page 447 : 2, 3, 9

Total /30

grade : 2, 3, 9

2/ curl $(f(\|r\|)r)$ where f is diff

10 pts

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(\|r\|)x & f(\|r\|)y & f(\|r\|)z \end{vmatrix}$$

3 pts

$$= \left(f'(\|r\|)\left(z\frac{\partial\|r\|}{\partial y} - y\frac{\partial\|r\|}{\partial z}\right), f'(\|r\|)\left(x\frac{\partial\|r\|}{\partial z} - z\frac{\partial\|r\|}{\partial x}\right), f'(\|r\|)\left(y\frac{\partial\|r\|}{\partial x} - x\frac{\partial\|r\|}{\partial y}\right)\right)$$

But $\|r\|^2 = x^2+y^2+z^2$

$\Rightarrow 2\|r\|\frac{\partial\|r\|}{\partial x} = 2x \Rightarrow \frac{\partial\|r\|}{\partial x} = \frac{x}{\|r\|}$ and similarly for y, z

3pts

So curl $(f(\|r\|)r)$

$$= \frac{f'(\|r\|)}{\|r\|}\left((zy-yz), (xz-zx), (yx-xy)\right)$$

2pts

$= 0$ unless, perhaps, $r = (0,0,0)$, where the above fmlas for $\frac{\partial\|r\|}{\partial x}$ etc fail.

2pts

OR use fmlas given : $\nabla\times(\varphi F) = (\nabla\varphi)\times F + \varphi(\nabla\times F)$ and $r\times r = 0$.

and $\nabla\times r = 0$, $\nabla f(\|r\|) = \frac{f'(\|r\|)}{\|r\|}r$

8pts, + 2 pts for $r=0$ case

3/ ~~Since~~ Want $\nabla \times (A \times r)$

10 pts

We're not given fmlas for this, so do directly:

$r = (x, y, z) \qquad A = (a, b, c)$] Setup, 2 pts

So $A \times r = \begin{vmatrix} i & j & k \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy, cx - az, ay - bx)$

+ 2 pts

and $\operatorname{curl}(A \times r) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix}$ + 3 pts

$= (2a, 2b, 2c) = 2A.$ + 3 pts

9/ Sps ~~exists~~ there is μ nonzero scalar field

so $\mu F = \nabla \varphi$. Show $F \perp \operatorname{curl} F$.

10 pts

Well $\nabla \times (\mu F) = \nabla \mu \times F + \mu \nabla \times F$] 2 pts

$\Rightarrow \nabla \times F = \frac{1}{\mu} \left(\nabla \times (\mu F) - \nabla \mu \times F \right)$ b/c $\mu \neq 0$

2 pts + 2 pts

and $\mu F = \nabla \varphi \Rightarrow \nabla \times (\mu F) = 0$] + 2 pts

So $\nabla \times F = -\frac{1}{\mu} (\nabla \mu) \times F$ is $\perp$ to F.] 2 pts pg 462 1, 2, 6, 7 Total /20

Grade 1, 6, 7

1/ S is surface of unit cube $[0,1]^3 \subset \mathbb{R}^3$ oriented outward. $F(x,y,z) = (x^2, y^2, z^2)$. Evaluate $\iint_S F \cdot dS$ by the divergence thm & directly.

Out of 10 pts

a) By div thm : $\underline{\operatorname{div} F = 2x + 2y + 2z}$ 2 pts

$$\iint_S F \cdot dS = \iiint \operatorname{div} F \, dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\, dx\,dy\,dz$$

$$= 2\left(\int_0^1 x\,dx + \int_0^1 y\,dy + \int_0^1 z\,dz\right)$$ b/c the other two integrations are over unit square of const.

$$= 6\int_0^1 x\,dx \quad \text{by symmetry}$$

$$= 3x^2\Big|_0^1 = 3.$$ + 3 pts

b) Directly. There are surfaces with normal $\pm i$ with $x = 0, 1$ and $(y,z) \in [0,1]^2$, which give integrals $\iint x^2\,dy\,dz$, thus contribute

$$\int_0^1\int_0^1 dy\,dz - \int_0^1\int_0^1 0\,dy\,dz = 1$$ 2 pts

Symmetrically those with normals $\pm j$ for $y = 0, 1$ and $(x,z) \in [0,1]^2$ give

$$\int_0^1\int_0^1 (1-0)\,dx\,dz = 1$$ 1 pt

and with normals $\pm k$, $z=0,1$ and $(x,y) \in [0,1]^2$

which give $\int_0^1\int_0^1 (1-0)\,dx\,dy = 1$ ~~pt~~ 1 pt

The sum of these is $\iint_S F.dS = 3$. 1 pt

$\boxed{6}$ $\iint_S f \frac{\partial g}{\partial n}\, dS = \iint_S (f\nabla g).n\, dS$ 2 pts

$= \iiint_V \operatorname{div}(f\nabla g)\, dx\,dy\,dz$ + 2 pts

$= \iiint_V (\nabla f.\nabla g + f\nabla^2 g)\, dx\,dy\,dz$

+ 2 pts

10 pts combined

$\boxed{7}$ $\iint_S f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\, dS$ + 2 pts

$= \iiint_V (f\nabla^2 g + \nabla f.\nabla g - g\nabla^2 f - \nabla f.\nabla g)\, dx\,dy\,dz$

by q 6

$= \iiint_V (f\nabla^2 g - g\nabla^2 f)\, dx\,dy\,dz$ + 2 pts pg 101 1, 2, 4, 7, 9, 11, 12.

grade: 1, 2, 11, 12.

1/ a) If $Tx = \lambda x$ then $(aT)x = a(Tx) = a(\lambda x) = (a\lambda)x$ 2 pts

5 pts b) If $T_1 x = \lambda_1 x$, $T_2 x = \lambda_2 x$ then $(aT_1 + bT_2)x$

$$= a\lambda_1 x + b\lambda_2 x$$

$$= (a\lambda_1 + b\lambda_2)x$$ 2 pts

So x is an evect with eval $a\lambda_1 + b\lambda_2$ 1 pt

1 pt

2/ If $Tx = \lambda x$ then $T^2 x = T(Tx) = T(\lambda x) = \lambda(Tx) = \lambda(\lambda x) = \lambda^2 x$

5 pts Inductively sps $T^{n-1}x = \lambda^{n-1}x$, then $T^n x = T(T^{n-1}x) = \lambda^{n-1}Tx = \lambda^n x$ +1 pt

so $T^n x = \lambda^n x$ by induct with base case $n=0$ $(T^0 x = x)$. +1 base case

Then $\sum_1^n (a_j T^j)x = \sum_1^n a_j(T^j x) = \sum_1^n a_j(\lambda^j x) = \left(\sum_1^n a_j \lambda^j\right)x$ 2 pts

So with $p(t) = \sum_1^n a_j t^j$ see that $P(T)x = P(\lambda)x$.

11/ Sps $Tx = \lambda x$, $Ty = \mu y$ and $T(ax+by) = \gamma(ax+by)$.

10 pts Then $T(ax+by) = aTx + bTy = a\lambda x + b\mu y$, so

$$\gamma ax + \gamma by = a\lambda x + b\mu y$$

$$\Rightarrow (\gamma - \lambda)ax + (\gamma - \mu)by = 0$$ +3

Now evects corresp distinct evals are lin indep, so we +2

must have $(\gamma - \lambda)a = 0 \Rightarrow \gamma = \lambda$ OR $a = 0$ +2

and $(\gamma - \mu)b = 0 \Rightarrow \gamma = \mu$ OR $b = 0$ +2

Since $\lambda \neq \mu$ we can't have both $\gamma = \lambda$ and $\gamma = \mu$,

and therefore one of $a = 0$ or $b = 0$ is true. +1

12) Sps $T: S \to V$ is such that $v \in S \Rightarrow \exists\, \gamma$ with $Tv = \gamma v$.

10 pts. Take $x, y \in S$. We have $\boxed{\begin{aligned} Tx &= \lambda x \\ Ty &= \mu y \\ T(x+y) &= \gamma(x+y) \end{aligned} \Bigg\} \text{ for some } \lambda, \mu, \gamma}$. +3

But ex 11, with $a \neq 0$, $b \neq 0$, yields $\lambda = \gamma = \mu$. / +3

Thus for $x, y \in S$ the eval of T is the same. That is,
+3

$\exists\, c$ such that $Tx = cx \;\; \forall x \in S$, and so $T = cI$.

+1

pg 107 : 1, 6, 7, 9, 13, 14

grade : 1, 7 a), b), 14.

1/ a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has eval 1 and all vects are evects. +2

dim E(1) = 2.

10 pts

b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eval 1 b/c $\det(\lambda I - A) = (\lambda-1)^2$.

+2

We have $I - A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has null vect $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (so evect of A), so dim E(1) = 1

c) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $\det(\lambda I - A) = (\lambda-1)^2$ so eval is 1.

$(I - A) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has null vect $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so this is +2

evect of A and dim(E(1)) = 1

d) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\det(\lambda I - A) = (\lambda-1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2)$

evals are $\lambda = 0$, $\lambda = 2$. +2

a null vect for $-A = 0I - A$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so is evect of A with eval 0

a null vect for $2I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so is evect of A with eval 2.

Thus dim E(0) = 1, dim E(2) = 1 +2

7/ a) $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$ $\det(\lambda - A) = (\lambda-1)^3$ so only eval is 1.

+1

10 pts

nullsps of $I - A = \begin{bmatrix} 0 & 0 & 0 \\ +3 & 0 & 0 \\ -4 & 7 & 0 \end{bmatrix}$ from row reduct: $\begin{bmatrix} 3 & 0 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

~~[illegible]~~ find null sps is 1 dim'l with spanning vect $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\text{So only evect is } \begin{bmatrix}0\\0\\1\end{bmatrix}}$ and $\boxed{\dim(E(1)) = 1}$ +1

+2

b) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}$ $\det(\lambda I - A) = \det \begin{bmatrix} \lambda-2 & -1 & -3 \\ -1 & \lambda-2 & -3 \\ -3 & -3 & \lambda-20 \end{bmatrix}$

row reduct

Expansion or row reduct or a mix is fine.

$= \det \begin{bmatrix} \lambda-2 & -1 & -3 \\ -1 & \lambda-2 & -3 \\ 0 & -3\lambda+3 & \lambda-11 \end{bmatrix}$

$= (\lambda-2)\det \begin{bmatrix} \lambda-2 & -3 \\ 3(1-\lambda) & \lambda-11 \end{bmatrix} + \det \begin{bmatrix} -1 & -3 \\ 3(1-\lambda) & \lambda-11 \end{bmatrix}$

$= (\lambda-2)\left[(\lambda-2)(\lambda-11) + 9(1-\lambda)\right] + \left((11-\lambda) + 9(1-\lambda)\right)$

$= (\lambda-2)\left[\lambda^2 - 13\lambda + 22 - 9\lambda + 9\right] + (20 - 10\lambda)$

$= (\lambda-2)(\lambda^2 - 22\lambda + 31) + 10(2-\lambda)$

$= (\lambda-2)(\lambda^2 - 22\lambda + 21)$

$= (\lambda-2)(\lambda-21)(\lambda-1)$

+2

Evals $\boxed{1, 2, 21}$ Must each be dim 1

Evects: $\lambda = 1$ Nullsp of $-\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 19 \end{bmatrix}$ So $\boxed{\begin{bmatrix}1\\-1\\0\end{bmatrix} \text{ is evect eval } 1}$

+1

$\lambda = 2$ Nullsp of $-\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 3 & 18 \end{bmatrix}$

Can row reduce to $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ So $\boxed{\text{null vect } \begin{bmatrix}3\\3\\-1\end{bmatrix} \text{ is evect eval } 2}$ +1

$\lambda = 21$ Nullsp of $-\begin{bmatrix} -19 & 1 & 3 \\ 1 & -19 & 3 \\ 3 & 3 & -1 \end{bmatrix}$ row reduce to $\begin{bmatrix} 1 & -19 & 3 \\ 0 & 1-19^2 & 60 \\ 0 & 60 & -10 \end{bmatrix}$

and to $\begin{bmatrix} 1 & -19 & 3 \\ 0 & 6 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

~~eval~~ ~~for~~

null vect is $\begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$ so this is evect with eval 21.

+1

Have $\dim E(1) = \dim E(2) = \dim E(21) = 1$.

+1

14/ Prove

10 pts a) $tr(A+B) = tr(A) + tr(B)$

If $A = [a_{ij}]$, $B = [b_{ij}]$ then $A+B = [a_{ij}+b_{ij}]$

and $tr(A+B) = \sum_{j=1}^{n}(a_{jj}+b_{jj}) = \sum_1^n a_{jj} + \sum_1^n b_{jj} = tr(A)+tr(B)$

+2

b) $tr(cA) = c\,tr(A)$.

Well $tr(cA) = \sum_1^n ca_{jj} = c\sum_1^n a_{jj} = c\,tr(A)$

+2

c) $tr(AB) = tr(BA)$

+1 +1

$\boxed{AB = \left[\sum_{k=1}^n a_{jk}b_{kl}\right]}$ and $\boxed{BA = \left[\sum_{l=1}^n b_{kl}a_{lm}\right]}$

so $tr(AB) = \sum_{j=1}^n\sum_{k=1}^n a_{jk}b_{kj}$

$tr(BA) = \sum_{k=1}^n\sum_{l=1}^n b_{kl}a_{lk} = \sum_{j=1}^n\sum_{k=1}^n a_{jk}b_{kj} = tr(AB)$

+2

d) $tr(A^t) = tr(A)$

$A^t = [a_{ji}]$ so $tr(A^t) = \sum_1^n a_{jj} = tr(A)$. +2