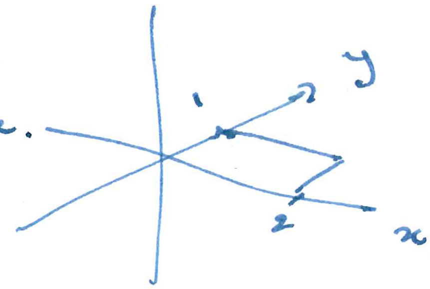
page 429 : 4, 5, 8, 10

grade : 4, 10

4/ Want area of $z^2 = 2xy$ above $x \geq 0,\ y \geq 0$ cut off by $x = 2,\ y = 1$. Set $z = f(x,y) = \sqrt{2xy}$.

So is graph above rectangle $0 \leq x \leq 2,\ 0 \leq y \leq 1$ in xy plane.



+2

Can then compute

$$\text{Area} = \int_0^2 \int_0^1 \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dy\, dx$$

$$= \int_0^2 \int_0^1 \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}}\ dy\, dx$$

+ 2

$$= \int_0^2 \int_0^1 \sqrt{\frac{2xy + y^2 + x^2}{2xy}}\ dy\, dx$$

Note $\sqrt{x^2 + 2xy + y^2} = \sqrt{(x+y)^2} = x + y$ b/c $x + y \geq 0$ in region

$$= \int_0^2 \int_0^1 \frac{x+y}{\sqrt{2xy}}\ dy\, dx$$

+ 2

$$= \int_0^2 \int_0^1 \frac{1}{\sqrt{2}}\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right) dy\, dx$$

$$= \int_0^2 \frac{1}{\sqrt{2}} \left[2\sqrt{y}\sqrt{x} + \frac{1}{\sqrt{x}}\frac{2}{3}y^{3/2}\right]_{y=0}^{y=1} dx$$

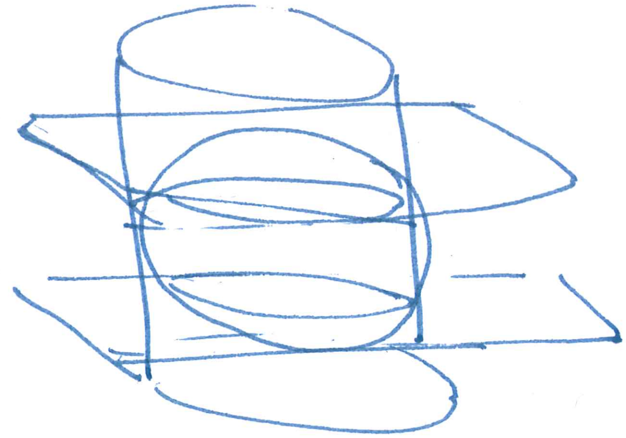
$$= \int_0^2 \frac{1}{\sqrt{2}}\left[2\sqrt{x} + \frac{2}{3\sqrt{x}}\right] dx$$

+ 2

$$= \frac{1}{\sqrt{2}}\left[2\,\frac{2}{3}x^{3/2} + \frac{2}{3}2\sqrt{x}\right]_0^2 = \frac{1}{\sqrt{2}}\left(\frac{4}{3}2\sqrt{2} + \frac{4}{3}\sqrt{2}\right) = \frac{12}{3} = 4$$

+2

10/ [redacted] Inscribe sphere in cylinder & slice both by parallel planes perpendicular to axis of cylinder. Show area of sphere & area of cylinder that are cut out are the same.



It is convenient to put the origin at the center of the sphere, ~~the~~ ~~of the heights~~ the cylindrical axis on the z axis & the planes parallel to the xy plane. If the z coords of the planes are a and b then the area of the cylinder cut out is $2\pi R|b-a|$, where R is radius of cylinder. Note that then R is also the radius of the sphere.

Now compute area of piece of sphere above $z=0$ & below $z=c$ by treating as graph of

$$z = \sqrt{R^2-(x^2+y^2)} = f(x,y)$$

above region $0 \le R^2-(x^2+y^2) \le c^2$

i.e. $R^2-c^2 \le (x^2+y^2) \le R^2$. Call this region A

The area of the piece of sphere is then

$$\iint_A \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy = \iint_A \sqrt{1+\left(\frac{-x}{z}\right)^2+\left(\frac{-y}{z}\right)^2}\, dx\,dy$$

$$= \iint_A \sqrt{\frac{z^2+x^2+y^2}{z^2}}\, dx\,dy$$

$$= \iint_A \frac{R}{z}\, dx\, dy$$ which is convenient in polars

$$= \int_0^{2\pi} \int_{\sqrt{R^2-c^2}}^{R} \frac{R}{\sqrt{R^2-r^2}}\, r\, dr\, d\theta$$

let $u = R^2 - r^2$
$du = -2r\,dr$
u is from c^2 to 0, so reverse order of integral & cancel $-$ve

$$= 2\pi \int_0^{c^2} \frac{R}{\sqrt{u}} \frac{du}{2}$$

$$= \pi R\, 2\sqrt{u}\Big|_0^{c^2} = 2\pi R c$$

This is now sufficient to get our result, b/c if a, b both > 0 then area between the planes is $2\pi R(b-a)$ if $b > a$ and $2\pi R(a-b)$ if $a > b$, so $2\pi R|b-a|$.

The same applies if a, b both < 0.

~~[illegible]~~ then if a and b have opposite signs we split at $z = 0$ and write as $2\pi R|b| + 2\pi R|a|$ which is $2\pi R|b-a|$ b/c a & b have opposite signs.

Remark: They could have done this, eg, in sph polars ~~[illegible]~~ or cylindrical polars.

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grade : 7

7/ Compute $\iint_S xz\, dy\wedge dz + yz\, dz\wedge dx + x^2\, dx\wedge dy$

S is surface $x^2+y^2+z^2 = a^2$ oriented outwards.

It is convenient to use spherical polar param.

$$x = a\sin\varphi\cos\theta$$

$$y = a\sin\varphi\sin\theta$$

$$z = a\cos\varphi$$

If in sph

as this param gives whole sphere in suitable way, except content 0 set at z axis.

Then recall $dy\wedge dz = \begin{vmatrix} \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta}\end{vmatrix} = \begin{vmatrix} +a\cos\varphi\sin\theta & a\sin\varphi\cos\theta \\ -a\sin\varphi & 0\end{vmatrix}$

$$= \boxed{a^2\sin^2\varphi\cos\theta} \quad +2$$

$dz\wedge dx = \begin{vmatrix} \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta} \\ \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta}\end{vmatrix} = \begin{vmatrix} -a\sin\varphi & 0 \\ a\cos\varphi\cos\theta & -a\sin\varphi\sin\theta\end{vmatrix}$

$$= \boxed{a^2\sin^2\varphi\sin\theta} \quad +2$$

$dx\wedge dy = \begin{vmatrix} \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta}\end{vmatrix} = \begin{vmatrix} a\cos\varphi\cos\theta & -a\sin\varphi\sin\theta \\ a\cos\varphi\sin\theta & a\sin\varphi\cos\theta\end{vmatrix}$

$$= \boxed{a^2\sin\varphi\cos\varphi} \quad +2$$

Note: It is important that with $r = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$

Have $\frac{\partial r}{\partial\varphi}\times\frac{\partial r}{\partial\theta} = \begin{vmatrix} i & j & k \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0\end{vmatrix}$

$$= (a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi)$$

is the outward unit normal because it is

$= a \sin\varphi \; r(\varphi, \theta)$

and $0 \le \varphi \le \pi$ so $\sin\varphi \geq 0$.

(Else we would need the opposite orientation).

So $\displaystyle\iint_S xz\, dy\wedge dz + yz\, dz\wedge dx + x^2\, dx\wedge dy$

$$= \int_0^{\pi}\int_0^{2\pi} \Big(a^2 \sin\varphi\cos\theta\cos\varphi\; a^2\sin^2\varphi\cos\theta + a^2\sin\varphi\sin\theta\cos\varphi\; a^2\sin^2\varphi\sin\theta$$
$$+ a^2\sin^2\varphi\cos^2\theta\; a^2\sin\varphi\cos\varphi \Big)\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} a^4 \sin^3\varphi\cos\varphi\,(\cos^2\theta + \sin^2\theta + \cos^2\theta)\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} a^4 \sin^3\varphi\cos\varphi\,(1+\cos^2\theta)\, d\theta\, d\varphi \quad +2$$

$$= \int_0^{\pi} a^4 \sin^3\varphi\cos\varphi\, d\varphi \int_0^{2\pi} (1+\cos^2\theta)\, d\theta$$

$$= \int a^4 u^3\, du \int_0^{2\pi} (1+\cos^2\theta)\, d\theta$$

$$= a^4 \frac{\sin^4\varphi}{4}\Big|_0^{\pi} \Big(\quad \Big)$$

$$= 0 \Big(\quad \Big) \qquad +2$$

don't need to compute this.

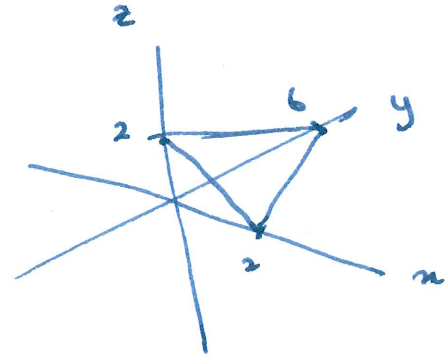
$= 0$

page 442 : 2, 4, 6, 8

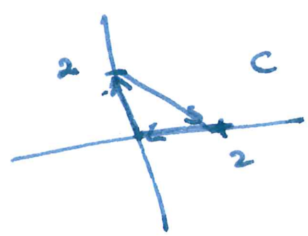
grade : 4, 6

4/ $F(x,y,z) = (xz, -y, x^2 y)$

Integrate $\iint_S \mathrm{curl}\, F \cdot dS$ over S the 3 faces of tetrahedron $x \geq 0, y \geq 0, z \geq 0$, $3x + y + 3z \leq 6$ that are not in the xz plane, with outward normal.



Options :
Use Stokes to convert to integral around bdy in xz plane



Choice of normal says orientation is clockwise

Then $\iint_S (\mathrm{curl}(F)) \cdot dS = \oint_C F \cdot d\alpha$

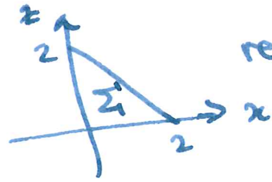
Can param C as
$\alpha_1(t) = t(0,0,1) \quad 0 \leq t \leq 2$
$\alpha_2(t) = (0,0,2) + t(1,0,-1) \quad 0 \leq t \leq 2$
$\alpha_3(t) = (2,0,0) + t(-1,0,0) \quad 0 \leq t \leq 2$

Then $F(\alpha_1(t)) \cdot \frac{d\alpha_1}{dt} = x^2 y = 0$ (b/c $y = 0$ on α_1)
$F(\alpha_2(t)) \cdot \frac{d\alpha_2}{dt} = xz - x^2 y = xz = t(2-t)$
$F(\alpha_3(t)) \cdot \frac{d\alpha_3}{dt} = -xz = 0$ (b/c $z = 0$ on α_3)

So have $\int_0^2 t(2-t)\, dt = t^2 - \frac{t^3}{3} \Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.

Alternatively, use Stoke's twice to get

that $\iint_S \text{curl } F \cdot dS = \iint_{\Sigma} \text{curl } F \cdot dS$

with Σ is 

region in xz plane

having inward (ie. +ve y dirn) unit normal, ie.

normal is vector j.

Then $\text{curl } F \cdot j = \frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(x^2 y)$

$= x - 2xy$ but we are on the xz plane, so $y = 0$.

Thus $\iint_{\Sigma_1} (\text{curl } F) \cdot dS = \int_0^2 \int_0^{2-x} x \, dz \, dx$

$= \int_0^2 x(2-x) \, dx$

$= \frac{4}{3}$ as before.

6/ Verify $\int_C (y+z)\,dx + (z+x)\,dy + (x+\cancel{z}\,y)\,dz = 0$

for C the curve of intersection of $x^2+y^2 = 2y$

and $y = z$.

The suggestion was to use Stokes theorem.

An obvious choice of surface is the plane $y=z$, + 2

and the region is then above the disc inside

$x^2+y^2=2y$

i.e. $x^2+(y-1)^2 = 1$ in xy plane. + 2

Now if $F = (P,Q,R)$ ~~then~~ and C is param

by $(x(t), y(t), z(t)) = \alpha(t)$ then clearly

$$\int_C F\cdot d\alpha = \int_C P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}$$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

So that ~~curl~~ the field being used is

$(y+z,\ z+x,\ x+\cancel{z}\,y)$ + 2

and its curl is
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+\cancel{z}\,y \end{vmatrix}$$

$$= (\cancel{2}\,0, 0, 0)$$ + 2

So $\int_C (y+z)\,dx + (z+x)\,dy + (x+z)\,dz$

$$= \iint_S (\cancel{2}\,0, 0, 0)\cdot dS = 0$$ + 2

~~We can param surf as graph $z=y$ above x,y plane, so then $r(x,y) = (x,y,y)$ $\frac{\partial r}{\partial x}\times\frac{\partial r}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1\end{vmatrix} = (0,-1,1)$~~ +2

~~and so the normal to the surface~~ +2

~~is $\perp$ to $(-1,0,0)$ and~~

$$\cancel{\iint_S (-1,0,0) \cdot dS = 0.}$$