Math 2142 Midterm: Things you should definitely know.

Note: This is not an exhaustive list of topics we covered in class, but if you have a solid grasp of these topics and can do problems with them you should be fine on the midterm.

1. Existence and uniqueness of solutions for first order linear IVPs $y' + P(x)y = Q(x)$, $y(a) = b$, and how to solve them by the integrating factor method (Theorem 8.3 on page 310)
2. Existence and uniqueness of solutions for second order constant coefficient linear IVPs $y'' + ay' + by = R(x)$, $y(a) = b$, $y'(a) = c$, and how to solve them using general solution of homogeneous equation (Theorem 8.7 page 326) and a particular solution (Theorem 8.8 page 330). Also how to obtain particular solution using Wronskian method (Theorem 8.9 page 330) and various "guessing" methods (Section 8.16).
3. Concept of integral curve and direction field (pages 341 and 343) and how to draw them, especially how to draw direction field from equation and obtain rough sketch of integral curves from direction field.
4. How to recognize and solve first order separable equations (Theorem 8.10 page 345), including how to determine the domain.
5. How to recognize and solve first order homogeneous equations by a substitution $y = xv$ or $y = \frac{x}{v}$. Including how homogeneity is seen on direction field (Section 8.25).
6. The arithmetic of complex numbers. Field properties (Sections 9.2 and 9.4)
7. Representing complex numbers on the complex plane. (page 363)
8. Modulus and argument of a complex number, complex numbers in polar co-ordinates. Conversion from rectangular $x + iy$ co-ordinates to polar co-ordinates and vice-versa. (Section 9.5)
9. The complex exponential and writing polar co-ordinates as $re^{i\theta}$ for complex numbers. The use of this version of polar form for problems involving products, quotients and roots. (end of page 367, beginning of 368.)
10. Definition of derivatives and integrals of complex-valued functions (page 369) and Theorems 9.5 and 9.6 on page 370.
11. Definition of limit of sequence on page 379. Computation of limits from definition.
12. Monotonic bounded sequences converge (Theorem 10.1 page 381).
13. Definition of convergent series (page 384). Must be confident at converting convergence problem into limit of partial sums.
14. This is not specifically in the book, but you should understand how convergence of a sequence or a series depends only on the "tail", meaning that the beginning of the sequence or series is not relevant. You should be able to make this precise in practice, for example by showing that convergence of $\sum_N^\infty a_n$ is equivalent to convergence of $\sum_1^\infty a_n$.
15. Linearity of convergent series, Theorem 10.2 on page 385, and its application to proving divergence in Theorem 10.3.
16. Telescoping series (Section 10.7). Recognizing these (sometimes you can spot it by computing the first few terms and seeing the cancelation happen), and computing the sum of series of this type (Theorem 10.4 page 386).
17. Geometric series (Section 10.8). Recognizing these, knowing when they are convergent and divergent, and computing sum of such series (see Theorem 10.5).
18. Terms converging to zero are a necessary but not sufficient condition for convergence of series (Theorem 10.6). Should know both how to use this to prove a series diverges, and an example which shows the condition is not sufficient for convergence.

19. Comparison tests (regular and limit comparison, Theorem 10.7, and especially Theorems 10.8 and 10.9). You should know not only what these say (including the conditions you need to check in order to apply them), but should also be proficient at using them in problems. You should also know some series to test/compare against, such as the geometric series and those in Examples 1 and 2 on page 398.

20. Integral test (Theorem 10.11 page 397). You should know the conditions needed to apply this, how the proof is connected to pictures of the integrals (page 397) and know how to apply this theorem in problems. A useful rule of thumb is that if you have a series for which the n^{th} terms does not "look like" a geometric series term r^n (so root and ratio tests fail), and is not a simple comparison with n^{-s} for some $s > 0$, then you should think of applying the integral test. Don't forget to check that the function is positive and decreasing, at least for large values of n.

21. The root and ratio tests (Theorems 10.12 and 10.13 on page 400). You should know what these say about when a series converges or diverges, including the conditions under which they can be applied. You should also know what they don't say, in the sense of when they are inconclusive, including examples that demonstrate that they are inconclusive in these cases. You should have at least an intuitive understanding that they are ways of comparing the given series to a geometric series.

22. You should know the Leibniz rule (Theorem 10.14 on page 404) and how to apply it in problems, as well as have some idea about the picture (page 404) which explains the proof.

23. You should know the definition of absolute convergence and conditional convergence (page 406), and Theorem 10.15 on page 406 which says that absolute convergence implies convergence. In particular you should be comfortable with applying convergence tests for positive series (eg comparison, root, ratio, integral tests) to the absolute values of terms from a non-positive sequence to prove absolute convergence.

24. You should be able to recognize both types of improper integrals and write down what it means for an improper integral to converge. Take special care with integrals that have two or more points where they are improper, or that are improper at two sides of a finite point.

25. You should be comfortable with using comparison on integrals with positive functions, as in Theorems 10.23, 10.24 and 10.25 on page 418. You should also be able to do comparison on absolute values of non-positive functions (eg complex-valued functions) to prove convergence of improper integrals.

26. You should have a firm grasp on how the convergence or divergence of $\int_1^\infty f(x)\,dx$ and $\sum_n f(n)$ are related. As importantly, you should know how they are not related. You should know examples that show the need for f to be positive, and examples that show the need for f to be decreasing. Some such properties are explored in Exercises 20–25 on page 421.

27. Definition of pointwise convergence of a sequence or series of functions (page 422), and examples illustrating that the pointwise limit of continuous (or even differentiable) functions is not necessarily continuous (or differentiable). Also an example showing that $f_n \to f$ pointwise does not imply $\int f_n \to \int f$.

28. Definition of uniform convergence (page 424) and have a sense of what it means geometrically. Know that the uniform limit of continuous functions is continuous Theorem 11.1 pg 424 (and the series version of this, Theorem 11.2 pg 425), and that you can pull a uniform limit (for a sequence or a series) inside an integral as in Theorems 11.3 and 11.4 on page 426. An example showing that if $f_n \to f$ uniformly then it is does not imply $f_n' \to f'$ (i.e. derivatives don't have to converge).

29. Know and be able to use the Weierstrass M-test in problems, Theorem 11.5 page 427. (This is important!)

30. You should be able to write down the general form of a power series (pg 428) and know that its region of (absolute) convergence is a disc in $\mathbb{C}$ or an interval in $\mathbb{R}$ (depending which of these domains is being used), centered at the center of the series. This disc could have zero radius (i.e. be a point) or infinite radius (i.e. be all of $\mathbb{C}$ or $\mathbb{R}$). The series diverges outside the radius of convergence and can converge or diverge (absolutely or conditionally) on the circle having this radius. See Theorem 11.7 on page 429.

31. Be able to compute the radius of convergence of a power series using the ratio or root test, or say something about the region of convergence if you know whether the series converges or diverges at a given point.

32. Know that a power series converges uniformly on any disc smaller than the disc of convergence and that has the same center point.

33. Know that a power series can be both integrated term by term and differentiated term by term (thus it is better than just uniformly convergent) as in Theorems 11.8 and 11.9 on page 432. In particular a power series is infinitely differentiable.

34. Be able to use the uniqueness theorem 11.10 on page 434 in problems, e.g. in solving differential equations by power series as on page 439.

35. Know definition of Taylor series as an obvious generalization of Taylor polynomials, and that if a function has a power series then that series is the Taylor series (page 434). You should know some basic series, like those for $\frac{1}{1-x}$, e^x, $\sin x$ and $\cos x$ and how to get new ones from old ones using differentiation, integration, and algebra (adding series, multiplying by scalars. (All of this, and the uniqueness statement amounts to the fact that inside the region of convergence, power series behave just as if they were polynomials.)

36. Be able to solve differential equations by power series (page 439).

37. Know the general form of a vector in $\mathbb{R}^n$ and how to convert a vector equation into n scalar equations, as well as how to add and subtract vectors, multiply by scalars.

38. Be able to interpret vectors geometrically as points in space, and be able to work with vector addition geometrically (e.g. by knowing that the vector which take you from A to B is $B - A$.)

39. Definition of vectors being parallel, understand geometrically why it is the right definition.

40. Know how to compute the length of a vector, and use it to find the distance between two points.

41. Be able to compute the dot product of two vectors, and the algebraic properties (Theorem 12.2 page 451) of the dot product.

42. Know both the Cauchy-Schwarz inequality (Theorem 12.3 page 452) and the Triangle inequality (Theorem 12.5 page 454) for vectors, and be able to draw a diagram illustrating the triangle inequality. You should also know the conditions under which these inequalities become equalities (i.e. what equality in these means implies about the vectors).

43. Definition of the angle between two vectors and be able to compute it (see eqn 12.9 page 458).

44. Know what the projection of one vector on another means geometrically and be able to compute it, as well as break a vector A into a piece in the direction of another vector B and a piece perpendicular to B. (see page 458.)

45. Definition of linear span (section 12.12 page 462) and linear independence (section 12.13 page 463 especially the criteria immediately after the definition).

46. There are some theorems about linear span and linear independence (the workhorse being Theorem 12.8 page 464), and some geometrical meaning. What I want you to understand out of this is that the span $L(S)$ of a set S of vectors looks like a vector space itself (eg, the span of two vectors in $\mathbb{R}^3$ looks like a plane, so like a copy of $\mathbb{R}^2$ in $\mathbb{R}^3$, if the vectors are linearly independent). How "big" $L(S)$ is depends both on how many vectors there are in S and whether there is any redundancy in S. Linear independence means that there is no redundancy – every vector in $L(S)$ can be written in just one unique way as a linear combination of elements of S – while linear dependence means that any vector can be written in multiple ways as a linear combination. This fact is more or less summarized in the concept of a basis (page 466) and Theorem 12.10, which says that the size of a linearly independent set having span all of $\mathbb{R}^n$ is n; this is also the minimum size of a set that spans $\mathbb{R}^n$ and the maximum size of a linearly independent set in $\mathbb{R}^n$.

47. Definition of a line as a set of points $\{P + tA : t \in \mathbb{R}\}$ where P is a vector giving a point on the line and $A \neq 0$ is the vector direction of the line and its geometric content. (page 472.)

48. There are a bunch of properties in Theorems 13.1–13.5. Most of these are kind of obvious if you understand the content of the definition of a line (e.g. replace A with a parallel vector $B = \alpha A$ and the line doesn't change). You should be able to use these as a result of understanding the geometric meaning of the definition of the line.

49. You should be able to find the vector equation of a line through a point with a given direction, or the equation of a line given two points on the line. Also you should be able to convert these to two scalar parametric equations or a Cartesian equation (eqn 13.2 and 13.3 on page 475).

50. In $\mathbb{R}^2$ you should be able to see geometrically why you can identify a line using a point P on it and a vector N perpendicular to it, as $(X - P) \cdot N = 0$. (Theorem 13.6 page 476.) Be able to get Cartesian equation of line using this, or find N using Cartesian equation.

51. The distance from a point to a line is the distance to the closest point on the line, which is found by going from the point to the line in the direction N perpendicular to the line. You should be able to compute this distance (Theorem 13.6 page 476).

52. Definition of a plane as $\{P + rA + sB : r, s \in \mathbb{R}\}$ where A and B are linearly independent directions and P is on the plane, and the geometric meaning of this. Theorems 13.7–13.11 contain things that are obvious if you understand the geometry of this definition, and you should be able to use them (e.g. if change P for some other point Q on the plane in this formula, then get the same plane; likewise if switch A and B with A' and B' such that the linear span doesn't change then the plane is the same).

53. Be able to get vector equation of plane from a point and two spanning directions, or from 3 given points on the plane, and be able to convert it into two scalar equations or a Cartesian equation (page 481).

54. In $\mathbb{R}^3$ see geometrically that a plane is determined by one point P on the plane and a vector N perpendicular to the plane, as $(X - P) \cdot N = 0$ (Theorem 13.15 page 493). Be able to compute this vector using cross product of two vectors parallel to plane, find Cartesian equation of plane from P and N and read N off Cartesian equation of plane (Section 13.16 page 494).

55. Understand distance between a point and a plane as the distance to the closest point on the plane, that it occurs by going from point to plane in direction N perpendicular to plane. Be able to compute it (Theorem 13.16 page 494).

56. Be able to compute $A \times B$ using determinants (page 487).

57. Know how to do algebra with cross product (Theorem 13.12 page 483), especially fact that $A \times B = 0$ iff A and B are linearly dependent.

58. Know geometric interpretation of cross product $A \times B$ (Theorem 13.13 page 484), that it is perpendicular to A and to B and in direction from right hand rule (page 485) and has length $\|A\|\|B\| \sin \theta$, which is area of parallelogram defined by A and B (page 485).

59. Be able to compute scalar triple product using determinants (page 488).

60. Know how triple product is related to linear dependence (Theorem 13.14 page 489) and its geometric meaning as volume of parallelpiped determined by three vectors.

61. Know that can change order of vectors in scalar triple product cyclically, or can change dot and cross, without changing answer (equations 13.9 and 13.10 page 490).

62. Note: You do not need to know Crammer's rule or conics.

63. Need to know how to compute limits, derivatives and integrals of vector-valued functions, and how they interact with the algebra of vectors and dot and cross products (Theorems 14.1–14.4 page 513-514).

64. Be able to give an accurate (not word-for-word) statement of the content of, and use, FTC I and II (Theorems 14.5 and 14.6 page 515).

65. Know Theorem 14.8 page 515 and be able to interpret it geometrically (see also Theorem 14.13 page 533).

66. Know X' is tangent direction to curve and be able to find equation of tangent line to curve at a point on curve (page 518).

67. Be able to compute velocity, acceleration, and speed.

68. Be able to compute unit tangent vector $T(t)$ and normal $N(t)$ and know what they mean geometrically.

69. Know the acceleration has a component in the direction T due to change in speed and component in direction N due to change in direction of motion. (Theorem 14.9 page 527.)

70. You should know the conceptual definition of arc length as a limit of lengths of polygons, and know it is additive (this is intuitively obvious about length), but I won't be testing your ability to work with this.

71. I will want you to be able to compute the length of a curve and the distance along a curve by integrating the speed (Theorem 14.13 page 533), and to be able to reparametrize a curve in terms of the arclength, as well as know this is the unit speed parametrization.

72. Understand curvature as rate of change $\frac{dT}{ds}$ of direction with respect to arclength (unit speed), and that it is $\frac{\|T'\|}{\|X'\|}$. (Equation 14.20 on page 537.) Know that for a circle it is $\frac{1}{\text{radius}}$.

73. Know how curvature appears in normal component of acceleration (equation 14.22 of Theorem 14.14 on page 538).

74. Don't need to know vector motion in polar coordinates.