Math 2110 Midterm 2

Instructions: You may not refer to any notes or your textbook. No calculators are permitted. You have from 10:00am until 10:50am to complete the test.

1. Find the limit or show it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - xy^2}{x^3 + 2y^3}$$

Solution: By trying a few lines through $(0,0)$ we can see it does not exist.

$$\text{On } x = 0: \quad \lim_{y\to 0} \frac{x^2y - xy^2}{x^3 + 2y^3} = \lim_{y\to 0} \frac{0}{2y^3} = 0$$

$$\text{On } y = -x: \quad \lim_{(x,-x)\to(0,0)} \frac{x^2(-x) - x(-x)^2}{x^3 + 2(-x)^3} = \lim_{x\to 0} \frac{-2x^3}{-x^3} = 2$$

2. Find the equation of the tangent plane to $2x^2y^5 + (zx)^2 - x^4yz = 16$ at the point $(2,0,2)$.

Solution: The surface is a level surface of $F(x,y,z) = 2x^2y^5 + (zx)^2 - x^4yz$, so ∇F is perpendicular to the surface and therefore to the tangent plane. We can then use the equation $n \cdot (r - r_0)$ for the plane with $r_0 = \langle 2,0,2\rangle$ and $n = \nabla F(r_0)$. Now

$$\nabla F(r_0) = \langle 4xy^5 + 2xz^2 - 4x^3yz, 10x^2y^4 - x^4z, 2zx^2 - x^4y\rangle = \langle 16, -32, 16\rangle = 16\,\langle 1, -2, 1\rangle$$

so the equation of the plane is

$$0 = \langle 1, -2, 1\rangle\Big(\langle x, y, z\rangle - \langle 2, 0, 2\rangle\Big) = x - 2 - 2y + z - 2 = x - 2y + z - 4$$

3. Find the maximum value of the z-coordinate on the curve obtained by intersecting $x^2 + y^2 = 5$ and $x - 2y + z = 4$.

Solution: We want to maximize $f(x,y,z) = z$ with $x^2+y^2 = 1$ and $x-2y+z = 4$. One way is to use Lagrange, which gives

$$\begin{aligned} 0 &= 2x\lambda + \mu \\ 0 &= 2y\lambda - 2\mu \\ 1 &= \mu \\ x^2 + y^2 &= 5 \\ x - 2y + z &= 4 \end{aligned}$$

using $\mu = 1$ and substituting into the other equations gives

$$\begin{aligned} 0 &= 2x\lambda + 1 \\ 0 &= 2y\lambda - 2 \\ x^2 + y^2 &= 1 \\ x - 2y + z &= 4 \end{aligned}$$

so that $\frac{-1}{2x} = \lambda = \frac{1}{y}$, so $y = -2x$. Substituting into $x^2 + y^2 = 5$ gives $5x^2 = 5$, so $x = 1$ and $y = -2$ or $x = -1$, $y = 2$. Finally from the last equation $z = 2y - x + 4$, so at $(x, y) = (1, -2)$ we get $z = 4 - 5 = -1$, and at $(x, y) = (-1, 2)$ we get $z = 5 + 4 = 9$. The second one has largest z value, $z = 9$.

4. You are walking in the hills and the height is described by a function

$$H(s,t) = \frac{2s + s^2 + 5}{1 + t^2}$$

   (a) You want to cross between the hills at the saddle point. What are the s-t coordinates of this point? What is the height at the saddle?

   (b) If you walk from the point $\left(0, 1, \frac{5}{2}\right)$ toward the point $\left(1, 2, \frac{8}{5}\right)$, what is the slope of the hill in the direction you are walking?

Solution:

(a) Find the critical points.

$$\nabla H = \left\langle \frac{2 + 2s}{1 + t^2}, \frac{-2t(2s + s^2 + 5)}{(1 + t^2)^2} \right\rangle$$

There are no points where ∇H does not exist. We get two equations from $\nabla H = \langle 0, 0 \rangle$. The first component gives $2 + 2s = 0$, so $s = -1$. The second gives $-2t(2s + s^2 + 5) = 0$, and since $s = -1$ it becomes $-8t = 0$, so $t = 0$. The critical point is at $(-1, 0)$.

We need to know that this critical point is a saddle, so we need to check if $H_{ss}H_{tt} - (H_{st})^2 < 0$. Computing at $(-1, 0)$,

$$\begin{aligned} H_{ss}(-1,0) &= \frac{2}{1 + t^2} = 2 \\ H_{st}(-1,0) &= \frac{-2t(2 + 2s)}{(1 + t^2)^2} = 0 \\ H_{tt}(-1,0) &= (2s + s^2 + 5)\left(\frac{-2}{(1 + t^2)^2} + \frac{(-2t)(-2)(2t)}{(1 + t^2)^3}\right) = 4(-2) = -8 \\ H_{ss}H_{tt} - (H_{st})^2 &= 2(-8) - 0 = -16 < 0 \end{aligned}$$

so the point is a saddle point. The height at the saddle is $H(-1, 0) = \frac{4}{1} = 4$.

(b) From $(0, 1)$ the direction to $(1, 2)$ is $\langle 1, 2 \rangle - \langle 0, 1 \rangle = \langle 1, 1 \rangle$. The slope in this direction is the directional derivative of the height, so is

$$D_{\langle 1,1 \rangle}H(0,1) = \nabla H(1,1) \cdot \frac{\langle 1, 1 \rangle}{|\langle 1, 1 \rangle|} = \left\langle \frac{2}{2}, \frac{(-2)(5)}{4} \right\rangle \cdot \frac{\langle 1, 1 \rangle}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \langle 2, -5 \rangle \cdot \langle 1, 1 \rangle = \frac{-3}{2\sqrt{2}}$$

Formula Sheet: Math 2110

- Projections of **b** onto **a**

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$
$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|}$$

- Cross product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Theorem on cross and dot products of vectors **a**, **b**, **c**.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

- Volume of parallelpiped determined by **a**, **b**, and **c** is $|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|$.
- Derivatives of dot and cross products

$$\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$
$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

- Curvature is given by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

- The binormal is $\mathbf{B} = \mathbf{T}\times\mathbf{N}$.
- Acceleration is $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where

$$a_T = \frac{dv}{dt} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$
$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$