular note the order n linear differential equation description on page 146 (equation 6.7).

46. The linear algebra methods of Chapter 6 apply only to linear differential equations. The heart of the theory is that the set of solutions to a homogeneous linear differential equation of order n is a dimension n subspace of the differentiable functions on an interval (Theorem 6.4 page 147). Hence if we can find n linearly independent solutions then every solution is a linear combination of these (Theorem 6.5 page 148). The solution to an inhomogeneous linear differential equation of order n is the sum of any particular solution with the general solution of the corresponding homogeneous equation (Theorem 6.10 page 156). It may help to think of $n = 2$, where the solutions to a homogeneous equation are a plane through the origin are a plane in the space of differentiable functions, and the solutions to an inhomogeneous equation are a plane not through the origin: a particular solution takes you to some point on this plane, and to get to all

other points you add a solution of the homogeneous equation, just as you would with vectors describing a plane in Euclidean space. Higher order equations have higher dimension solution sets.

47. As a result of this theory we can solve a linear differential equation by solving the homogeneous equation (i.e. finding a basis for the n dimensional solution space) and finding a particular solution. Our main methods for doing this are applicable to the case of constant coefficient linear differential equations.

48. To find a solution to a homogeneous constant coefficient linear differential equation we write it as a differential operator applied to a function f. We may factorize the operator just as we would factorize a polynomial (this is justified by Theorem 6.6 page 149), and the solution of any factor applied to f is a solution of the whole equation (Theorem 6.7 page 151). Moreover we can write down the solution to any of these simple differential equations: $(D-\alpha)^k f = 0$ has linearly independent solutions $e^{\alpha x}, xe^{\alpha x}, \ldots, x^{k-1}e^{\alpha x}$ (Theorem 6.9 page 152). Solutions for different factors are linearly independent, there are n factors (in $\mathbb{C}$, counting multiplicity) so they give a basis for solutions to the differential equation and allow us to write the general solution.

49. One thing to note in the above. If the coefficients in the differential equation are real numbers then any complex root occurs along with its complex conjugate, and we may obtain real solutions by taking the real and imaginary parts of the complex exponentials that occur in the above solution. Specifically, a basis for the solution space of $(D-a-bi)^k(D-a+bi)^k f = 0$ is $e^{ax}\cos(bx), e^{ax}\sin(bx), xe^{ax}\cos(bx), xe^{ax}\sin(bx), \ldots, x^{k-1}e^{ax}\cos(bx), x^{k-1}e^{ax}\sin(bx)$.

50. Finding a solution to an inhomogeneous constant coefficient linear differential equation is a matter of finding a particular solution and the general solution of the corresponding homogeneous equation. We have done the latter. For the former we have two methods. One is called the "variation of parameters" or "Wronsky" method. It is in Theorem 6.11 page 160. It is completely general, but sometimes has a lot of computation in it. The other is called the "annihilator method" (and is related to the "guessing" method we had earlier). The idea is that if the inhomogeneous term in the equation can be killed using a constant coefficient linear differential operator then by applying this operator to both sides of the equation one gets a homogeneous constant coefficient linear differential equation, for which we can get a basis of solutions by our earlier method. We then substitute a general linear combination of these basis elements into the original inhomogeneous equation and solve to find the coefficients that give a solution. It is hard to explain this method except by examples, which are in Section 6.14 starting on page 163. Note also Table 6.1 page 166.

51. We had a general linear algebra approach for linear differential equations, but only had techniques to make it all work for the constant coefficient case. This is just reality – d.e.s are too complicated for fully general methods to exist. In particular if the equation does not have constant coefficients then it may not be possible to factorize the operator, and even if you can factorize you might not

get solutions by applying one factor at a time (try to come up with examples for yourself – I think I gave one in class).

52. There was one class of non-constant coefficient linear differential equations for which we said something. Recall that a function is called analytic if it has a power series expansion. Now if we have linear differential equation in which all of the coefficient functions are analytic, then we can try replacing them with their power series expansions and plugging in a power series with unknown coefficients for the solution. This may allow us to solve for the unknown coefficients and give a solution to the equation (as a power series). Theorem 6.13 on page 169 gives one setting where this works ($n = 2$ and homogeneous equation). In fact this is true more generally but we did not cover that.

53. As an example of the power series method we looked at the Legendre equation (section 6.18), the Legendre polynomials (section 6.19) and the Rodrigues formula (section 6.20). You do not need to memorize anything about these.

54. The only thing we did from Sections 6.22–6.24 was the Gamma function; you do not have to know the formula for this.

55. One other thing from Chapter 6. There is a method to solve first order linear IVPs of the type $y' + P(x)y = Q(x)$, $y(a) = b$, by the integrating factor method (Theorem 6.1 on page 143). This is from earlier in the course, but comes back in Chapter 7.

56. Chapter 7 deals with first order linear systems of differential equations. This is more general than Chapter 6 in that one can have several equations happening at the same time and interacting with one another. It is also possible to convert an order n linear equation into a first order linear system (see page 191 for an example – you should know how to do this).

57. The main point of Chapter 7 is that a first order linear system can be written as a differential equation for a vector, with a coefficient that is a matrix. (equation 7.6 page 192). We can then hope to solve it by studying the calculus of matrices.

58. Calculus operations (derivatives, integrals, limits) are done term by term in the matrix (or vector). Derivatives and integrals are linear, and the product rule applies for derivatives (though the order is important, which it was not in the scalar case) see page 193.

59. Series of matrices are defined. Convergence is that each location in the matrix is a convergent series of numbers (page 194).

60. A norm is defined on matrices (page 195) by adding the absolute values of each term in the matrix. It has the usual norm properties, and also $||AB|| \leq ||A||||B||$.

61. There is a convergence test for series of matrices using the norm. If the series of the norms converges then so does the series of matrices (Theorem 7.2 page 195).

62. Using a power series one can define the exponential e^{tA} for a matrix A (page 197). It commutes with A, is invertible and has inverse e^{-tA} (Theorem 7.4 page 198) and the solution of the vector differential equation $F' = AF$, $F(0) = B$ is $F = e^{tA}B$ when A is a constant matrix (Theorem 7.5 page 199).

63. The index laws for exponentials of matrices are valid if the matrices commute (Theorem 7.6 page 199). They usually fail if the matrices don't commute (try to find an example).

64. From this we have an alternative method for solving constant coefficient linear differential equations by exponentiating matrices (Theorem 7.13 on page 213).

65. Exponentiating matrices is not always easy. It is easy if the matrix is diagonal because then we just get the exponential of each term on the diagonal (page 201). It is also true that $e^{tS^{-1}AS} = S^{-1}e^{tA}S$, so if we can diagonalize the matrix A via $S^{-1}e^{tA}S$ then it is easy to compute the exponential (make sure you know how to do this – example on page 201-202).

66. In general we have to be able to compute powers of A to compute e^{tA}. It is a fact that all powers of A can be written as linear combinations of the powers $I = A^0, A^1, \dots, A^{n-1}$ when A is size $n \times n$. This comes from the Cayley-Hamilton theorem 7.8 page 203) which says that $f(A) = 0$ if f is the characteristic polynomial of A.

67. A good basis for computing e^{tA} in terms of powers of A is found like this. Factorize the characteristic polynomial $f(\lambda)$ as $\prod_1^n(\lambda - \lambda_j)$ (with repeated factors if needed). Let $P_k(A) = \prod_1^k(A - \lambda_j I)$ for each $k = 1, \dots, n$ and $P_0(A) = I$. Then one can give a formula for e^{tA} in terms of the polynomials $P_k(A)$ with coefficients that are obtained fairly easily (see Theorem 7.9 page 206). There are simpler versions for particular sorts of matrices (Theorems 7.10, 7.11, 7.12 on pages 209-210); the first two might be worth knowing but the third is too complicated to be useful and none of them are critical if you know how to use the main theorem 7.9.

68. Sections 7.18–7.23 are basically concerned with proving Theorem 7.17 and its generalization Theorem 7.19. These are very important theorems, as they are basically the foundation on which our whole treatment of differential equations is based (they prove Theorem 6.3 page 147, which is where all our work started!). However the theory is quite difficult, and is hard to test. What I want you to know from this theory is just the method of successive approximations given in Section 7.21. In essence this is that you have to solve an equation $Y'(t) = A(t)Y$ with initial condition $Y(0) = B$. You do so by finding a sequence of functions $Y_0(t), Y_1(t), Y_2(t), \dots$. The function $Y_0(t)$ is a first guess, usually the constant function B. One gets $Y_1(t)$ by setting $Y_1' = A(t)Y_0(t)$ and integrating. Similarly $Y_2(t)$ is from $Y_2'(t) = A(t)Y_1(t)$ and in general $Y_k'(t) = A(t)Y_{k-1}(t)$. Under some conditions this sequence of functions will converge; it is not crucial for you to know the conditions or how they operate. Since there will not be returned homework on this topic there will not be an exam question about it.