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Grade 2, 10, 12, 15

2/ $y'' - 4y = e^{2x}$

is $(D^2-4)\,y = e^{2x}$

$(D-2)(D+2)\,y = e^{2x}$

Factor + 2

Now $(D-2)\,e^{2x} = 0$ so must have $(D-2)^2(D+2)\,y = 0$ + 2

whence soln is $ae^{2x} + be^{-2x} + cxe^{2x} = y$ + 2

but 1st two terms solve $(D^2-4)\,y = 0$, so can get coeff of 3rd term via $(D^2-4)\,cxe^{2x}$ + 2

$= (D+2)\,c\,e^{2x}$

$= c4e^{2x}$ But need $= e^{2x}$

$\Rightarrow c = \frac{1}{4}$.

or diff whole thing to get c.

So gen soln is $ae^{2x} + be^{-2x} + \frac{1}{4}xe^{2x}$. + 2.

10/ $y^{(4)} - y = x^2e^{-x}$

$\Rightarrow (D^4-1)\,y = x^2e^{-x}$

$\Rightarrow (D^2-1)(D^2+1)\,y = x^2e^{-x}$

$\Rightarrow (D-1)(D+1)(D-i)(D+i)\,y = x^2e^{-x}$.

+ 2

Know $(D+1)^3x^2e^{-x} = 0$, so $(D-1)(D+1)^4(D-i)(D+i)\,y = 0$ + 2

giving $y = a_1\cos x + a_2\sin x + a_3e^{x} + a_4e^{-x} + a_5xe^{-x} + a_6x^2e^{-x}$
$+ a_7x^3e^{-x}$ + 2

Now the terms with coeffs a_1, a_2, a_3, a_4 are annihilated by (D^4-1),
so it suffices to find coeffs a_5, a_6, a_7 by computing

$(D^4-1)\, xe^{-x} = (D-1)(D^2+1)\, e^{-x} = (D-1)\, 2e^{-x} = \cancel{\ } -4e^{-x}$

$(D^4-1)\, x^2e^{-x} = (D-1)(D^2+1)\, 2xe^{-x} = \cancel{\ } \; 12e^{-x} - 8xe^{-x}$

$(D^4-1)\, x^3e^{-x} = (D-1)(D^2+1)\, 3x^2e^{-x} = -12x^2e^{-x} + 36xe^{-x} - 24e^{-x}$

(Note: An easy way to do these is to note that

$(D-1) = (D+1-2)$, $D^2+1 = (D+1)^2 - 2(D+1) + 2$ b/c know

$(D+1)\, p(x)e^{-x} = p'(x)e^{-x}$.)

+2 (or diff whole thing - ugh!)

Clearly need $-12a_7 = 1$, so $a_7 = -\frac{1}{12}$

and $36a_7 - 8a_6 = 0$ so $a_6 = \frac{36a_7}{8} = -\frac{3}{8}$

and $-24a_7 + 12a_6 - 4a_5 = 0$ so $a_5 = \frac{1}{4}\left(-12\left(\frac{3}{8}\right) + \frac{24}{12}\right)$

$= \frac{1}{4}\left(-\frac{9}{2} + 2\right)$

$= \cancel{\ } -\frac{5}{8}$

So that gen soln is

$a_1\cos x + a_2\sin x + a_3e^{x} + a_4e^{-x} - \frac{5}{8}xe^{-x} - \frac{3}{8}x^2e^{-x} - \frac{1}{12}x^3e^{-x}$.

+2

12) A const coeff diff op with char poly p_A.

a) Consider $A(y) = e^{\alpha x}$

~~[illegible]~~

~~[illegible]~~ If α is not a ~~[illegible]~~ zero of p_A then for $y_1 = \frac{e^{\alpha x}}{p_A(\alpha)}$

have $Ay_1 = \frac{p_A(\alpha)e^{\alpha x}}{p_A(\alpha)} = e^{\alpha x}$. +2

b) Write $p_A(r) = \sum_{j=0}^{n} a_j (r-\alpha)^j$. This is possible +1

with $n = \deg(p_A)$. Then $p_A(r)$ has a single root at $r = \alpha$ iff $a_0 = 0$, $a_1 \neq 0$. +1 Now look at

$$A x e^{\alpha x} = \sum_{0}^{n} a_j (D-\alpha)^j (x e^{\alpha x})$$

We know $(D-\alpha)^j x e^{\alpha x} = \begin{cases} 0 & \text{if } j \geq 2 \\ e^{\alpha x} & \text{if } j = 1 \end{cases}$ +1

and since $a_0 = 0$ we get

$$A x e^{\alpha x} = \sum_{1}^{n} a_j (D-\alpha)^j (x e^{\alpha x})$$

$$= a_1$$

But $p_A'(r) = \sum_{1}^{n} j a_j (r-\alpha)^j \Rightarrow p_A'(\alpha) = a_1 \neq 0$ +1

So $A\left(\frac{x e^{\alpha x}}{p_A'(\alpha)}\right) = e^{\alpha x}$ as required.

c) Similarly if $p_A(r)$ has a zero of mult k at α then $a_0, a_1, \dots, a_{k-1}$ are all zero and $a_k \neq 0$ +1 and $A = \sum_{k} a_j (D-\alpha)^j$.

We know $(D-\alpha)^j x^k e^{\alpha x} = \begin{cases} 0 & j > k \\ k!\, e^{\alpha x} & \text{if } j = k \end{cases}$ +1

So that $A(x^k e^{\alpha x}) = \sum_{k}^{n} a_j (D-\alpha)^j x^k e^{\alpha x}$

$$= a_k k!\, e^{\alpha x}$$ +1

Whence $\frac{x^k e^{\alpha x}}{a_k k!} = y_1$ has $A y_1 = e^{\alpha x}$. (Note $a_k \neq 0$ from above)

But also $p_A^{(k)}(r) = k!\,a_k \neq 0$, so] +1

$y_1 = \dfrac{x^k e^{\alpha x}}{p_A^{(k)}(\alpha)}$ is a soln of $Ay = e^{\alpha x}$.

15/ Let $Qy(x) = x\,y(x)$ if y is a C^∞ funct of x.
Let $Iy = y$ be identity op.

a) $(DQ - QD)y = D(xy) - xDy = xy' + y - xy'$
$= y$

So $(DQ - QD) = I$ +2

b) $D^2Q - QD^2 = D(DQ - QD) + (DQ - QD)D$
$= DI + ID = 2D$. +2

c) $D^3Q - QD^3 = D(D^2Q - QD^2) + (DQ - QD)D^2$
$= D(2D) + ID^2$ + 2
$= 3D^2$.

Guess & Say "base case" +2

d) Guess $D^nQ - QD^n = nD^{n-1}$ True for $n=1$ (eg part a)
(and 2,3 parts b,c)
Base Case

Induction: $D^{n+1}Q - QD^{n+1}$
$= D(D^nQ - QD^n) + (DQ - QD)D^n$
$= D(nD^{n-1}) + ID^n$
$= (n+1)D^n$. So our fmla is true by induct.

+2 Induct step.