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Grade 2, 4, 8

2/ a) $f(x) = 1 - \frac{x}{2}\log\frac{1+x}{1-x}$. $|x| < 1$

Convenient to note $\frac{d}{dx}\log\frac{1+x}{1-x} = \frac{d}{dx}(\log 1+x - \log 1-x)$

$$= \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}.$$

then $f'(x) = -\frac{1}{2}\log\frac{1+x}{1-x} - \frac{x}{1-x^2}$

$$f''(x) = -\frac{1}{1-x^2} - \frac{1}{1-x^2} + \frac{x(-2x)}{(1-x^2)^2}$$

$$= \frac{-2}{1-x^2} - \frac{2x^2}{(1-x^2)^2}$$

So $(1-x^2)f''(x) - 2x f'(x) + 2f(x)$ Legendre with $\alpha = 1$

$$= \cancel{-2} \cancel{-\frac{2x^2}{1-x^2}} + \cancel{\frac{2x}{2}\log\frac{1+x}{1-x}} + \cancel{\frac{2x^2}{1-x^2}} + \cancel{2} \cancel{- x\log\frac{1+x}{1-x}}$$

$= 0$ Verify diff eqn successfully 5 pts.

A reasonable argument using odd & even parts of power series to see that f is a multiple of u. 3 pts

(b) We want to write this as an odd & even part. The easy way is to say $f(x) = a u_1(x) + b u_2(x)$ b/c all solns are lin comb of u_1 & u_2. Now $f(0) = 1$ and $a u_1(0) + b u_2(0) = a$. Also $f'(0) = 0$ and $a u_1'(0) + b u_2'(0)$ is ~~[illegible]~~ $= b$ (this is b/c can diff series term by term).

So $a = 1$ and $b = 0$ whence

2pts $\underline{f(x) = u_1(x)}$.

Finding that the multiplicative factor is 1, so $f = u_1$

Note: Alternatively, verify (easily) that $f(-x) = f(x)$, $u_1(-x) = u_1(x)$ and $u_2(-x) = u_2(x)$ to conclude f is a multiple of u_1 then use value @ 0 to find $u_1 = f$.

4/ $y'' - 2xy' + 2\alpha y = 0$ Hermite eqn

$$\text{if } y = \sum_0^\infty a_j x^j \qquad \left. \begin{aligned} y'' &= \sum_2^\infty a_j\, j(j+1) x^{j-2} \\ &= \sum_0^\infty a_{j+2} (j+2)(j+1) x^j \end{aligned} \right] +1$$

$$+1 \left[y' = \sum_1^\infty j a_j x^{j-1} \Rightarrow -2xy' = \sum_1^\infty -2j a_j x^j \right.$$

$$\left. \begin{aligned} \text{Thus eqn is } &\sum_1^\infty \left((j+2)(j+1) a_{j+2} - 2j a_j + 2\alpha a_j \right) x^j \\ &+ (2a_2 + 2\alpha a_0) = 0 \end{aligned} \right] +2$$

So $a_2 = -\alpha a_0$

And for $j \geq 1$

$$\left. \begin{aligned} (j+2)(j+1)\, a_{j+2} &= -2(\alpha - j)\, a_j \\ \Rightarrow \quad a_{j+2} &= \frac{-2(\alpha - j)}{(j+2)(j+1)}\, a_j \end{aligned} \right] \text{Recursion for } a_{j+2} \text{ in terms of } a_j \quad +2$$

$$\Rightarrow \quad = (-1)^2 \frac{2(\alpha - j)}{(j+2)(j+1)} \cdot \frac{2(\alpha - j + 2)}{j(j-1)}\, a_{j-2}$$

etc

Inductively..

Inductively see a_j depends on a_0 if j even
on a_1 if j odd

and have
$$a_{2m} = (-1)^m \frac{2^m \alpha(\alpha-2)\cdots(\alpha-2(m-1))}{(2m)!} a_0 \qquad m \geq 1$$

Fmla for coeffs +2 (1 each)

$$a_{2m+1} = (-1)^m \frac{2^m (\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!} a_1 \qquad m \geq 1$$

So there is an odd & an even soln,

$$u_1 = \sum_{m=0}^{\infty} a_{2m} x^{2m}, \quad u_2 = \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1}$$

with coeffs as detailed above.

Finally we note that u_1 is a polynomial iff $\exists\ m$ such that one of the factors $\alpha, (\alpha-2), \ldots, (\alpha-2(m-1))$ is zero, as then all a_{2j} are zero for $j \geq m$.

Hence u_1 is a polynomial iff α is an even integer.

Similarly u_2 is a polynomial iff α is an odd integer.

(In particular it cannot be the case that both are polys).

+2

8/ a) $P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$

$$= \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n\left((x+1)^n(x-1)^n\right)$$

$$= \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}\left(\left(\frac{d}{dx}\right)^k(x+1)^n\right)\left(\left(\frac{d}{dx}\right)^{n-k}(x-1)^n\right)$$

$$= \frac{1}{2^n n!}\left[\sum_{k=1}^{n}\binom{n}{k}\left(\left(\frac{d}{dx}\right)^k(x+1)^n\right)\left(\left(\frac{d}{dx}\right)^{n-k}(x-1)^n\right) + (x+1)^n\left(\frac{d}{dx}\right)^n(x-1)^n\right]$$

Now $\left(\frac{d}{dx}\right)^{n-k}(x-1)^n = \left(\frac{n!}{k!}\right)(x-1)^k$ and $\left(\frac{d}{dx}\right)^n(x-1)^n = n!$

So
$$= \frac{1}{2^n n!}\left[\sum_{k=1}^{n}\binom{n}{k}\frac{n!}{k!}(x-1)^k\left(\frac{d}{dx}\right)^k(x+1)^n + n!(x+1)^n\right]$$

$$= \frac{1}{2^n}(x+1)^n + (x-1)\underbrace{\sum_{k=1}^{n}\binom{n}{k}\frac{n!}{k!}(x-1)^{k-1}\left(\frac{d}{dx}\right)^k(x+1)^n}_{\text{This is evidently a polynomial b/c } k-1 \geq 0 \text{ for } k \geq 1.}$$

3 pts.

b) $P_n(1) = \frac{2^n}{2^n} + 0\,Q_n(1) = 1$] +2

By performing same computation as above but swapping the factors $(x+1)$ and $(x-1)$ we find

$$P_n(x) = \frac{1}{2^n}(x-1)^n + (x+1)\tilde{Q}_n(x) \quad \text{for a poly } \tilde{Q}_n$$

So then $P_n(-1) = \frac{(-2)^n}{2^n} + 0\,\tilde{Q}_n(-1) = (-1)^n$] +2

c) Notice that the solns u_1 and u_2 for Legendre eqn in book

- Span the (2 dim) soln set
- have property that u_1 is a polynomial iff ~~[illegible]~~ $\alpha \in 2\mathbb{N}$ and u_2 is a polynomial iff $\alpha \in 2\mathbb{N}+1$.

Now sps for $\alpha = n$ ~~had~~ have two solns $P_n(x)$, $Q_n(x)$ that were polys & have $P_n(1) = Q_n(1) = 1$. If they are distinct then must ~~have~~ be lin indep, b/c lin dependence would say $P_n = kQ_n$ for some $k \in \mathbb{R}$

but $P_1(1) = Q_n(1) \Rightarrow k = 1$, contradicting them being distinct.

Then P_n and Q_n would be a basis for the soln set for the Legendre eqn with $\alpha = n$, b/c this set is 2 dim & they are lin indep. But then both u_1 and u_2, being in this soln set, would be polys in contradiction to the fact that u_1 is poly $\Leftrightarrow \alpha \in 2\mathbb{N}$

u_2 is poly $\Leftrightarrow \alpha \in 2\mathbb{N}+1$

So there can't be a second poly soln Q_n with $Q_n(1) = 1$.

3 pts, 1 for each cpt of soln below:

1/ can't have both u_1 & u_2 poly

2/ If have ~~[illegible]~~ two poly solns which agree at 1 then they are lin dep

3/ Space is 2 dim, so then all solns poly. Contradict