Pg 211, 2, 5, 6, 8, 12

Grade: 2, 8.

2) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

$$\mathrm{Det}(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda + 1 \end{vmatrix} = (\lambda-1)(\lambda+1) - 4 = \lambda^2 - 5 = (\lambda - \sqrt{5})(\lambda + \sqrt{5})$$

$$\lambda_1 = \sqrt{5}, \quad \lambda_2 = -\sqrt{5}$$

2 pts

Can either diagonalize or use ~~[illegible]~~. Lagrange!

EITHER 6pts

Lagrange:
$$L_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 1+\sqrt{5} & 2 \\ 2 & -1+\sqrt{5} \end{bmatrix} = \frac{1}{2\sqrt{5}}(A + \sqrt{5}\, I)$$

$$L_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \frac{-1}{2\sqrt{5}} \begin{bmatrix} 1-\sqrt{5} & 2 \\ 2 & -1-\sqrt{5} \end{bmatrix} = \frac{-1}{2\sqrt{5}}(A + \sqrt{5}\, I)$$

And
$$e^{tA} = e^{\sqrt{5}t} L_1 + e^{-\sqrt{5}t} L_2 = \frac{1}{2\sqrt{5}}\left(e^{\sqrt{5}t} - e^{-\sqrt{5}t}\right) A + \frac{I}{2}\left(e^{\sqrt{5}t} + e^{-\sqrt{5}t}\right)$$

OR 6pts

Diagonalize:
$$\begin{bmatrix} \sqrt{5}-1 & -2 \\ -2 & \sqrt{5}+1 \end{bmatrix} \begin{bmatrix} 2 \\ \sqrt{5}-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ so evect } \begin{bmatrix} 2 \\ \sqrt{5}-1 \end{bmatrix}$$

Similarly evect for $-\sqrt{5}$ is $\begin{bmatrix} 2 \\ -(\sqrt{5}+1) \end{bmatrix}$

Then let $U = \begin{bmatrix} 2 & 2 \\ \sqrt{5}-1 & -(\sqrt{5}+1) \end{bmatrix}$ Have $\begin{bmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{bmatrix} = U^{-1} A U$

so
$$e^{tA} = U \begin{bmatrix} e^{\sqrt{5}t} & 0 \\ 0 & e^{-\sqrt{5}t} \end{bmatrix} U^{-1}$$

$= \ldots$ (compute). = Same as

Could write answer as

$$e^{tA} = \cosh(\sqrt{5}t)\, I + \frac{1}{\sqrt{5}} \sinh(\sqrt{5}t)\, A$$

2 pts.

8/ Solve $Y' = AY$ for $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, $Y(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

5pts

Soln is $e^{tA} Y(0)$ $\Big]$ 2pts

$$= \cosh(\sqrt{5}t)\, I\, Y_0 + \frac{1}{\sqrt{5}} \sinh(\sqrt{5}t)\, A\, Y_0$$

$$= \cosh(\sqrt{5}t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \frac{1}{\sqrt{5}} \sinh(\sqrt{5}t) \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 - c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1\left(\cosh(\sqrt{5}t) + \frac{1}{\sqrt{5}} \sin(\sqrt{5}t)\right) + \frac{2c_2}{\sqrt{5}} \sinh(\sqrt{5}t) \\ c_2\left(\cosh(\sqrt{5}t) - \frac{1}{\sqrt{5}} \sinh(\sqrt{5}t)\right) + \frac{2}{\sqrt{5}} c_1 \sinh(\sqrt{5}t) \end{bmatrix}.$$

Either one, 3pts

pg 215 3, 4, 5, 8, 11

Grade : 3, 4 ~~5~~ , 11

3 ~~a~~/ ~~Ex~~ 1. Have A is $n \times n$ const matrix

B are n-vectr

α scalar.

a) Consider $z'(t) = Az + e^{\alpha t} C$

It has soln $z(t) = e^{\alpha t} B$

$\Leftrightarrow \alpha e^{\alpha t} B = z'(t) = Az + e^{\alpha t} C$

$= A e^{\alpha t} B + e^{\alpha t} C$

Now $e^{\alpha t} \neq 0$ so this is

$\Leftrightarrow (\alpha I - A) B = C \quad (*)$ 3 pts

b) If α is not an eval of A then can solve $(*)$ b/c $(\alpha I - A)$ invertible, so given C $\exists$ B satisfying $(*)$, and so $e^{\alpha t} B$ solves the d.e with init condt $e^0 B = B$. 3 pts

c) If α is not an eval then get soln ~~$Y(t) = e^{tA}$~~

$Y(t) = e^{tA}(Y(0) - B) + e^{\alpha t} B$

$\Leftrightarrow A e^{tA}(Y(0) - B) + \alpha e^{\alpha t} B = Y'(t) = AY(t) + e^{\alpha t} C$ 4 pts

$\Leftrightarrow A e^{tA}(Y(0) - B) + \alpha e^{\alpha t} B = A e^{tA}(Y(0) - B) + A e^{\alpha t} B + e^{\alpha t} C$

$\Leftrightarrow e^{\alpha t}(\alpha I - A) B = e^{\alpha t} C$

$\Leftrightarrow (\alpha I - A) B = C$ So is soln iff $B = (\alpha I - A)^{-1} C$

4/ $\quad Y' = AY + e^{2t} C$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Applying method of prev question

Have soln by taking $\alpha = 2$,

$$(2I - A)B = C$$

so $B = (2I-A)^{-1}C = \begin{bmatrix} -1 & -1 \\ -2 & 0 \end{bmatrix}^{-1} C = -\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad 2 \text{ pts}$$

And then soln is

$$Y(t) = e^{tA}(Y(0) - B) + e^{2t} B$$

$$= e^{tA}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + e^{2t}\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= e^{tA}\left(\frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad + 2 \text{ pts}$$

But e^{tA} must be computed. Evals from $(3-\lambda)(2-\lambda) - 2 = 0$

$$\Leftrightarrow \lambda^2 - 5\lambda + 6 - 2 = 0$$

$$\Leftrightarrow (\lambda - 4)(\lambda - 1) = 0$$

So $\lambda_1 = 1, \ \lambda_2 = 4$ $\quad] + 2$ pts

Can use Lagrange:

$$e^{tA} = e^{\lambda_1 t}\frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} + e^{\lambda_2 t}\frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$$

$$= \frac{e^t}{6}\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} + \frac{e^{4t}}{-6}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

+ 2 pts (or another method to get same)

So $e^{tA}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{e^t}{6}\begin{bmatrix} 2 \\ -4 \end{bmatrix} + -\frac{e^{4t}}{6}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2e^t + e^{4t} \\ -4e^t + e^{4t} \end{bmatrix}$

So soln is $Y(t) = \frac{1}{\ }\begin{bmatrix} e^{4t} + 2e^t + 6e^{2t} \\ \ldots \end{bmatrix}$ $\quad$ + 2 pts

11/ Solve $y'(t) = AY(t) + Q(t)$

for $A = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix}$, $Q(t) = \begin{bmatrix} 7e^t - 27 \\ -3e^t + 12 \end{bmatrix}$, $Y(0) = \frac{1}{422}\begin{bmatrix} -1007 \\ (707)2 \end{bmatrix}$

Equ is $\left(e^{-tA} y(t)\right)' = e^{-tA} Q(t)$

so soln $Y(t) = e^{tA} Y(0) + e^{tA} \int_0^t e^{-sA} Q(s)\, ds$.

Convenient to notice $Q(s) = e^s \begin{bmatrix} 7 \\ -3 \end{bmatrix} + \begin{bmatrix} -27 \\ 12 \end{bmatrix} = e^{sI}\begin{bmatrix} 7 \\ -3 \end{bmatrix} + \begin{bmatrix} -27 \\ 12 \end{bmatrix}$

so that $e^{-sA} Q(s) = e^{s(I-A)} \begin{bmatrix} 7 \\ -3 \end{bmatrix} + e^{-sA} \begin{bmatrix} -27 \\ 12 \end{bmatrix}$

2pts

Now $e^{tA} \int_0^t e^{-sA} Q(s)\, ds$

$$= e^{tA}\left((I-A)^{-1}\left(e^{t(I-A)} - I\right) \begin{bmatrix} 7 \\ -3 \end{bmatrix} - A^{-1}\left(e^{-tA} - I\right) \begin{bmatrix} -27 \\ 12 \end{bmatrix}\right)$$

$$= \left(e^t - e^{tA}\right)(I-A)^{-1} \begin{bmatrix} 7 \\ -3 \end{bmatrix} + \left(I - e^{tA}\right) A^{-1} \begin{bmatrix} -27 \\ 12 \end{bmatrix}$$

(I have used that A & I commute a few times)

2pts

So compute $(I-A)^{-1}\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & -1 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 31 \\ -4 \end{bmatrix}$

and $A^{-1}\begin{bmatrix} -27 \\ 12 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -3 & 1 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} -27 \\ 12 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 93 \\ -6 \end{bmatrix}$

So we have

$$Y(t) = e^{tA} Y(0) + \left(e^t - e^{tA}\right)\begin{bmatrix} 31/26 \\ -4/26 \end{bmatrix} + \left(I - e^{tA}\right)\begin{bmatrix} 93/17 \\ -6/17 \end{bmatrix}$$

2pts

$$= \frac{e^t}{26}\begin{bmatrix} 31 \\ -4 \end{bmatrix} + \frac{1}{17}\begin{bmatrix} 93 \\ -6 \end{bmatrix} + e^{tA}\left(\frac{1}{422}\begin{bmatrix} -1007 \\ 1414 \end{bmatrix} - \frac{1}{26}\begin{bmatrix} 31 \\ -4 \end{bmatrix} + \frac{1}{17}\begin{bmatrix} 93 \\ -6 \end{bmatrix}\right)$$

$$= \frac{e^t}{26}\begin{bmatrix} 31 \\ -4 \end{bmatrix} + \frac{1}{17}\begin{bmatrix} 93 \\ -6 \end{bmatrix} + \frac{e^{tA}}{442}\begin{bmatrix} 884 \\ 1326 \end{bmatrix}$$

$$= \frac{e^t}{26}\begin{bmatrix} 31 \\ -4 \end{bmatrix} + \frac{1}{17}\begin{bmatrix} 93 \\ -6 \end{bmatrix} + e^{tA}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Finally we compute e^{tA}.

2 pts

Char poly is $(\lambda+5)(\lambda+3)+2 = \lambda^2+8\lambda+17$

So $\lambda = \dfrac{-8 \pm \sqrt{64-4(17)}}{2}$

$= -4 \pm \frac{1}{2}\sqrt{-4} = -4 \pm i$

Let $\lambda_1 = -4+i$, $\lambda_2 = -4-i$. Using Lagrange interpolants:

2 pts

$$e^{tA} = e^{t(-4+i)}\left(\frac{A-(-4-i)I}{-4+i-(-4-i)}\right) + e^{t(-4-i)}\left(\frac{A-(-4+i)I}{-4-i-(-4+i)}\right)$$

$$= \frac{1}{2i}e^{-4t}e^{it}\left(A+(4+i)I\right) - \frac{1}{2i}e^{-4t}e^{-it}\left(A+(4-i)I\right)$$

$$= e^{-4t}\left((A+4I)\left(\tfrac{1}{2i}\right)(e^{it}-e^{-it}) + I\left(\tfrac{1}{2}\right)(e^{it}+e^{-it})\right)$$

$$= e^{-4t}\left((A+4I)\sin t + I\cos t\right)$$

So $e^{tA}\begin{bmatrix}2\\3\end{bmatrix} = e^{-4t}\left(\begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\sin t + \begin{bmatrix}2\\3\end{bmatrix}\cos t\right)$

$$= e^{-4t}\left(\begin{bmatrix}-5\\7\end{bmatrix}\sin t + \begin{bmatrix}2\\3\end{bmatrix}\cos t\right)$$

and $y(t) = \dfrac{e^t}{26}\begin{bmatrix}31\\-4\end{bmatrix} + \dfrac{1}{17}\begin{bmatrix}93\\-6\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix}\cos t + (e^{-4t}\sin t)\begin{bmatrix}-5\\7\end{bmatrix}$.

Remark: May have done by subst for $e^{-sA}Q(s)$ in integral & compute out. This is longer but still correct. Up to 6 pts for valid reasoning of this kind.