Math 2110 Midterm 1

Instructions: You may not refer to any notes or your textbook. No calculators are permitted. You have from 10:00am until 10:50am to complete the test.

1. (a) Give a vector equation of the line containing the points $(1, -3, 2)$ and $(4, 1, 0)$
   (b) Find the intersection of the line with the plane $x + y + z = 10$
   (c) Is the point $(10, 9, -4)$ on this line?

Solution:

(a) Setting $\mathbf{r}_0 = \langle 1, -3, 2\rangle$ and $\mathbf{v} = \langle 4, 1, 0\rangle - \langle 1, -3, 2\rangle = \langle 3, 4, -2\rangle$, we have the vector equation

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 1, -3, 2\rangle + t\langle 3, 4, -2\rangle = \langle 3t + 1, 4t - 3, 2 - 2t\rangle$$

(b) At the intersection point both equations must be true. Thinking of $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ we can substitute the formulas for $x(t)$, $y(t)$ and $z(t)$ into the equation of the plane to find

$$10 = x(t) + y(t) + z(t) = 3t + 1 + 4t - 3 + 2 - 2t = 5t$$

so $t = 2$. Substituting back into the equation of the line gives the intersection point $\mathbf{r}(2) = \langle 7, 5, -2\rangle$.

(c) To check if the point is on the line we can find if there is a t value so $r(t) = (10, 9, -4)$. For this to happen we need all of the equations

$$\begin{aligned} 3t + 1 &= 10 \\ 4t - 3 &= 9 \\ 2 - 2t &= -4 \end{aligned}$$

to be true at the same t value. This occurs with $t = 3$, so the point is on the line. Alternatively you could check that the direction vector from $(1, -3, 2)$ to $(10, 9, -4)$ is $\langle 9, 12, -6\rangle$, which is equal $3\langle 3, 4, -2\rangle$, so is in the same direction as the line.

2. For the curve $\mathbf{r}(t) = \langle 2t^3, 1 - t^3, -2t^3\rangle$, $t \geq 0$:
   (a) Reparametrize the curve with respect to arc-length, starting at $t = 0$ and moving in direction of increasing t.
   (b) Find the distance along the curve from $(2, 0, -2)$ to $(16, -7, 16)$
   (c) Find the curvature of the curve at $(2, 0, -2)$
   (d) What sort of curve is this?
   (e) Find the normal plane to the curve at $(2, 0, -2)$.

Solution:

(a) We first need the arc-length function $s(t)$, which is $s(t) = \int_0^t |\mathbf{r}'(u)|\, dt$. Start by finding

$$\mathbf{r}'(t) = \langle 6t^2, -3t^2, -6t^2\rangle = 3t^2\langle 2, -1, -2\rangle$$

so $|\mathbf{r}'(t)| = 3t^2\sqrt{4 + 1 + 4} = 9t^2$. Then

$$s(t) = \int_0^t 9u^2\, du = 3t^3.$$

We can substitute directly $t^3 = \frac{s}{3}$ into $\mathbf{r}(t)$ to get the arc-length parametrization

$$\mathbf{r}(s) = \langle \frac{2s}{3}, 1 - \frac{s}{3}, \frac{-2s}{3} \rangle. \tag{1}$$

(b) First find the t values corresponding to the points. Since $(2, 0, -2)$ is $\mathbf{r}(1)$, and $(16, -7, -16)$ is $\mathbf{r}(2)$, the distance between them on the curve is

$$\int_1^2 |\mathbf{r}'(t)| = \int_1^2 9t^2\, dt = 3t^3\Big|_1^2 = 3(8-1) = 21$$

or you can use previous part to say that the distance from $t = 0$ to $t = 1$ points is $s(1) = 3$, while from $t = 0$ to $t = 2$ is $s(2) = 24$, so from $t = 1$ to $t = 2$ points is $s(2) - s(1) = 21$.

(c) To get the curvature we first need the unit tangent $\mathbf{T}(t)$. Using the previous parts

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 6t^2, -3t^2, -6t^2 \rangle}{9t^2} = \frac{1}{3}\langle 2, -1, -2 \rangle$$

for $t > 0$. In order to use $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ we compute $\mathbf{T}'(t)$. But from the previous expression for $\mathbf{T}(t)$ we see $\mathbf{T}(t)$ is constant so $\mathbf{T}'(t) = \langle 0, 0, 0 \rangle$ and $|\mathbf{T}'(t)| = 0$ for all t (except perhaps $t = 0$, where the formula for $\mathbf{T}(t)$ might not be valid). Now the point $(2, 0, -2)$ corresponds to $t = 1$, so $\kappa(1) = 0$ is the curvature there.

(d) The curve is a line. You can see this either by using the fact that its curvature is zero (it does not curve) or by seeing that the arc-length parametrization (1) is

$$\mathbf{r}(s) = \langle 0, 1, 0 \rangle + s\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \rangle = \mathbf{r}_0 + s\mathbf{v}$$

with $\mathbf{r}_0 = \langle 0, 1, 0 \rangle$ and $\mathbf{v} = \frac{1}{3}\langle 2, -1, -2 \rangle$.

(e) The normal plane to a curve is perpendicular to the tangent. From an earlier part, the tangent is $\frac{1}{3}\langle 2, -1, -2 \rangle$, which is parallel to $\mathbf{n} = \langle 2, -1, -2 \rangle$. The point $\mathbf{r}_1 = \langle 2, 0, -2 \rangle$ must be on the plane, so the equation of the plane is

$$0 = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = \langle 2, -1, -2 \rangle \cdot (\langle x, y, z \rangle - \langle 2, 0, -2 \rangle) = 2(x-2) - y - 2(z+2) = 2x - y - 2z - 8$$