pg 195 2, 3, 5, 7, 9, 10

Grade 2b, 3, 5, 10

2b) if $P(x), Q(x)$ matrix valued diff functs, $P = [p_{ij}(x)]$, $Q = [q_{ij}(x)]$

5pts

$$[PQ]_{ij} = \sum_{k=1}^{n} p_{ik}(x) q_{kj}(x)$$ ijth cpt of PQ from defn of matrix product

go to cpts. Write matrix product using cpts +2

$$\Rightarrow [PQ]'_{ij} = \sum_{k=1}^{n} \left(p_{ik}(x) q'_{kj}(x) + p'_{ik}(x) q_{kj}(x) \right)$$ using usual product rule for derivs

+1

$$= [PQ']_{ij} + [P'Q]_{ij}$$ ijth cpts of PQ' & $P'Q$.

recognize as cpt of product +1

$$\Rightarrow PQ = PQ' + P'Q.$$ (since equality for each entry)

answer +1.

3) if $P(x)$ is diff matrix-valued funct

10 pts

$$(P^2)' = PP' + P'P$$ by 2b above +2

$$(P^3)' = P(P^2)' + P'P^2$$
$$= P(PP' + P'P) + P'P^2$$
$$= P^2P' + PP'P + P'P^2$$ +2

Inductively, if $(P^k)' = \sum_{j=0}^{k-1} P^{k-j-1} P' P^j$ true for $k = 2$ as above

Inductive step ~~proof~~ of proof +3

then $$(P^{k+1})' = P(P^k)' + P'P^k$$
$$= P\left(\sum_{j=0}^{k-1} P^{k-1-j} P' P^j\right) + P'P^k$$
$$= \left(\sum_{j=0}^{k-1} P^{k-j} P' P^j\right) + P^0 P' P^k$$
$$= \sum_{j=0}^{k} P^{k-j} P' P^j$$

Correct fmla +3

$$\boxed{\text{So } (P^k)' = \sum_{j=0}^{k-1} P^{k-1-j} P' P^j}$$ for all k by induct.

5/ Sps $P'(t) = 0$ for all $t \in (a,b)$

5pts Write $P(t) = [p_{ij}(t)]$, then $p_{ij}'(t) = 0$ on (a,b)

$\Rightarrow$ $p_{ij}(t) = a_{ij}$ a const indep of t

deal with prob one entry of matrix at a time +2

$\Rightarrow$ $P(t) = [a_{ij}] = A$ is a const matrix.

Conclusion +1

Use of deriv = 0 $\Rightarrow$ funct is const for scalar functs +2

10/ Let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

10pts

Then prove by induct that $D^k = \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k)$, b/c true for $k=1$ and if true for k then

$$D^{k+1} = D D^k = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,\mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k)$$
$$= \mathrm{diag}(\lambda_1^{k+1}, \ldots, \lambda_n^{k+1}).$$

+3

Thus
$$\sum_{k=0}^{n} \frac{D^k}{k!} = \sum_{k=0}^{n} \mathrm{diag}\left(\frac{\lambda_1^k}{k!}, \ldots, \frac{\lambda_n^k}{k!}\right)$$
$$= \mathrm{diag}\left(\sum_{k=0}^{n} \frac{\lambda_1^k}{k!}, \ldots, \sum_{k=0}^{n} \frac{\lambda_n^k}{k!}\right)$$

+2

and each entry has $\sum_0^n \frac{\lambda_j^k}{k!} \to e^{\lambda_j}$ as $n \to \infty$, +2

So $\sum_{k=0}^{\infty} \frac{D^k}{k!}$ converges, b/c each entry cnvges, +2

and limit is $\mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$ by taking limit of each entry. +1 Correct limit from limits of entries pg 205 : 2, 4, 5, 8, 11

Grade : 2, 5, ~~11~~

2/ $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ evals are 1 and 2 (b/c triangular)

10 pts

so char poly $(\lambda-1)(\lambda-2)$

Cayley-Hamilton : $(A-I)(A-2I) = 0$

$\Rightarrow A^2 - 3A + 2I = 0$

$\Rightarrow \boxed{A^2 = 3A - 2I}$ +2

One can use this to look for a pattern :

$A^3 = 3A^2 - 2A = 3(3A-2I) - 2A = 7A - 6I$

$A^4 = 7A^2 - 6A = 7(3A-2I) - 6A = 15A - 14I$

$A^5 = 15A^2 - 14A = 15(3A-2I) - 14A = 31A - 30I$

It looks like $\boxed{A^k = (2^k-1)A - (2^k-2)I}$ $k \geq 2$ +3 for formula

this is true for $k = 2, 3, 4, 5$ so far.

Prove by induct : if true for k then

$A^{k+1} = (2^k-1)A^2 - (2^k-2)A$

$= (2^k-1)(3A-2I) - (2^k-2)A$

$= (3 \cdot 2^k - 3 - 2^k + 2)A - (2^{k+1}-2)I$

$= (2^{k+1}-1)A - (2^{k+1}-2)I$

so the formula is proved.

+2 for any decent pf of formula for A^k

Then $e^{tA} = \sum_0^\infty \frac{t^k A^k}{k!}$

$= \sum_0^\infty \left(\frac{t^k(2^k-1)A}{k!} - \frac{t^k(2^k-2)}{k!} I \right)$

$$= \left(\sum_{k=0}^{\infty} \frac{t^k 2^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k}{k!}\right) A - \left(\sum_{0}^{\infty} \frac{t^k 2^k}{k!} - 2\sum_{0}^{\infty} \frac{t^k}{k!}\right) I$$

+ 2 for writing e^{tA} in terms of A and I

$$= (e^{2t} - e^{t})A + (2e^{t} - e^{2t})I.$$

+ 1 for simplifying as exponentials.

5/ a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has char eqn $\lambda^2 + 1 = 0$

$\Rightarrow A^2 = -I$ by Cayley Hamilton

$\Rightarrow A^{2k} = (-1)^k I$] + 2

and $A^{2k+1} = (-1)^k A$] + 1

Option!
10 pts total,
6 for part a,
4 for part b.

So $e^{tA} = \sum_{0}^{\infty} \frac{t^k A^k}{k!} = \boxed{\sum_{k=0}^{\infty} \frac{t^{2k} A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1} A^{2k+1}}{(2k+1)!}}$ (breaking as odd and even) + 1

At this point they don't have Putzer method or similar. So they must do either this way (find powers by Cayley-Hamilton)

OR

$$= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} A$$ + 1

$$= \left(\frac{e^{it} + e^{-it}}{2}\right) I + \left(\frac{e^{it} - e^{-it}}{2i}\right) A$$

$$= \cos t\, I + \sin t\, A$$

+ 1 (either one)

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

b) If $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = aI + bA$ where A as in part (a)

4 pts

There is a notation problem here b/c book uses A for two things in same problem

then since A and I commute,

Using commutation +1 to get e^{sum} = product +1

$$e^{tB} = e^{atI} e^{tbA}$$

$$= \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix} \quad \text{by part (a)}$$

$$= e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}. \quad +2$$

OPTION 2

(10 pts, 6 part a, 4 part b)

a) Diagonalize. +1 [Evals are $\pm i$ b/c char poly $\lambda^2+1=0$

Evects from $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

evects in change of basis matrix +2

Set $\boxed{U = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}}$ then $U^{-1}AU = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = D$

and $\boxed{U^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}}$ +1

Then $e^{tA} = U e^{tD} U^{-1}$

$$= U \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} U^{-1} \quad +1$$

$$= \begin{bmatrix} i & i \\ -1 & +1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \frac{1}{2i} \begin{bmatrix} 1 & -i \\ 1 & +i \end{bmatrix}$$

$$= \frac{1}{2i} \begin{bmatrix} ie^{it} & ie^{-it} \\ -e^{it} & +e^{-it} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & +i \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} ie^{it} + ie^{-it} & -i^2e^{it} + i^2e^{-it} \\ -e^{it} + e^{-it} & ie^{it} + ie^{-it} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^{it} + e^{-it}) & \frac{1}{2i}(e^{it} - e^{-it}) \\ -\frac{1}{2i}(e^{it} - e^{-it}) & \frac{1}{2}(e^{it} + e^{-it}) \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$ +1

Part b as before, or they might do

Char poly is $(\lambda - a)^2 + b^2 = 0$

$\Rightarrow \lambda = a \pm ib$] +1

Find evects, which are same as in part a)

+1

Change basis.

Get $\quad U e^{tD} U^{-1}$ for $D = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$

Then $\quad e^{tD} = e^{at}\begin{bmatrix} e^{ibt} & 0 \\ 0 & e^{-ibt} \end{bmatrix}$ +1

And so $\quad U e^{tD} U^{-1} = e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$

+1.