Math 2110 Final

Instructions: You may not refer to any notes or your textbook. No calculators are permitted. You have from 3:30pm until 5:30pm to complete the test.

[5 points] 1. Find the equation of the tangent plane to the function

$$z = xy^3 - 2xy + x + y$$

at the point $(2, -1)$.

Solution: We use the equation for a tangent plane. First compute

$$f_x(2,-1) = y^3 - 2y + 1 = -1 + 2 + 1 = 2,$$
$$f_y(2,-1) = 3xy^2 - 2x + 1 = 3(2)(-1)^2 - 4 + 1 = 3.$$

The value of z at $(2, -1)$ is $z = 2(-1)^3 - 2(2)(-1) + 2 - 1 = 3$, so the equation of the tangent plane is

$$z = 2(x-2) + 3(y+1) + 3 = 2x + 3y + 2.$$

[5 points] 2. Given functions $f(x, y)$, $x(s, t)$ and $y(s, t)$, suppose that at some point we have $f_s = 2$, $f_t = 1$, $x_s = 3$, $x_t = 1$, $y_s = 5$, and $y_t = 2$. Find the values of f_x and f_y at this point.

Solution: From the chain rule, at the point in question:

$$2 = f_s = f_x x_s + f_y y_s = 3f_x + 5f_y,$$
$$1 = f_t = f_x x_t + f_y y_t = 1f_x + 2f_y.$$

The first equation minus 3 times the second equation gives $-1 = -f_y$, so $f_y = 1$. Substituting $f_y = 1$ in the first equation gives $2 = 3f_x + 5$, so $f_x = -3$, and $f_x = -1$.

[5 points] 3. Compute

$$\iint_R 10\sqrt{y}\,dA$$

where R is the region between the curves $y = 2x$, $y = 1$ and $y = x^2$.

Solution: The region is of type II, and is between $y = 0$ and $y = 1$ and has $\frac{y}{2} \leq x \leq \sqrt{y}$. Thus we compute

$$\begin{aligned}
\iint_R 10\sqrt{y}\,dA = \int_0^1 \int_{y/2}^{\sqrt{y}} 10\sqrt{y}\,dx\,dy &= \int_0^1 10\sqrt{y}\left(\sqrt{y} - \tfrac{y}{2}\right)dy \\
&= \int_0^1 10y - 5y^{3/2}\,dy \\
&= \left[5y^2 - 2y^{5/2}\right]_0^1 \\
&= 5 - 2 = 3.
\end{aligned}$$

[7 points] 4. Compute $\iiint_E 4(x^2+y^2+z^2)\,dV$ where E is the solid region above the $x-y$ plane and between the spheres of radius 1 and radius 2 around the origin in 3-dimensional space.

Solution: It is convenient to use spherical polar coordinates, in which $x^2+y^2+z^2=\rho^2$, and the region between the spheres is given simply by $1\le\rho\le 2$. The part of this region above the $x-y$ plane has $0\le\phi\le\frac{\pi}{2}$ and $0\le\theta\le 2\pi$.

Therefore the integral is

$$\begin{aligned}\int_0^{\pi/2}\int_0^{2\pi}\int_1^2 4\rho^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/2}\int_0^{2\pi}\tfrac{4}{5}\Big[\rho^5\Big]_1^2\,d\theta\,d\phi\\ &= \int_0^{\pi/2}\tfrac{124}{5}(2\pi)\sin\phi\,d\phi\\ &= \tfrac{248\pi}{5}\Big[-\cos\phi\Big]_0^{\pi/2}\\ &= \tfrac{248\pi}{5}(-0+1)=\tfrac{248\pi}{5}.\end{aligned}$$

[8 points] 5. Find the maximum and minimum values of

$$f(x,y)=x^3+xy^2+y$$

on the disc $x^2+y^2\le 1$.

Solution: We first look for critical points $\nabla f=0$, by setting

$$\begin{aligned}0&=\frac{\partial f}{\partial x}=3x^2+y^2,\\ 0&=\frac{\partial f}{\partial y}=2xy+1.\end{aligned}$$

From the first equation $x=y=0$, so the second equation can't be true, and there are no critical points. Next we restrict to the boundary, which is the circle of radius 1 and can be parameterized by $(\cos t,\sin t)$, $0\le t<2\pi$. Substituting this into f we get

$$f(\cos t,\sin t)=\cos^3 t+\cos t\sin^2 t+\sin t=\cos t(\cos^2 t+\sin^2 t)+\sin t=\cos t+\sin t.$$

To find the maximum and minimum we look at the derivative with respect to t, which is $-\sin t+\cos t$. It is zero when $\sin t=\cos t$, so $t=\pi/4$ or $t=5\pi/4$. This gives

$$\begin{aligned}f\Big(\cos\tfrac{\pi}{4},\sin\tfrac{\pi}{4}\Big)&=\tfrac{1}{\sqrt2}+\tfrac{1}{\sqrt2}=\sqrt2\\ f\Big(\cos\tfrac{5\pi}{4},\sin\tfrac{5\pi}{4}\Big)&=\tfrac{-1}{\sqrt2}+\tfrac{-1}{\sqrt2}=-\sqrt2\end{aligned}$$

[10 points] 6. Set up an integral in rectangular coordinates (x, y, and z, in an order of your choice) for the volume of the part of the region between the sphere $x^2+y^2+z^2=1$ and the ellipsoid $\frac{x^2}{4}+y^2+z^2=1$ that has $x\ge 0$.
(DO NOT EVALUATE THIS INTEGRAL.)

Solution: Only two choices of integration order give just one integral, and both have the dx integral in the innermost position. The two surfaces intersect when $x = 0$ and $y^2 + z^2 = 1$, so the outer integral is over the disc of radius 1 in the $y - z$ plane. The inner (x) integral has as its lower endpoint the hemispherical surface $x = \sqrt{1 - y^2 - z^2}$, and as its upper endpoint the part of the ellipsoid with $x \geq 0$, which is $x = 2\sqrt{1 - y^2 - z^2}$. Thus the two possible answers are

$$\int_{-1}^{1} \int_{y=-\sqrt{1-z^2}}^{y=\sqrt{1-z^2}} \int_{x=\sqrt{1-y^2-z^2}}^{x=2\sqrt{1-y^2-z^2}} dx\, dy\, dz$$

and

$$\int_{-1}^{1} \int_{z=-\sqrt{1-y^2}}^{z=\sqrt{1-y^2}} \int_{x=\sqrt{1-y^2-z^2}}^{x=2\sqrt{1-y^2-z^2}} dx\, dz\, dy.$$

[10 points] 7. Let $F(x, y) = \langle y^2 \sin(xy), xy \sin(xy) - \cos(xy)\rangle$. Compute $\int_C F.dr$ for the curve C which starts at $(0, 0)$, travels in a straight line to $(1, 1)$, and then travels in a straight line from $(1, 1)$ to $(0, 2)$.

Solution: It is possible to do this by parameterizing the curve, but it is a bit messy. A better way is to say $F = \langle P, Q\rangle$ for $P = y^2 \sin(xy)$ and $Q = xy \sin(xy) - \cos(xy)$, and to check that

$$\frac{\partial P}{\partial y} = 2y \sin(xy) + y^2 x \cos(xy)$$
$$\frac{\partial Q}{\partial x} = y \sin(xy) + xyy \cos(xy) + y \sin(xy) = 2y \sin(xy) + y^2 x \cos(xy)$$

so that the field F is conservative. Then we can either find a scalar function f such that $\nabla f = F$ or use the fact that it is path-independent to take a path from $(0, 0)$ to $(0, 2)$ on which the integral is simpler. A simpler choice of path is the straight line $r(t) = \langle 0, 2t\rangle$ with $0, \leq t \leq 1$. Then $r'(t) = \langle 0, 2\rangle$ and the integral is

$$\int_0^1 F(r(t)).r'(t)\, dt = \int_0^1 2\Big(0 \sin(0) - \cos(0)\Big)\, dt = \int_0^1 -2\, dt = -2.$$

If instead you choose to find f, then you would set

$$f(x, y) = \int y^2 \sin(xy)\, dx + g(y) = -y \cos(xy) + g(y)$$

and since $xy \sin(xy) - \cos(xy) = f_y = xy \sin(xy) - \cos(xy) + g'(y)$ it follows that $g(y)$ is a constant that may be taken to be zero. Then the integral is

$$\int_C F.dr = \int_C \nabla f.dr = f(0, 2) - f(0, 0) = -2 \cos(0) - 0 \cos(0) = -2.$$

Formula Sheet: Math 2110

- Projections of **b** onto **a**

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$
$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|}$$

- Cross product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Theorem on cross and dot products of vectors **a**, **b**, **c**.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

- Volume of parallelpiped determined by **a**, **b**, and **c** is $|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|$.
- Derivatives of dot and cross products

$$\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$
$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

- Curvature is given by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

- The binormal is $\mathbf{B} = \mathbf{T}\times\mathbf{N}$.
- Acceleration is $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where

$$a_T = \frac{dv}{dt} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$
$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

- The discriminant in the second derivative test for two-variable functions is given by

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

- Green's theorem: For a smooth positively oriented simple closed curve C enclosing a region D

$$\int_C P\,dx + Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$