1. Let \( f(x) \) be defined by

\[
    f(x) = \begin{cases} 
        \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0; \\
        \frac{1}{2} & \text{if } x = 0. 
    \end{cases}
\]

Is \( f(x) \) continuous? **Justify** your assertion using the \( \varepsilon - \delta \) definition of continuity.

2. Use the \( \varepsilon - \delta \) definition of continuity to prove that \( f(x) = x^2 \) is continuous at every point in the interval \([-10, 10]\).

3. (a) Prove that \( \sin x \leq x \) if \( x \in [0, \frac{\pi}{2}] \). (Hint: Express \( \sin x \) as an integral.)
   (b) Prove that \( \sin x \leq x \) for all \( x \geq 0 \).
   (c) Prove that \( |\sin x| \leq |x| \) for all \( x \in \mathbb{R} \).

4. Compute the following integrals
   (a) \( \int_0^1 f(x) \, dx \) where \( f \) is a continuous function satisfying \( f(1 - x) = -f(x) \) for \( x \in [0, 1] \).
   (b) \( \int_1^5 f(x) \, dx \) if \( f \) is periodic of period 1 and we know \( \int_0^2 f(2x) \, dx = 3 \).

5. Suppose that \( f \) and \( g \) are monotonic functions on \([0, 2]\) such that \( f(x) \leq g(x) \) for all \( x \in [0, 1] \) and \( \int_0^2 f(x) \, dx > \int_0^2 g(x) \, dx \).
   (a) Show that there is a number \( c \in [1, 2] \) for which \( \int_c^1 f(x) - g(x) \, dx = 0 \).
   (b) Is there necessarily a point \( d \) where \( f(d) = g(d) \)? Either prove that there is, or give a counterexample.

6. The following inductive proof must be wrong, because it “proves” that all positive integers are equal. What is the error in the argument?

   **For positive integers \( a \) and \( b \), define**

   \[
   \max(a, b) = \begin{cases} 
       a & \text{if } a > b \\
       b & \text{if } b > a \\
       a = b & \text{if } a = b
   \end{cases}
   \]

   and let \( \Lambda_n \) be the statement “If \( \max(a, b) = n \) then \( a = b = n \)”.

   Let us prove that \( \Lambda_n \) holds for all \( n \geq 1 \), using induction:

   For the base case, observe that if \( n = 1 \) and \( a, b \) are positive integers with \( \max(a, b) = 1 \), then \( a = b = 1 \).

   For the inductive step, suppose \( \Lambda_n \) is true. Let \( a, b \) be positive integers with \( \max(a, b) = n + 1 \). Then set \( a' = a - 1 \), \( b' = b - 1 \). We have \( \max(a', b') = n \), so by the truth of \( \Lambda_n \) we know \( a' = b' = n \), from which \( a = b = n + 1 \). Thus \( \Lambda_{n+1} \) is true.

   Hence by induction \( \Lambda_n \) is true for all \( n \). But then any two positive integers \( a \) and \( b \) such that \( \max(a, b) = n \) satisfy \( a = b = n \), so \( a = b \) and so are equal to each other.