Math 2144Q - Advanced Calculus IV - Practice Final

Instructions:

- You may not refer to any notes or your textbook. No calculators, cellphones, or other electronic devices are permitted.
- You have from 12:30pm until 1:45pm to complete the test.
- In every question **you must justify your answers**.
- In any question, you may use the result of an earlier part to solve a later part, even if you did not solve the earlier part. For example, you may use the result of 3(a) to solve 3(c) even if you did not do 3(a).

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1. Let $F(x, y, z) = (3y, -xz, yz^2)$. Compute $\iint_S (\nabla \times F) \cdot dS$ where S is the surface $2z = x^2 + y^2$ below the plane $z = 2$, oriented downwards.

Solution: One can do this by parametrizing the surface as a graph, finding the relevant Jacobian, substituting and integrating. Or one can use Stokes' theorem. I think the latter is easier. Let C be the boundary curve $z = 2$ and $2x = x^2 + y^2$, so $x^2 + y^2 = 4$. This is a circle radius 2 around the z axis on the plane $z = 2$. Orient it counterclockwise when seen from above (i.e. from a point with $z > 2$) so its orientation is opposite that of the surface, and parametrize it by $\alpha(\theta) = (2\cos\theta, 2\sin\theta, 2)$. Stokes' theorem says $\iint_S (\nabla \times F) \cdot dS = -\int_C F \cdot d\alpha$. We compute $F(\alpha(\theta)) = (6\sin\theta, -4\cos\theta, 8\sin\theta)$ and $d\alpha = (-2\sin\theta, 2\cos\theta, 0)$. Then

$$\iint_S (\nabla\times F)\cdot dS = -\int_C F\cdot d\alpha = \int_0^{2\pi} 12\sin^2\theta + 8\cos^2\theta \, d\theta = \int_0^{2\pi} 8+4\sin^2\theta \, d\theta = \int_0^{2\pi} 10-2\cos 2\theta \, d\theta = 20\pi$$

2. Let $f(x, y, z) = x^2 + 3y^2 + 2z^2 - 2xy + 2xz$. Find and classify all critical points of f.

Solution: Compute the first and second partial derivatives. Note that the order of the second partials does not matter because these are continuous (since f is a polynomial).

$$\frac{\partial f}{\partial x} = 2x - 2y + 2z \qquad \frac{\partial f}{\partial y} = 6y - 2x \qquad \frac{\partial f}{\partial z} = 4z + 2x$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = 6 \qquad \frac{\partial^2 f}{\partial z^2} = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2 \qquad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 0 \qquad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 2$$

Setting the equations in the first line to be zero we find the critical point(s) have $x = 3y = -2z$ and $0 = 2x - 2y + 2z = 6y - 2y - 3y = y$, so $x = y = z = 0$ and the only critical point is $(0, 0, 0)$. From the remaining equations we find that the Hessian is

$$H = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 6 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

Then the characteristic equation is

$$\begin{aligned} p(\lambda) = \begin{vmatrix} \lambda - 2 & 2 & -2 \\ 2 & \lambda - 6 & 0 \\ -2 & 0 & \lambda - 4 \end{vmatrix} &= (\lambda - 4) \begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 6 \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 \\ \lambda - 6 & 0 \end{vmatrix} \\ &= (\lambda - 4)(\lambda^2 - 8\lambda + 12 - 4) - 4(\lambda - 6) \\ &= \lambda^3 - 12\lambda^2 + 36\lambda - 8 \end{aligned}$$

It does not look easy to find the roots. Let us consider another option. Maybe we can say something about the location of the roots. Observe that $p'(\lambda) = 3\lambda^2 - 24\lambda + 36 = 3(\lambda^2 - 8\lambda + 12) = 3(\lambda - 2)(\lambda - 6)$, so $p(\lambda)$ is increasing on $(-\infty, 2) \cup (6, \infty)$ and decreasing on $(2, 6)$. Thus $p(\lambda)$ has a max of $p(2) = 24$ then a min of $p(6) = -8$ and then $p(\lambda) \to \infty$ as $\lambda \to \infty$. Note also that $p(0) = -8$. The intermediate value theorem then gives that there is a root in $(0, 2)$, another in $(2, 6)$ and a third in $(6, \infty)$. This accounts for all 3 roots and they are all positive. So all are positive and the critical point is a minimum.

3. Six students are each given a real 3×3 matrix (different students get different matrices). They compute the eigenvalues and eigenvectors, using the standard basis, and write them in a table, in which v_j is the eigenvector corresponding to λ_j. The results are:

NAME	λ_1	v_1	λ_2	v_2	λ_3	v_3
Amy	1	$(1,0,1)$	-1	$(0,1,0)$	0	$(1,1,0)$
Bob	$\frac{1}{\sqrt{2}}(1-i)$	v_1	$\frac{1}{\sqrt{2}}(1+i)$	v_2	i	v_3
Carrie	3	$(0,1,1)$	-1	$(1,1,1)$		
Devon	2	$(1,-2,0)$	2	$(0,1,-1)$	-4	$(0,0,1)$
Elizabeth	0	$(2,-1,-1)$	-1	$(-4,2,2)$	1	$(2,0,0)$
Frank	i	$(0,1,i)$	$-i$	$(0,1,-i)$	1	$(1,0,0)$

In each of the following **you must justify your answer**

(a) Which student(s) (if any) can you be sure made a mistake? (i.e. put a line in the table which is impossible.)

(b) Which student(s) (if any) were given a matrix that could be diagonalized? (Omit those listed in part a.)

(c) Which student(s) (if any) were given a unitary matrix? (Omit those listed in part a.)

(d) Write down the matrix for Amy's transformation.

Solution:

(a) B is impossible because the matrix is real but the eigenvalues are not in complex conjugate pairs; for example the determinant would be $(1/2)(1-i)(1+i)i = i$, which cannot be the case for a matrix with real entries. E is impossible because $(-4,2,2) = -2(2,-1,-1)$ so these cannot be eigenvectors with distinct eigenvalues.

(b) We can omit B and E, because these students made errors. If the eigenvectors form a basis the matrix can be diagonalized. This occurs for A, D, F and not for C.

(c) A unitary matrix must have eigenvalues with absolute value (as complex numbers) equal to 1. Both B and F have this property, but B had an error so we are left with F as a possibility. Now a unitary matrix also has orthogonal eigenvectors, and for F v_3 is obviously orthogonal to v_1 and v_2, and the complex dot product $v_1 \cdot v_2 = 1 + i^2 = 0$, so the eigenvectors are orthogonal. This is enough to say the matrix is unitary (because, for example, we can normalize the eigenvectors, put them as columns of a matrix U which is then unitary, and see the matrix is $U\Lambda U^{-1}$ for Λ the matrix with the eigenvalues on the diagonal). So F got a unitary matrix.

(d) If Λ is the matrix with diagonal entries $1, -1, 0$ and we label the matrix in row A by A then $\Lambda = T^{-1}AT$ where T has columns the eigenvectors of A. So

$$A = T\Lambda T^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

where we made a subsidiary calculation of T^{-1} which I did not write out. You can check easily that the eigenvectors and values are correct if you like.

4. Find the general solution of the differential equation $y''' - 4y' = 2\cosh(2x)$.

Solution: The equation $y''' - 4y' = 2\cosh 2x$ is $(D^2 - 4)Dy = 2\cosh 2x$. The corresponding homogeneous equation $0 = (D^2-4)Dz = (D+2)(D-2)Dz$ has solutions e^{2x}, e^{-2x}, 1. We apply an annihilator method. Recall that $2\cosh 2x = (e^{2x} + e^{-2x})$, so is annihilated by $D^2 - 4$. Then $(D^2 - 4)^2 Dy = 0$, or $(D+2)^2(D-2)^2 Dy = 0$. The usual basis for the solution set is $e^{2x}, xe^{2x}, e^{-2x}, xe^{-2x}, 1$. Evidently the e^{2x}, e^{-2x} and 1 are not useful in finding a particular solution because they are solutions of the homogeneous solution. So we consider $y_1 = axe^{2x} + bxe^{-2x}$. Then $y_1' = ae^{2x} + 2xae^{2x} + be^{-2x} - 2bxe^{-2x}$, $y_1'' = 4ae^{2x} + 4xae^{2x} - 4be^{-2x} + 4bxe^{-2x}$ and finally $y_1''' = 12ae^{2x} + 8xe^{2x} + 12be^{-2x} - 8bxe^{-2x}$. Thus $y_1''' - 4y_1' = 8ae^{2x} + 8be^{-2x}$. We want this equal to $(e^{2x} + e^{-2x})$, so must have $8a = 8b = 1$, therefore $a = 1/8$ and $b = 1/8$; we can then recognize that $y_1 = (x\cosh 2x)/4$. This gives us the general solution

$$y(x) = \frac{x}{4}\cosh 2x + Ae^{2x} + Be^{-2x} + C.$$

5. Solve the simultaneous differential equations $2y' - 16z + 6y = -2$ and $z' = 3z + 2y + 1$ with $y(0) = 1$, $z(0) = -1$.

Solution: Write them as a system by setting $Y = \left[\begin{smallmatrix} y \\ z \end{smallmatrix}\right]$, so that

$$Y' = \begin{bmatrix} -3 & 8 \\ 2 & 3 \end{bmatrix} Y + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = AY + Q$$

We know a general solution to this problem when $Y(0)$ is known: it is $Y(x) = e^{xA}Y(0) + e^{xA}\int_0^x e^{-tA}Q(t)\,dt$. In this instance Q is constant, so may be moved out of the right side of the integration to obtain $e^{xA}\int_0^x e^{-tA}\,dtQ$. Since we know the derivative of e^{-tA} is $-e^{-tA}A$ and can see A is invertible we can perform the integration to obtain $\int_0^x e^{-tA}\,dt = (I - e^{-xA})A^{-1}$. We use this to simplify our solution to

$$Y(x) = e^{xA}Y(0) + e^{xA}\int_0^x e^{-tA}Q(t)\,dt = e^{xA}Y(0) + (e^{xA} - I)A^{-1}Q = e^{xA}(Y(0) + A^{-1}Q) - A^{-1}Q.$$

It remains to compute $A^{-1} = (-1/25)\left[\begin{smallmatrix} 3 & -8 \\ -2 & -3 \end{smallmatrix}\right] = A/25$ and thus $A^{-1}Q = (1/25)\left[\begin{smallmatrix} 11 \\ 1 \end{smallmatrix}\right]$, and to compute e^{xA}. One convenient way to get e^{xA} is to say $A^2 = 5^2 I$, so that $A^{2k} = 5^{2k}I$ and $A^{2k+1} = 5^{2k}A$. Then

$$\begin{aligned} e^{tA} &= \sum_{j=0}^{\infty} \frac{1}{j!} t^j A^j \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} t^{2k} A^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} t^{2k+1} A^{2k+1} \\ &= I \sum_{k=0}^{\infty} \frac{1}{(2k)!} t^{2k} 5^{2k} + \frac{A}{5} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} t^{2k+1} 5^{2k+1} \\ &= I \cosh 5t + \frac{A}{5} \sinh 5t \end{aligned}$$

where we used the fact that cosh has the even terms of the power series for the exponential and sinh has the odd terms – this is obvious from $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$. We then obtain

$$\begin{aligned} Y(x) = e^{xA}(Y(0) + A^{-1}Q) - A^{-1}Q &= (Y(0) + A^{-1}Q)\cosh 5x + \frac{1}{5}A(Y(0) + A^{-1}Q)\sinh 5x - A^{-1}Q \\ &= \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{25}\begin{bmatrix} 11 \\ 1 \end{bmatrix}\right)\cosh 5x + \frac{1}{5}\left(\begin{bmatrix} -11 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)\sinh 5x - \frac{1}{25}\begin{bmatrix} 11 \\ 1 \end{bmatrix} \\ &= \frac{12}{25}\begin{bmatrix} 3 \\ -2 \end{bmatrix}\cosh 5x - \frac{12}{5}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sinh 5x - \frac{1}{25}\begin{bmatrix} 11 \\ 1 \end{bmatrix} \end{aligned}$$

6. Suppose that A is an $n \times n$ matrix having n distinct eigenvalues. Let $p(\lambda)$ be the characteristic polynomial of A. Without using the Cayley-Hamilton theorem, show that $p(A) = 0$ (note that both sides of this equation are matrices).

Solution: Since all eigenvalues are distinct we have $p(\lambda) = \prod_{k=1}^{n}(\lambda - \lambda_k)$, where the λ_k are the eigenvalues. We also then know that the corresponding eigenvectors v_k are a linearly independent set, hence a basis. Observe that $(A - \lambda_j I)v_j = Av_j - \lambda_j v_j = 0$. Now consider $p(A)v_j$. We can write the factors in $p(A) = \prod_{k=1}^{n}(A - \lambda_k I)$ in any order, because A and I commute, so we think of putting the $(A - \lambda_j)$ at the far right, so that $p(A) = q_j(A)(A - \lambda_j I)$ for some polynomial $q_j(A)$. But then $p(A)v_j = q_j(A)(A - \lambda_j I)v_j = q_j(A)0 = 0$. So $p(A)v_j = 0$ for any j. Now for any vector v we can write it in terms of the eigenvector basis as $v = \sum_j a_j v_j$. We have $p(A)v = p(A)\sum_j a_j v_j = \sum_j a_j p(A)v_j = \sum_j 0 = 0$. However the only matrix that multiplies every vector to give zero is the zero matrix, so $p(A) = 0$.