Math 2144Q - Advanced Calculus IV
Practice Midterm

Instructions:

- You may not refer to any notes or your textbook. No calculators, cellphones, or other electronic devices are permitted.
- You have from 12:30pm until 1:45pm to complete the test.
- In every question **you must justify your answers**.

[10 points] 1. Let S be the sphere radius 1 around the origin oriented outward. Compute

$$\iint_S \left(2yz + z^3 - x\right) dy \wedge dz + \left(x^2 + y - 2z\right) dz \wedge dx + \left(z + x^2 y + \cos x\right) dx \wedge dy$$

Solution: You could parametrize using spherical polar coordinates with fixed radius and $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, but it is easier to use the divergence theorem. Recall this says that $\iint_S F \cdot dS = \iiint_R \nabla \cdot F \, dxdydz$ where R is the region inside (orientable) boundary surface S and S has outward normal. Moreover $\iint_S Ady \wedge dz + Bdz \wedge dy + Cdx \wedge dy = \iint_S F \cdot dS$ for $F = (A, B, C)$. Substituing the formulas for A, B, C in the question we find $\nabla \cdot F = -1 + 1 + 1 = 1$, so that the integral is $\iiint_R dxdydz = 4\pi/3$ because it is the volume of a ball of unit radius.

[10 points] 2. One eigenvalue of the matrix A below is 5. Find all eigenvalues and eigenvectors and determine whether there is a matrix C such that $C^{-1}AC$ is diagonal. If there is no such C, explain why not.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$\begin{aligned}(\lambda - 4)\begin{vmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 2 \end{vmatrix} - \begin{vmatrix} -3 & -1 \\ \lambda - 2 & 0 \end{vmatrix} &= (\lambda - 4)(\lambda^2 - 3\lambda + 2 - 9) - (\lambda - 2) \\ &= \lambda^3 - 7\lambda^2 + 3\lambda + 28 - \lambda + 2 \\ &= (\lambda - 5)(\lambda^2 - 2\lambda - 6)\end{aligned}$$

so the eigenvalues are $5, 1 + \sqrt{7}, 1 - \sqrt{7}$. With 3 distinct eigenvalues and a 3 dimensional space there is a basis of eigenvectors and we can diagonalize (so C exists). It is fairly easy to spot that the eigenvector corresponding to 5 is $(1, 1, 1)^t$. The other two are from the nullspaces of the two matrices (corresponding to the choice of $\pm\sqrt{7}$) which we row reduce simultaneously below

$$\begin{pmatrix} \mp\sqrt{7} & 3 & 1 \\ 3 & 1 \mp \sqrt{7} & 0 \\ 1 & 0 & 3 \mp \sqrt{7} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 \mp \sqrt{7} \\ 0 & 3 & -6 \pm 3\sqrt{7} \\ 0 & 1 \mp \sqrt{7} & -9 \pm 3\sqrt{7} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 \mp \sqrt{7} \\ 0 & 1 & -2 \pm \sqrt{7} \\ 0 & 1 \mp \sqrt{7} & -9 \pm 3\sqrt{7} \end{pmatrix}.$$

In the matrix at the right the bottom row is a multiple of the middle row so reduces to a row of zeros at the next step. Thus the eigenvector $(x, y, z)^t$ corresponding to $\lambda = 1 \pm \sqrt{7}$ has $x = (-3 \pm \sqrt{7})z$ and $y = (2 \mp \sqrt{7})z$, yielding eigenvectors $(-3 \pm \sqrt{7}, 2 \mp \sqrt{7}, 1)$. Then the matrix C is

$$\begin{pmatrix} 1 & -3+\sqrt{7} & -3-\sqrt{7} \\ 1 & 2-\sqrt{7} & 2+\sqrt{7} \\ 1 & 1 & 1 \end{pmatrix}$$

[10 points] 3. Compute the surface area of the part of the cone $x^2 + y^2 = z^2$ that lies between $z = 0$ and $x = 3 - 2z$.

Solution: The intersection of the surface and $x = 3 - 2z$ is a curve with equation $x^2 + y^2 = (3 - x)^2/4$, which simplifies to $3x^2 + 6x + 4y^2 = 9$, or, more usefully, $3(x + 1)^2 + 4y^2 = 12$ (which is an ellipse centered at $(-1, 0)$ and semimajor axis 2, semiminor axis $\sqrt{3}$). The region has $0 \leq z \leq (3 - x)/2$, so it is inside the ellipse; we call the region in the xy-plane inside the ellipse T. Now we can parametrize as a graph $z = f(x, y) = \sqrt{x^2 + y^2}$ (note that $z \geq 0$ so we only needed the positive root). The partial derivatives are $\partial f/\partial x = x/z$, $\partial f/\partial y = y/z$, so the area is from

$$\iint_S dS = \iint_T \sqrt{1 + (x/z)^2 + (y/z)^2}\, dxdy = \iint_T \sqrt{2}\, dxdy$$

This is $\sqrt{2}$ times the area of the ellipse, so if you know the area formula for an ellipse you can use it to get $\sqrt{2}\pi 2\sqrt{3} = 2\sqrt{6}\pi$. Otherwise you should integrate over T, by doing one of

$$\int_{-} 3^1 \int_{-\sqrt{3-3(x+1)^2/4}}^{\sqrt{3-3(x+1)^2/4}} dy\, dx \quad \text{or} \quad \int_{\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{4-4y^2/3}-1}^{\sqrt{4-4y^2/3}-1} dxdy \quad \text{or} \quad \int_0^{2\pi} \int_0^1 2\sqrt{3}s\, dsd\theta$$

where the last is from setting $x = 2s\cos\theta$, $y = \sqrt{3}s\sin\theta$, so the Jacobian is $\begin{vmatrix} 2\cos\theta & -2s\sin\theta \\ \sqrt{3}\sin\theta & \sqrt{3}s\cos\theta \end{vmatrix} = 2\sqrt{3}$.