Math 2144Q - Advanced Calculus IV Midterm

Instructions:

- You may not refer to any notes or your textbook. No calculators, cellphones, or other electronic devices are permitted.
- You have from 12:30pm until 1:45pm to complete the test.
- In every question **you must justify your answers**.
- If asked to prove something almost identical to a theorem in the text, you should not use that theorem in your proof.

[10 points] 1. Compute $\iint_S \frac{x}{\sqrt{x^2+y^2}}\, dS$, where S is the parabolic surface $x^2+y^2+z=4$ that lies in the region $x \geq 0, z \geq 0$.

Solution: The surface can be parametrized as a graph $z = f(x,y) = 4 - x^2 - y^2$. Note that $\frac{\partial f}{\partial x} = -2x$ and $\frac{\partial f}{\partial y} = -2y$. The projection of the region then satisfies, from $z \geq 0$, that $x^2+y^2 \leq 4$ and also the given $x \geq 0$. This region is half-disc T in the xy-plane. Then we can rewrite the integral using the parametrization and change to polar coordinates to evaluate:

$$\iint_S \frac{x}{\sqrt{x^2+y^2}}\, dS = \iint_T \frac{x}{\sqrt{x^2+y^2}} \sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}\, dxdy = \int_0^2 \int_{-\pi/2}^{\pi/2} \frac{r\cos\theta}{r} \sqrt{1+4r^2}\, d\theta\, rdr$$
$$= \int_1^{17} \sqrt{u}\frac{du}{8}\Big[\sin\theta\Big]_{-\pi/2}^{\pi/2} = \frac{1}{6}((17)^{3/2}-1).$$

[10 points] 2. For the following matrix A, find all eigenvalues and eigenvectors, and if possible find a matrix C such that $C^{-1}AC$ is diagonal. If it is not possible, explain why this is the case.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Solution: The characteristic polynomial is $(\lambda-3)((\lambda-2)(\lambda-3))-(\lambda-2) = (\lambda-2)(\lambda^2-6\lambda+9-1) = (\lambda-2)^2(\lambda-4)$ so the eigenvalues are 2 and 4, with 2 having multiplicity two. We can find eigenvectors by determining,for $\lambda = 4$, the nullspace of

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

which is one-dimensional and spanned by $(1,1,1)^t$, and, for $\lambda = 2$, the nullspace of

$$\begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix},$$

which is two dimensional and spanned by $(0,1,0)^t$ and $(1,0,-1)^t$ (though linear combinations of these also work). Then we can take

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

to get the desired diagonalization. Note that other choices of basis for the $\lambda = 2$ eigenspace are equally valid and give correct answers for C.

[10 points] 3. Let S be the surface $5 - z = x^2 + y^2$ with $z \geq 0$. Let $\mathbf{F} = (y^2 z - y + x \sin z, x + y^2 \log(1+z), x^2 y - z^3 + \cos x)$. Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

where $\mathbf{n}$ denotes the unit normal which has a negative component in the z-direction.

Solution: From Stoke's theorem the integral is $\int_C F \cdot d\alpha$ where α parametrizes the boundary curve C with the correct orientation. Since the the surface is a graph $z = 5-(x^2+y^2)$ and $z \geq 0$ the curve C is the intersection of the graph with the xy-plane, which is the circle $x^2 + y^2 = 5$. The normal is downward and the right-hand rule gives that C is oriented clockwise. We can parametrize this way as $\alpha(\theta) = \sqrt{5}(\cos\theta, -\sin\theta, 0)$, which gives $\alpha'(\theta) = -\sqrt{5}(\sin\theta, \cos\theta, 0)$. Now compute $F(\alpha(\theta)) = (\sqrt{5}\sin\theta, \sqrt{5}\cos\theta, -5^{3/2}\cos\theta\sin\theta + \cos(\sqrt{5}\cos\theta))$, so that

$$\int_C F \cdot d\alpha = \int_0^{2\pi} (-5\sin^2\theta - 5\cos^2\theta) \, d\theta = -5 \int_0^{2\pi} d\theta = -10\pi.$$