# Math 2110 Practice Final

Instructions: You may not refer to any notes or your textbook. No calculators are permitted. You have from 3:30pm until 5:30pm to complete the test.

1. Suppose that $f(x,y) = x^3y + xy^2 - y$, but $x(s,t) = s\cos t$ and $y(s,t) = ts$. Compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $(s,t) = (3,\pi)$.

**Solution:** We use the chain rule:

$$f_s = f_x x_s + f_y y_s,$$
$$f_t = f_x x_t + f_y y_t.$$

Now $f_x = 3x^2y + y^2$, and $f_y = x^3 + 2xy - 1$. At the point $(s,t) = (3,\pi)$ we have $x = 3\cos\pi = -3$ and $y = 3\pi$, so $f_x = 81\pi + 9\pi^2$ and $f_y = -28 - 18\pi$. Also $x_s = \cos t = -1$ at $t = \pi$, and $x_t = -s\sin t = -3\sin\pi = 0$. Similarly $y_s = t = \pi$ and $y_t = s = 3$. Substituting we get

$$f_s = f_x x_s + f_y y_s = (81\pi + 9\pi^2)(-1) + (-28 - 18\pi)(\pi) = -27\pi^2 - 109\pi,$$
$$f_t = f_x x_t + f_y y_t = (81\pi + 9\pi^2)(0) + (-28 - 18\pi)(3) = -54\pi - 84.$$

2. Set up two iterated integrals (in different orders) for the volume of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(1,4,0)$. **(DO NOT EVALUATE THESE INTEGRALS.)**

**Solution:** Several choices for the order are available. The object has 4 sides, each of which is a plane. We can get the equations of these planes by using that each plane contains 3 of the vertices. The planes are then

Contains $(0,0,0)$, $(1,0,0)$, $(1,0,1)$: Plane is $y = 0$ (the $x - z$ plane)
Contains $(0,0,0)$, $(1,0,0)$, $(1,4,0)$: Plane is $z = 0$ (the $x - y$ plane)
Contains $(1,0,0)$, $(1,0,1)$, $(1,4,0)$: Plane is $x = 1$
Contains $(0,0,0)$, $(1,0,1)$, $(1,4,0)$: Plane has normal $\langle -4, 1, 4\rangle$ so is -4x+y+4z=0

In the all cases we can find the above by computing the normal using the cross product of two vectors in the plane and using the equation $(r - r_0)\cdot n = 0$.

From the above we can make the outer integrals in the $x-z$ plane and therefore over the triangle with vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$. In this case the inner integral is over $y$, and runs from $y = 0$ to $y = 4x - 4z$ (from the last of the faces). The possibilities are

$$\int_0^1 \int_0^{1-x} \int_0^{4x-4z} dy\,dz\,dx$$
$$\int_0^1 \int_0^{1-z} \int_0^{4x-4z} dy\,dx\,dz$$

We could also make the outer integrals in the $x - y$ plane, over the triangle with vertices $(0,0,0)$, $(1,0,0)$, $(1,4,0)$. The inner integral would be with respect to $z$ from $z = 0$ to the last face $z = x - \frac{y}{4}$. This gives possibilities

$$\int_0^1 \int_0^{4x} \int_0^{x-y/4} dz\,dy\,dx$$
$$\int_0^4 \int_0^{x/4} \int_0^{x-y/4} dz\,dx\,dy$$

Finally, we could integrate over the $y-z$ plane. In this case the triangle we integrate over is obtained by projecting the third face above to the $y-z$ plane so has vertices $(0,0,0)$, $(0,0,1)$, $(0,4,0)$. The inner integral is with respect to $x$ from the fourth face $x = \frac{y}{4} + z$ to $x = 1$. This gives possibilities

$$\int_0^1 \int_0^{4-4z} \int_{y/4+z}^1 dx\,dy\,dz$$
$$\int_0^4 \int_0^{1-y/4} \int_{y/4+z}^1 dx\,dz\,dy$$

3. Find the distance from $(8,2,3)$ to the plane $x + 2y - 2z = 7$.

**Solution:** The normal to the plane is $\mathbf{n} = \langle 1,2,-2\rangle$. A point in the plane is $(7,0,0)$, so a vector from a point in the plane to $(8,2,3)$ is

$$\mathbf{v} = \langle 8,2,3\rangle - \langle 7,0,0\rangle = \langle 1,2,3\rangle.$$

Now we know that the scalar projection of $\mathbf{v}$ in the direction of $\mathbf{n}$ has magnitude the distance from the plane to the point. This scalar projection is

$$\mathrm{Comp}_{\mathbf{n}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}|} = \frac{1 + 2(2) - 2(3)}{\sqrt{9}} = \frac{-1}{3}$$

so the distance is $\frac{1}{3}$.

4. Compute

$$\iint_R \tfrac{3}{2} e^{(y^{3/2})} dA$$

where $R$ is the region between the curves $x = 0$, $y = x^2$ and $y = 4$. (Hint: The order of integration may affect whether you can do the integral!)

**Solution:** This region can be written as either type I or type II. As a type I it is $0 \le x \le 2$ and $x^2 \le y \le 4$. As a type II it is between $y = 0$ and $y = 4$ and has $0 \le x \le \sqrt{y}$. This gives the integrals

$$\int_0^2 \int_{x^2}^4 \tfrac{3}{2} e^{(y^{3/2})}\,dy\,dx$$
$$\int_0^4 \int_0^{\sqrt{y}} \tfrac{3}{2} e^{(y^{3/2})}\,dx\,dy$$

and we notice that we don't know how to do the innermost integral in the first case. In the second case we can do the innermost integral, and the outer one can be done by the substitution $u = y^{3/2}$, $du = \frac{3}{2}y^{1/2}dy$:

$$\int_0^4 \tfrac{3}{2}\sqrt{y}e^{(y^{3/2})}\,dy = \int_0^8 e^u\,du = \Big[e^u\Big]_0^8 = e^8 - 1$$

where the change in endpoints is as follows, $y = 0$ becomes $u = 0^{3/2} = 0$ and $y = 4$ becomes $u = 4^{3/2} = 2^3 = 8$.

5. Find all local maximum, minimum and saddle points of

$$f(x,y) = 4x^3 + xy^2 + 2x^2y - 9x$$

**Solution:** Compute critical points, where $\nabla f = 0$, by setting

$$0 = \frac{\partial f}{\partial x} = 12x^2 + y^2 + 4xy - 9,$$
$$0 = \frac{\partial f}{\partial y} = 2xy + 2x^2 = 2x(y+x)$$

From the second equation, either $x = 0$ or $y = -x$. Substituting $x = 0$ into the first equation gives $y^2 = 9$ so $y = \pm 3$. This gives points $(0,3)$ and $(0,-3)$. Substituting $y = -x$ into the first equation gives $12x^2 + x^2 - 4x^2 - 9 = 0$, so $9x^2 = 9$, so $x = \pm 1$. This gives points $(1,-1)$ and $(-1,1)$.

Next we need to test what sorts of points these are. We use the discriminant.

$$D(x,y) = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = (24x+4y)(2x) - (2y+4x)^2 = 48x^2 + 8xy - 4y^2 - 16xy - 16x^2 = 32x^2 - 8xy - 4y^2$$

Thus $D(0,3) = D(0,-3) = -36$, so both $(0,3)$ and $(0,-3)$ are saddle points. Also $D(1,-1) = D(-1,1) = 36$, and $f_{xx}(1,-1) = 20$ so there is a minimum at $(1,-1)$, and $f_{xx}(-1,1) = -20$ so there is a maximum at $(-1,1)$.

6. Evaluate

$$\int \left(y + 7e^{\sqrt{x}}\right) dx + \left(3x + 7\cos(y^2)\right) dy$$

over the positively oriented boundary of the region between $y = x^2$ and $y^2 = x$.

**Solution:** This integral looks ridiculously hard to do by parametrization, so we try by Green's theorem. According to this theorem,

$$\int \left(y + 7e^{\sqrt{x}}\right) dx + \left(3x + 7\cos(y^2)\right) dy = \iint_R (3-1)\, dA$$

where $R$ is the region between those curves. Now this region is type I with $0 \le x \le 1$ and $x^2 \le y \le \sqrt{x}$, so we have

$$\iint_R 2\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 2\, dy\, dx = \int_0^1 2\sqrt{x} - 2x^2\, dx = \left[\tfrac{4}{3}x^{3/2} - \tfrac{2}{3}x^3\right]_0^1 = \tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}.$$

7. Compute $\iiint_E \sqrt{x^2+y^2}\, dV$ where $E$ is the region between $z = 45 - 4x^2 - 4y^2$ and $z = x^2 + y^2$.

**Solution:** We can do this in any coordinate system, but it looks like it would be easiest in cylindrical polar coordinates, because then the region is between $z = 45 - 4r^2$ and $z = r^2$, and the integrand is $r = \sqrt{x^2+y^2}$. The intersection of these surfaces is at $5r^2 = 45$, so $r = 3$, and we see that the points in $E$ all have $r \le 3$, thus we may make the triple integral into a double integral over the disc of radius 3 and have $r^2 \le z \le 45 - 4r^2$. In cylindrical polars this is

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^3\int_{r^2}^{45-4r^2} r\,dz\,rdr\,d\theta$$
$$= \int_0^{2\pi}\int_0^3 r^2(45-5r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}\Big[15r^3 - r^5\Big]_0^3\,d\theta$$
$$= 2\pi\left(153^3 - 3^5\right) = 2\pi 3^4(5-3) = 4\pi 81 = 324\pi.$$

## Formula Sheet: Math 2110

- Projections of **b** onto **a**

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$
$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|}$$

- Cross product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Theorem on cross and dot products of vectors **a**, **b**, **c**.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

- Volume of parallelpiped determined by **a**, **b**, and **c** is $|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|$.
- Derivatives of dot and cross products

$$\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$
$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

- Curvature is given by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

- The binormal is $\mathbf{B} = \mathbf{T}\times\mathbf{N}$.
- Acceleration is $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where

$$a_T = \frac{dv}{dt} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$
$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

- The discriminant in the second derivative test for two-variable functions is given by

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$