

# UNIMODULAR FOURIER MULTIPLIERS FOR MODULATION SPACES

ÁRPÁD BÉNYI, KARLHEINZ GRÖCHENIG, KASSO A. OKOUDJOU\*,  
AND LUKE G. ROGERS

ABSTRACT. We investigate the boundedness of unimodular Fourier multipliers on modulation spaces. Surprisingly, the multipliers with general symbol  $e^{i|\xi|^\alpha}$ , where  $\alpha \in [0, 2]$ , are bounded on all modulation spaces, but, in general, fail to be bounded on the usual  $L^p$ -spaces. As a consequence, the phase-space concentration of the solutions to the free Schrödinger and wave equations are preserved. As a byproduct, we also obtain boundedness results on modulation spaces for singular multipliers  $|\xi|^{-\delta} \sin(|\xi|^\alpha)$  for  $0 \leq \delta \leq \alpha$ .

## 1. INTRODUCTION AND MOTIVATION

A Fourier multiplier is a linear operator  $H_\sigma$  whose action on a test function  $f$  on  $\mathbb{R}^d$  is formally defined by

$$(1) \quad H_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(\xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

The function  $\sigma$  is called the symbol or multiplier. Using the inverse Fourier transform, one can also rewrite the operator as a convolution operator

$$H_\sigma(x) = \check{\sigma} * f(x)$$

where  $\check{\sigma}$  is the (distributional) inverse Fourier transform of  $\sigma$ .

Fourier multipliers arise naturally in the formal solution of linear PDEs with constant coefficients and in the convergence of Fourier series. For this reason, a fundamental problem in the study of Fourier multipliers is to relate the boundedness properties of  $H_\sigma$  on certain function spaces to properties of the symbol  $\sigma$ . On  $L^2$ ,  $L^1$  and  $L^\infty$  this is relatively straightforward, while the full resolution of this problem for general  $L^p$ -spaces is an analytic gem known as the Hörmander-Mihlin multiplier theorem [17, 20]. A detailed exposition of the theory of Fourier multipliers may be found in [12, 25].

In this paper we study unimodular Fourier multipliers, in particular the multipliers with symbol  $e^{it|\xi|^\alpha}$  for  $t \in \mathbb{R}$  and  $\alpha \in [0, 2]$ . To understand the problem, consider first a general unimodular multiplier  $\sigma(\xi) = e^{im(\xi)}$  for some real-valued function  $m$ .

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If  $|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-\alpha}$  for all  $\xi \neq 0$  and  $|\alpha| \leq [d/2] + 1$ , then  $e^{im}$  is bounded on  $L^p$  by Mihlin's condition, see [12, p. 367]. The case of the unimodular function  $e^{i|\xi|^\alpha}$  is more complicated. In addition to the singularity of the derivatives at the origin, the multiplier has large oscillations at infinity and thus possesses large derivatives. If  $\alpha > 1$ , this excludes the application of the multiplier theorems of Hörmander-Mihlin and their variations. Indeed, the multiplier may be unbounded. Specifically, if  $\alpha > 1$ , then the operator  $H_{e^{i|\xi|^\alpha}}$  is bounded on  $L^p(\mathbb{R}^d)$  if and only if  $p = 2$  as a consequence of [17, 18].

The cases  $\alpha = 1$  and  $\alpha = 2$  are particularly interesting and have been studied intensively in PDE, because they occur in the time evolution of the wave equation ( $\alpha = 1$ ) and the free Schrödinger operator ( $\alpha = 2$ ). The unboundedness of the multiplier on general  $L^p$  means that  $L^p$ -properties of the initial conditions are not preserved by the time evolution. There is an extensive literature about estimates on the wave operator; see, for example, [2, 19, 22, 29]. Little is known for  $\alpha \in (0, 1)$ , and many results are concerned almost exclusively with appropriate corrections of the symbol  $e^{i|\xi|^\alpha}$  with functions that are essentially smooth away from the origin, [16, 22, 31].

In view of the unboundedness of the multiplier  $e^{i|\xi|^\alpha}$  on  $L^p$  it is natural to question whether the  $L^p$ -spaces are really the appropriate function spaces for the study and understanding of these operators. We suggest that they are not, and propose that the so-called *modulation spaces* are a good alternative class for the study of unimodular Fourier multipliers.

To define the modulation spaces we fix a non-zero Schwartz function  $g$  and consider the short-time Fourier transform  $V_g f$  of a function  $f$  with respect to  $g$

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i \omega \cdot t} \overline{g(t-x)} f(t) dt.$$

The modulation space  $\mathcal{M}^{p,q}$  is the closure of the Schwartz class with respect to the norm

$$\|f\|_{\mathcal{M}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q}$$

(with appropriate modifications when  $p = \infty$  or  $q = \infty$ ). A priori it may seem that this norm depends on  $g$ , so it is worth noting that different choices of  $g$  give equivalent norms.

Since their introduction by Feichtinger [5], modulation spaces have become canonical for both time-frequency and phase-space analysis. Their many applications are surveyed in [7]; for the special case of  $\mathcal{M}^{\infty,1}$ , which is sometimes called the Sjöstrand class, see also [14, 15, 23, 30] and the references therein.

The reason for the ubiquity of modulation spaces is essentially that  $V_g f$  is a local version of the Fourier transform. In the terminology of physics, if  $x$  is the position and  $\omega$  the momentum of a physical state, then  $V_g f(x, \omega)$  is a measure of the amplitude of a state  $f$  at the point  $(x, \omega)$  in phase space. The modulation space norm  $\|\cdot\|_{\mathcal{M}^{p,q}}$  can then be understood as a measure for the phase space concentration of  $f$ . In this interpretation, boundedness of a Fourier multiplier on modulation spaces expresses

the conservation of phase-space properties, which is the natural extension of the energy conservation corresponding to the obvious  $L^2$ -boundedness.

An abstract characterization of all Fourier multipliers on modulation spaces was obtained in [10] (see also Theorem 3 below), however this characterization requires a deep understanding of Fourier multipliers on  $L^p$ -spaces and it is often very difficult to check that the given conditions are satisfied for a given symbol. One exception is the case where the multiplier has sufficiently many bounded derivatives ([10] Theorem 20) where the Hörmander-Mihlin theorem can be applied locally. We note that this result is not applicable to the multipliers  $e^{it|\xi|^\alpha}$ .

Our main result is that the unimodular multipliers discussed above are bounded on all modulation spaces.

**Theorem 1.** *If  $\alpha \in [0, 2]$ , then the Fourier multiplier  $H_{\sigma_\alpha}$  with  $\sigma_\alpha(\xi) = e^{i|\xi|^\alpha}$  is bounded from  $\mathcal{M}^{p,q}(\mathbb{R}^d)$  into  $\mathcal{M}^{p,q}(\mathbb{R}^d)$  for all  $1 \leq p, q \leq \infty$  and in any dimension  $d \geq 1$ .*

As a consequence, we will show that the Cauchy problems for the free Schrödinger equation and the wave equation with initial data in a modulation space satisfy an analog of the principle of conservation of energy. It is worth noting that modulation spaces were recently rediscovered and many of their properties reproved in [1] in a study of the non-linear Schrödinger equation and the Ginzburg-Landau equation. In particular, Corollary 1(a) was obtained in [1] by a different method than that used here.

**Corollary 1.** (a) *Let  $u(x, t)$  be the solution of the free Schrödinger equation  $iu_t = \Delta_x u$  and  $u(x, 0) = f(x)$ . If  $f \in \mathcal{M}^{p,q}$ , then  $u(\cdot, t) \in \mathcal{M}^{p,q}$  for all  $t > 0$ .*

(b) *Let  $u(x, t)$  be the solution of the wave equation  $iu_{tt} = \Delta_x u$  and  $u(x, 0) = f(x)$ ,  $u_t(x, 0) = g(x)$ . If  $f, g \in \mathcal{M}^{p,q}$ , then  $u(\cdot, t) \in \mathcal{M}^{p,q}$  for all  $t > 0$ .*

To put it more succinctly, we may say that *the phase-space concentration of an initial state is preserved under the time evolution of the free Schrödinger equation and the wave equation.* This result is in striking contrast to the behavior of these Cauchy problems with initial data in  $L^p$  spaces [17, 19, 27, 28, 29].

Our paper is organized as follows. In Section 2 we set up the notation and define the modulation spaces and some amalgam spaces needed for the multiplier theory. Section 3 is devoted to the abstract characterization of Fourier multipliers of modulation spaces given in [10]. Furthermore, several sufficient conditions for a Fourier multiplier to be bounded on the modulation spaces are provided. These conditions are then used to prove our main results, which are stated and proved in Section 4. Finally, Section 5 deals with the applications of our results to the analysis of the Cauchy problems associated to the Schrödinger and wave equations.

## 2. THE SHORT-TIME FOURIER TRANSFORM AND ASSOCIATED FUNCTION SPACES

**2.1. General notation.** For  $x, \omega \in \mathbb{R}^d$  the translation and modulation operators acting on a function  $f$  defined over  $\mathbb{R}^d$  are given respectively by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t),$$

where  $t \in \mathbb{R}^d$ . The Schwartz class of test functions will be denoted by  $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ , its dual is the space of tempered distributions  $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$  on  $\mathbb{R}^d$ . The Fourier transform of  $f \in \mathcal{S}$  is given by

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot \omega} dt, \quad \omega \in \mathbb{R}^d,$$

which is an isomorphism of the Schwartz space  $\mathcal{S}$  onto itself that extends to the tempered distributions by duality. The inverse Fourier transform is given explicitly by  $\check{f}(x) = \int_{\mathbb{R}^d} f(\omega) e^{2\pi i x \cdot \omega} d\omega$ , and we have  $(\hat{f})^\check{ } = f$ . The inner product of two functions  $f, g \in L^2$  is  $\langle f, g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t)} dt$ , and its extension to  $\mathcal{S}' \times \mathcal{S}$  will be also denoted by  $\langle \cdot, \cdot \rangle$ .

A key object in time-frequency analysis is the short-time Fourier transform (STFT), which in a sense is a ‘‘local’’ Fourier transform that has the advantage of displaying the frequency content of any function during various time intervals. More precisely, the STFT of a tempered distribution  $f \in \mathcal{S}'$  with respect to a window  $g \neq 0 \in \mathcal{S}$  is

$$(2) \quad V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i \omega \cdot t} \overline{g(t-x)} f(t) dt,$$

(where the integral version exists only for functions of polynomial growth). We will consistently use the following equivalent forms for the STFT

$$(3) \quad V_g f(x, \omega) = \widehat{f \cdot T_x \overline{g}}(\omega) = e^{-2\pi i x \cdot \omega} V_{\check{g}} \hat{f}(\omega, -x).$$

If  $g \in \mathcal{S}$  and  $f \in \mathcal{S}'$ , then  $V_g f$  is a continuous function of polynomial growth [13]. In a less obvious way, the STFT can be defined even when both  $f \in \mathcal{S}'(\mathbb{R}^d)$  and  $g \in \mathcal{S}'(\mathbb{R}^d)$  [11, Prop. 1.42].

The time-frequency content of a tempered distribution can be quantified by imposing a mixed-norm on its STFT. Throughout the paper, we let  $L^{p,q} = L^{p,q}(\mathbb{R}^d \times \mathbb{R}^d)$  be the spaces of measurable functions  $f(x, \omega)$  for which the mixed norm

$$\|f\|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q}$$

is finite. If  $p = q$ , we have  $L^{p,p} = L^p$ , the usual Lebesgue spaces.

We use the notation  $u \lesssim v$  to denote  $u \leq cv$ , for a universal (independent of  $u$  and  $v$ ) positive constant  $c$ . Similarly, we use the notation  $u \asymp v$  to denote  $cu \leq v \leq Cu$ , for some universal positive constants  $c, C$ .

## 2.2. Modulation spaces.

**Definition 1.** Given  $1 \leq p, q \leq \infty$ , and given a non-zero window function  $g \in \mathcal{S}$ , the modulation space  $\mathcal{M}^{p,q} = \mathcal{M}^{p,q}(\mathbb{R}^d)$  is the space of all distributions  $f \in \mathcal{S}'$  for which the following norm is finite:

$$(4) \quad \|f\|_{\mathcal{M}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q} = \|V_g f\|_{L^{p,q}},$$

with the usual modifications if  $p$  and/or  $q$  are infinite. When  $p = q$ , we will write  $\mathcal{M}^p$  for the modulation space  $\mathcal{M}^{p,p}$ .

This definition is independent of the choice of the window  $g$  in the sense of equivalent norms. Moreover, if  $1 \leq p, q < \infty$ , then  $\mathcal{M}^1$  is densely embedded into  $\mathcal{M}^{p,q}$ , as is the Schwartz class  $\mathcal{S}$ . If  $1 \leq p, q < \infty$ , then the dual of  $\mathcal{M}^{p,q}$  is  $\mathcal{M}^{p',q'}$ , where  $1 \leq p, q < \infty$  and  $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ . We refer to [5, 13] and the references therein for the precise details and the rich theory of modulation spaces.

Note that an application of Plancherel's theorem yields  $\mathcal{M}^2 = L^2$ . However, it can be shown that for  $p, q \neq 2$ ,  $\mathcal{M}^{p,q}$  does not coincide with any Lebesgue space. Instead, one may use the embeddings  $\mathcal{M}^p \subset L^p \subset \mathcal{M}^{p,p'}$ , if  $1 \leq p \leq 2$  and  $\mathcal{M}^{p,p'} \subset L^p \subseteq \mathcal{M}^p$  for  $p \geq 2$ . We will also use the fact that modulation spaces are invariant under dilation, i.e., if  $f \in \mathcal{M}^{p,q}$ , the  $f(t \cdot) \in \mathcal{M}^{p,q}$  for every  $t > 0$ , see for instance [13, Ch. 9].

**2.3. Wiener Amalgam Spaces.** If we reverse the order of integration in (4), then we obtain the Wiener amalgam spaces [6]. Let us write  $\mathcal{FL}^1$  for the space of all Fourier transforms of  $L^1$ , that is

$$\mathcal{FL}^1 = \{f \in L^\infty(\mathbb{R}^d) : \hat{f} \in L^1(\mathbb{R}^d)\},$$

with norm  $\|f\|_{\mathcal{FL}^1} = \|\hat{f}\|_{L^1}$ . This allows us to define an amalgam space that will be used frequently.

**Definition 2.** Fix  $g \in \mathcal{S}(\mathbb{R}^d)$ ,  $g \neq 0$ . Then the space  $W(\mathcal{FL}^1, \ell^\infty)$  consists of all functions  $\sigma \in L^\infty(\mathbb{R}^d)$  for which

$$\begin{aligned} \|\sigma\|_{W(\mathcal{FL}^1, \ell^\infty)} &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |V_g \sigma(x, \omega)| d\omega \\ (5) \qquad \qquad \qquad &= \sup_x \|\sigma T_x g\|_{\mathcal{FL}^1} < \infty. \end{aligned}$$

Roughly speaking,  $\sigma$  is in  $W(\mathcal{FL}^1, \ell^\infty)$  if  $\sigma$  is locally in the Fourier algebra  $\mathcal{FL}^1$  with the local norms being uniformly bounded. As in the case of modulation spaces, the definition of  $W(\mathcal{FL}^1, \ell^\infty)$  is independent of the test function  $g \in \mathcal{S}$ . We refer to [6] and the references therein for more details on these Wiener amalgam-type spaces.

For the abstract multiplier theorem of Feichtinger-Narimani (Theorem 3 below) we need one other Wiener amalgam space.

**Definition 3.** For fixed  $g \in \mathcal{S}(\mathbb{R}^d)$ ,  $g \neq 0$  the space  $W(M_{\mathcal{F}}(L^p), \ell^\infty)$  consists of all tempered distributions  $\sigma$  for which

$$(6) \qquad \qquad \qquad \|\sigma\|_{W(M_{\mathcal{F}}(L^p), \ell^\infty)} = \sup_{x \in \mathbb{R}^d} \|H_{\sigma T_x g}\|_{L^p \rightarrow L^p} < \infty.$$

In particular, each  $\sigma \in W(M_{\mathcal{F}}(L^p), \ell^\infty)$  coincides locally with a Fourier multiplier on  $L^p$ .

### 3. ABSTRACT MULTIPLIER THEOREMS ON MODULATION SPACES

The results of this paper were in part inspired by earlier work of three of the authors and Loukas Grafakos [3], in which an analogue of the classical Marcinkiewicz multiplier theorem was proven in the modulation space context. By an easy tensor

product argument, the proof of Theorem 1 of [3] (see also Corollary 19 of [10]) may be extended to show the following.

**Theorem 2.** *For any  $b = (b_1, b_2, \dots, b_d) \in (\mathbb{R}_+)^d$  with all  $b_j > 0$ , let  $Q_b = \prod_{j=1}^d (0, b_j)$ . For a bounded sequence  $\mathbf{c} = (c_n)_{n \in \mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d)$ , define the function*

$$\sigma_{b,\mathbf{c}} = \sum_{n \in \mathbb{Z}^d} c_n \chi_{n+Q_b},$$

where  $\chi_E$  is the indicator function of the set  $E$ . Then the operators  $H_{\sigma_{b,\mathbf{c}}}$  are bounded from  $\mathcal{M}^{p,q}$  into  $\mathcal{M}^{p,q}$ ,  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ , with a norm estimate

$$\|H_{\sigma_{b,\mathbf{c}}} f\|_{\mathcal{M}^{p,q}} \leq C(b, p, q) \|\mathbf{c}\|_{\ell^\infty} \|f\|_{\mathcal{M}^{p,q}}.$$

This theorem gives a concrete example of a multiplier that is not bounded on  $L^p$  for  $p \neq 2$  except in trivial cases. To see that it is a version of the usual Marcinkiewicz theorem, the reader should note that the proof of the one-dimensional case in [3] uses only that  $\sigma$  is bounded and of bounded variation, and that there is a uniform bound  $\int_{nb}^{(n+1)b} |d\sigma| \leq C$ ,  $n \in \mathbb{Z}$ , on the variation on  $b$ -length intervals. Careful examination of the proof also shows that the boundedness on modulation spaces is related to the localization of the multiplier in time and frequency. It was this idea that led to Theorem 5 below.

A key feature of modulation spaces that permits boundedness of a class of multipliers larger than that for  $L^p$  is the fact that multipliers for different locations act approximately independently. Feichtinger and Narimani [10] made this idea precise to give an abstract characterization of all Fourier multipliers on modulation spaces.

**Theorem 3** ([10] Theorem 17(1)). *A multiplier is bounded on  $\mathcal{M}^{p,q}$  if and only if  $\sigma \in W(M_{\mathcal{F}}(L^p), \ell^\infty)$ .*

Since this characterization rests on understanding all Fourier multipliers on  $L^p$  it is difficult to apply to a concrete function. Note, however, that Theorem 20 of [10] shows that a multiplier with  $\lfloor d/2 \rfloor + 1$  bounded derivatives on  $\mathbb{R}^d$  is bounded on the modulation spaces. This result could be used to prove that part of Theorem 1 which is established in Theorem 5 in the case  $\alpha < 1$ , but is not useful in general for the multipliers  $e^{it|\xi|^\alpha}$  for  $\alpha \in [0, 2]$  because there is a singularity at the origin and there are high frequencies (unbounded derivatives) far from the origin when  $\alpha > 1$ .

As an alternative to the abstractness of Theorem 3, we offer several sufficient conditions that are easier to verify in practice. In particular we will see in Theorem 5 that they are valid when the multiplier is well localized in time-frequency space. We remark that the conditions in the next result are far from being necessary.

**Lemma 1.** *The Fourier multiplier  $H_\sigma$  is bounded on all modulation spaces  $\mathcal{M}^{p,q}(\mathbb{R}^d)$  for  $d \geq 1$  and  $1 \leq p, q \leq \infty$  under each of the following conditions:*

- (i)  $\sigma \in W(\mathcal{FL}^1, \ell^\infty)$ .
- (ii)  $\sigma \in \mathcal{M}^{\infty,1}$ .
- (iii)  $\sigma \in \mathcal{FL}^1$ .

*Proof.* (i) Since  $\mathcal{FL}^1 \subseteq M_{\mathcal{F}}(L^p)$ , we have  $W(\mathcal{FL}^1, \ell^\infty) \subset W(M_{\mathcal{F}}(L^p), \ell^\infty)$  and the claim follows from Theorem 3. However, to keep the presentation self-contained, we sketch a simple and direct proof that is based on the convolution relations in [4].

Let  $g^*(x) = \overline{g(-x)}$ . Then we can write the modulus of the STFT as  $|V_g f(x, \omega)| = |f * M_\omega g^*(x)|$  and the modulation space norm as

$$\|f\|_{\mathcal{M}^{p,q}} = \left( \int_{\mathbb{R}^d} \|f * M_\omega g^*\|_{L^p}^q d\omega \right)^{1/q}.$$

Likewise

$$\begin{aligned} \|\sigma\|_{W(\mathcal{FL}^1, \ell^\infty)} &\asymp \sup_{\omega} \int_{\mathbb{R}^d} |\langle \sigma, M_{-x} T_\omega \hat{g} \rangle| dx \\ &= \sup_{\omega} \int_{\mathbb{R}^d} |\langle \check{\sigma}, T_x M_\omega g \rangle| dx = \sup_{\omega} \|\check{\sigma} * M_\omega g^*\|_{L^1}. \end{aligned}$$

Now we choose a window that factors as  $g_1 = g * g$  for some  $g \in \mathcal{S}$  and observe that  $M_\omega g_1 = M_\omega g * M_\omega g$ . The boundedness of  $H_\sigma$  for  $\sigma \in W(\mathcal{FL}^1, \ell^\infty)$  now follows from the following chain of inequalities, where we use repeatedly that the modulation space norm is independent of the window.

$$\begin{aligned} \|H_\sigma f\|_{\mathcal{M}^{p,q}}^q &= \int_{\mathbb{R}^d} \|(H_\sigma f) * M_\omega g_1^*\|_{L^p}^q d\omega \\ &= \int_{\mathbb{R}^d} \|\check{\sigma} * f * M_\omega g^* * M_\omega g^*\|_{L^p}^q d\omega \\ &\leq \int_{\mathbb{R}^d} \|\check{\sigma} * M_\omega g^*\|_{L^1}^q \|f * M_\omega g^*\|_{L^p}^q d\omega \\ &\leq \sup_{\omega} \|\check{\sigma} * M_\omega g^*\|_{L^1}^q \int_{\mathbb{R}^d} \|f * M_\omega g^*\|_{L^p}^q d\omega \\ &\asymp \|\sigma\|_{W(\mathcal{FL}^1, \ell^\infty)}^q \|f\|_{\mathcal{M}^{p,q}}^q. \end{aligned}$$

Statements (ii) and (iii) now follow from the embeddings  $\mathcal{FL}^1 \subset \mathcal{M}^{\infty,1} \subset W(\mathcal{FL}^1, \ell^\infty)$ . For the first embedding, if we let  $g \in \mathcal{S}$ ,  $g \neq 0$ , and  $f \in \mathcal{FL}^1$ , then by (3)

$$\begin{aligned} \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |V_g f(x, \omega)| d\omega &= \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |V_{\hat{g}} \hat{f}(\omega, -x)| d\omega \\ &= \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |\widehat{f T_\omega \hat{g}}(-x)| d\omega \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{f}(y) \overline{T_\omega \hat{g}(y)}| dy d\omega \\ &\leq \int_{\mathbb{R}^d} |\hat{f}| * |\hat{g}|(\omega) d\omega \\ &\leq \|\hat{f}\|_{L^1} \|\hat{g}\|_{L^1}. \end{aligned}$$

The second embedding is even easier:

$$\begin{aligned} \|f\|_{W(\mathcal{FL}^1, \ell^\infty)} &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |V_g f(x, \omega)| d\omega \\ &\leq \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |V_g f(x, \omega)| d\omega = \|f\|_{\mathcal{M}^{\infty,1}}. \end{aligned}$$

□

#### 4. UNIMODULAR FUNCTIONS AS FOURIER MULTIPLIERS

The primary goal of this section is to prove Theorem 1, showing that the multipliers  $e^{i|\xi|^\alpha}$  are bounded on all modulation spaces for  $\alpha \in [0, 2]$ . Except in a few special cases these multipliers fail to be bounded on  $L^p, p \neq 2$ .

As earlier noted, there are two main obstacles to overcome, namely the singularity at  $\xi = 0$  (except when  $\alpha$  is an even integer) and large oscillations at infinity (for  $\alpha > 1$ ) which give large derivatives and preclude application of the multiplier theorems of Hörmander-Mihlin and their variations. For clarity we will treat the singularity at the origin separately from the oscillations at infinity.

Our methods will also establish a stronger time-frequency property of the multipliers when  $\alpha \in [0, 1]$ , and will show that the singular multipliers  $\mathfrak{S}(e^{i|\xi|^\alpha} |\xi|^{-\delta}) = |\xi|^{-\delta} \sin |\xi|^\alpha$ ,  $\alpha \geq \delta \geq 0$ , are bounded on the modulation spaces.

In what follows,  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  will denote a test function such that

$$(7) \quad \chi(\xi) = \begin{cases} 0 & \text{if } |\xi| \geq 2 \\ 1 & \text{if } |\xi| \leq 1 \\ 0 \leq \chi(\xi) \leq 1 & \text{if } 1 \leq |\xi| \leq 2. \end{cases}$$

**4.1. The Singularity at the Origin.** For the purposes of establishing boundedness, the relevant feature of the singularity at the origin is its homogeneity.

**Theorem 4.** *Assume that  $\mu \in \mathcal{C}^{d+1}(\mathbb{R}^d \setminus \{0\})$  is homogeneous of order  $\alpha > 0$ , that is,  $\mu(s\xi) = s^\alpha \mu(\xi)$  for all  $s > 0$  and all  $\xi \neq 0$ . Then  $e^{i\mu} \chi \in \mathcal{FL}^1$ , and consequently  $H_{e^{i\mu} \chi}$  is bounded on all modulation spaces  $\mathcal{M}^{p,q}(\mathbb{R}^d)$ , as well as on all Lebesgue spaces  $L^p(\mathbb{R}^d)$  for  $1 \leq p, q \leq \infty$ .*

*Proof.* We expand  $e^{i\mu(\xi)} \chi$  as

$$e^{i\mu(\xi)} \chi(\xi) = \sum_{k=0}^{\infty} \frac{i^k}{k!} \mu(\xi)^k \chi(\xi)$$

and will show that  $\phi_k = \mu^k \chi \in \mathcal{FL}^1$  for all  $k \in \mathbb{N}$  with norm estimates sufficient to ensure convergence. To do so, define  $\psi(\xi) = \chi(\xi/2) - \chi(\xi)$ , so that  $\text{supp}(\psi) \subset \{\xi \in \mathbb{R}^d : 1 \leq |\xi| \leq 4\}$  and  $\sum_{j=1}^{\infty} \psi(2^j \xi) = \chi(\xi)$  for all  $\xi \neq 0$ . Using this and the homogeneity of  $\mu$  we further decompose  $\phi_k$  as

$$(8) \quad \phi_k(\xi) = \sum_{j=1}^{\infty} \mu(\xi)^k \psi(2^j \xi) = \sum_{j=1}^{\infty} 2^{-kj\alpha} \mu(2^j \xi)^k \psi(2^j \xi) = \sum_{j=1}^{\infty} 2^{-kj\alpha} \psi_k(2^j \xi)$$



where  $\psi_k = \mu^k \psi \in \mathcal{C}^{d+1}$  and has compact support. The Fourier transform of  $\psi_k(2^j \xi)$  is  $2^{-jd} \widehat{\psi}_k(2^{-j} \xi)$ , so that  $\|\psi_k(2^j \cdot)\|_{\mathcal{FL}^1} = \|\widehat{\psi}_k\|_{L^1} = \|\psi_k\|_{\mathcal{FL}^1}$  is independent of  $j$ . We estimate it in two parts

$$\|\widehat{\psi}_k\|_{L^1} = \int_{|\omega| \leq 1} |\widehat{\psi}_k(\omega)| d\omega + \int_{|\omega| \geq 1} |\widehat{\psi}_k(\omega)| d\omega = A + B.$$

To estimate  $A$ , observe that homogeneity of  $\mu$  ensures there is a constant  $C_0$  such that  $\mu^k$  is bounded by  $C_0^k 4^{k\alpha}$  on the support of  $\psi$ , and compute

$$(9) \quad A \leq v_d \|\widehat{\psi}_k\|_{L^\infty} \leq v_d \sup_{|\xi| \leq 4} |\mu(\xi)|^k \|\psi\|_{L^1} \leq v_d C_0^k 4^{k\alpha} \|\psi\|_{L^1} \lesssim C_0^k 4^{k\alpha}$$

where  $v_d$  is the volume of the unit ball in  $\mathbb{R}^d$ .

The estimate for  $B$  is a little more involved. Since  $\widehat{\psi}_k(\omega) = (2\pi i \omega)^{-\beta} \widehat{\partial^\beta \psi_k}(\omega)$  for all  $|\beta| \leq d+1$  we may use the pointwise estimate

$$\begin{aligned} |\widehat{\psi}_k(\omega)| &\leq \min_{|\beta| \leq d+1} |(2\pi i \omega)^{-\beta}| |\widehat{\partial^\beta \psi_k}(\omega)| \\ &\lesssim \max_{|\beta| \leq d+1} \|\partial^\beta \psi_k\|_{L^\infty} \min_{|\beta| \leq d+1} |\omega^{-\beta}|. \end{aligned}$$

Then we have

$$(10) \quad B \leq \max_{|\beta| \leq d+1} \|\partial^\beta \psi_k\|_{L^\infty} \int_{|\omega| \geq 1} \frac{1}{|\omega^\beta|} d\omega \leq \max_{|\beta| \leq d+1} \|\partial^\beta \psi_k\|_{L^1} \int_{|\omega| \geq 1} \min_{|\beta| \leq d+1} \frac{1}{|\omega^\beta|} d\omega.$$

By Leibniz's rule, the derivative  $\partial^\beta(\mu^k \psi)$  is a sum of  $\binom{k+|\beta|}{k}$  terms of the form  $(\partial^{\gamma_{k+1}} \psi) \prod_{j=1}^k \partial^{\gamma_j} \mu$  with  $\sum_{j=1}^{k+1} \gamma_j = \beta$ . Each of those involving  $\mu$  may be estimated using homogeneity, since

$$\partial^{\gamma_j} \mu(\xi) = s^{-\alpha} s^{|\gamma_j|} (\partial^{\gamma_j} \mu)(s\xi) \quad \forall s > 0, \xi \neq 0.$$

In particular we see that when  $|\gamma_j| \leq d+1$  and  $1 \leq |\xi| \leq 4$

$$|\partial^{\gamma_j} \mu^k(\xi)| \leq 4^{\alpha-|\gamma_j|} \sup_{|\xi|=1} |\partial^{\gamma_j} \mu(\xi)| \leq C_1 4^{\alpha-|\gamma_j|}$$

where  $C_1$  is a bound for the first  $d+1$  partial derivatives of  $\mu$  on the unit sphere. Such a bound exists because  $\mu \in \mathcal{C}^{d+1}(\mathbb{R}^d \setminus 0)$ . The derivatives of  $\psi$  are clearly bounded, so each term of  $\partial^\beta(\mu^k \psi)$  satisfies  $|\partial^{\gamma_{k+1}} \psi| \prod_{j=1}^k |\partial^{\gamma_j} \mu| \lesssim 4^{k\alpha}$ . Crudely estimating the number of these by  $\binom{k+|\beta|}{k} \leq C_2^k$  for a sufficiently large  $C_2 = C_2(d)$  we arrive at the bound  $\max_{|\beta| \leq d+1} \|\partial^\beta \psi_k\|_{L^1} \lesssim 4^{k\alpha} C_2^k$ .

It is not difficult to show that  $\min_{|\beta| \leq d+1} |\omega^\beta|^{-1}$  is integrable outside the unit sphere (see, e.g., [13, pp. 321]), so it follows from (10) that  $B \lesssim C^k 4^{k\alpha}$ . Combining this with (9) we obtain

$$\|\widehat{\psi}_k\|_{L^1} \lesssim C^k 4^{k\alpha}$$

and therefore from (8),

$$\|\phi_k\|_{\mathcal{FL}^1} = \|\widehat{\phi}_k\|_{L^1} \leq \sum_{j=1}^{\infty} 2^{-kj\alpha} \|\widehat{\psi}_k\|_{L^1} \lesssim C^k \frac{2^{k\alpha}}{1 - 2^{-k\alpha}}.$$

Consequently,

$$\|e^{i\mu}\chi\|_{\mathcal{FL}^1} \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|\phi_k\|_{\mathcal{FL}^1} \lesssim \sum_{k=0}^{\infty} \frac{2^{k\alpha} C^k}{k!} < \infty.$$

We have proved that the multiplier  $e^{i\mu(\xi)}\chi(\xi)$  is in  $\mathcal{FL}^1$  and thus by Lemma 1 it is bounded on all modulation spaces  $\mathcal{M}^{p,q}$ , and clearly also on all Lebesgue spaces  $L^p$ .  $\square$

**4.2. Large Oscillations at Infinity.** To deal with the oscillatory behavior of  $e^{i|\xi|^\alpha}$  at infinity, we use a time-frequency version of the stationary phase method [12, 26], the proof of which is reminiscent of the localization principle for oscillatory integrals of the first kind. A key feature of the modulation space case is that we may make a linear alteration of phase without affecting the norm in  $W(\mathcal{FL}^1, \ell^\infty)$ .

**Lemma 2.** *Assume that  $\alpha(x), \beta(x)$  are arbitrary (measurable) functions on  $\mathbb{R}^d$ . Set  $\tilde{\sigma}_x(\xi) = \sigma(\xi)e^{i(\alpha(x)+\xi\beta(x))}$ . Then*

$$(11) \quad \|\sigma T_x g\|_{\mathcal{FL}^1} = \|\tilde{\sigma}_x T_x g\|_{\mathcal{FL}^1} \quad \forall x \in \mathbb{R}^d.$$

Consequently,  $\|\sigma\|_{W(\mathcal{FL}^1, \ell^\infty)} = \sup_{x \in \mathbb{R}^d} \|\tilde{\sigma}_x T_x g\|_{\mathcal{FL}^1}$ .

*Proof.* We have

$$(\tilde{\sigma}_x T_x g)^\wedge(\omega) = e^{i\alpha(x)} \int_{\mathbb{R}^d} \sigma(\xi) g(\xi - x) e^{-2\pi i \xi(\omega - \frac{\beta(x)}{2\pi})} d\xi = e^{i\alpha(x)} V_g \sigma \left( x, \omega - \frac{\beta(x)}{2\pi} \right)$$

so  $\|\sigma\|_{W(\mathcal{FL}^1, \ell^\infty)} = \sup_x \|\tilde{\sigma}_x T_x g\|_{\mathcal{FL}^1}$  by the translation invariance of  $L^1(d\omega)$ .  $\square$

Our main result describing the behavior of multipliers with large oscillations is as follows.

**Theorem 5.** *For  $d \geq 1$ , let  $l = \lfloor d/2 \rfloor + 1$ . Assume that  $\mu$  is  $2l$ -times differentiable and  $\|\partial^\beta \mu\|_{L^\infty} \leq C$ , for  $2 \leq |\beta| \leq 2l$ , and some constants  $C$ . Then  $\sigma = e^{i\mu} \in W(\mathcal{FL}^1, \ell^\infty)$  and therefore  $H_\sigma$  is bounded on all modulation spaces  $\mathcal{M}^{p,q}$  for  $1 \leq p, q \leq \infty$ .*

*Proof.* The argument is most easily understood in the case  $d = 1$  where the assumption is that  $\mu'' \in L^\infty(\mathbb{R})$ . We give this proof first and then indicate the necessary modifications for general  $d$ .

Let  $g$  be a compactly supported test function in  $C^\infty$ . If we modify the phase  $\mu$  by subtracting the linear Taylor polynomial at  $x$

$$(12) \quad r_x(\xi) = \mu(\xi) - \mu(x) - \mu'(x)(\xi - x)$$

then by Lemma 2 we have

$$(13) \quad \begin{aligned} \|\sigma\|_{W(\mathcal{FL}^1, \ell^\infty)} &= \sup_{x \in \mathbb{R}} \|e^{ir_x} T_x g\|_{\mathcal{FL}^1} \\ &= \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{ir_x(\xi)} T_x g(\xi) e^{-2\pi i \xi \omega} d\xi \right| d\omega \\ &= \sup_{x \in \mathbb{R}} \int_{|\omega| \leq 1} \dots d\omega + \int_{|\omega| \geq 1} \dots d\omega = A + B. \end{aligned}$$

By pulling in the absolute values, the first term is readily estimated by

$$A \leq \sup_{x \in \mathbb{R}} \int_{|\omega| \leq 1} \left( \int_{\mathbb{R}} |g(\xi - x)| d\xi \right) d\omega \leq 2\|g\|_{L^1}.$$

For the estimate of  $B$  we write the exponential as  $e^{-2\pi i \xi \omega} = \frac{-1}{4\pi^2 \omega^2} \frac{d^2}{d\xi^2} e^{-2\pi i \xi \omega}$ . Using integration by parts, we obtain

$$(14) \quad B = \sup_{x \in \mathbb{R}} \int_{|\omega| \geq 1} \frac{1}{4\pi^2 \omega^2} \left| \int_{\mathbb{R}} \frac{d^2}{d\xi^2} \left( e^{ir_x(\xi)} T_x g(\xi) \right) e^{-2\pi i \xi \omega} d\xi \right| d\omega.$$

The second derivative in this integral is

$$\frac{d^2}{d\xi^2} [T_x g(\xi) e^{ir_x(\xi)}] = (T_x g''(\xi) + 2ir'_x(\xi) T_x g'(\xi) + ir''_x(\xi) T_x g(\xi) - (r'_x(\xi))^2 T_x g(\xi)) e^{ir_x(\xi)}.$$

However Taylor's theorem supplies bounds  $|r_x(\xi)| \lesssim \|\mu''\|_{\infty} |x - \xi|^2$  and  $|r'_x(\xi)| \leq \|\mu''\|_{\infty} |x - \xi|$ , and it is obvious that  $\|r''_x\|_{\infty} = \|\mu''\|_{\infty}$ . Since  $T_x g$  and its derivatives are supported in a fixed neighborhood of  $x$  we conclude that

$$\left| \int_{\mathbb{R}} \frac{d^2}{d\xi^2} \left( e^{ir_x(\xi)} T_x g(\xi) \right) e^{-2\pi i \xi \omega} d\xi \right| \leq C$$

and substituting into (14) we have the bound

$$B \leq \int_{|\omega| \geq 1} \frac{C}{4\pi^2 \omega^2} d\omega < \infty.$$

Combining the estimates for  $A$  and  $B$ , we have shown that  $e^{i\mu} \in W(\mathcal{FL}^1, \ell^{\infty})$ , and by Lemma 1 the associated Fourier multiplier is bounded on  $\mathcal{M}^{p,q}$  for  $1 \leq p, q \leq \infty$ . This concludes the proof for the case  $d = 1$ .

The proof for general  $d$  is very similar. We define

$$r_x(\xi) = \mu(\xi) - \mu(x) - \nabla \mu(x) \cdot (\xi - x).$$

and compute as in (13). Evidently the estimate for the first term becomes  $A \leq v_d \|g\|_{L^1}$ , where  $v_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . For the estimate of  $B$  we write the exponential as  $e^{-2\pi i \xi \cdot \omega} = \left( \frac{-1}{4\pi^2 |\omega|^2} \right)^l \Delta^l (e^{-2\pi i \xi \cdot \omega})$  and integrate by parts as before, finding that

$$B = \sup_{x \in \mathbb{R}^d} \int_{|\omega| \geq 1} \frac{1}{(4\pi^2 |\omega|^2)^l} \left| \int_{\mathbb{R}^d} \Delta^l \left( e^{ir_x(\xi)} T_x g(\xi) \right) e^{-2\pi i \xi \cdot \omega} d\xi \right| d\omega.$$

Since  $l = \lfloor d/2 \rfloor + 1$  we know that  $|\omega|^{-2l}$  is integrable outside a neighborhood of the origin, so the desired bound will follow if we can show  $\Delta^l (e^{ir_x(\xi)} T_x g(\xi))$  is uniformly bounded on  $\mathbb{R}^d$ .

Using Leibniz's rule and the chain rule, we write  $\Delta^l (e^{ir_x(\xi)} T_x g(\xi))$  as a linear combination of terms of the form

$$e^{ir_x} \partial^{\gamma_1} (T_x g) \partial^{\gamma_2} r_x \partial^{\gamma_3} r_x \dots \partial^{\gamma_m} r_x \quad \text{for} \quad \sum_{j=1}^m \gamma_j = 2l$$

It is immediate for multi-indices  $|\gamma_j| \geq 2$  that  $|\partial^{\gamma_j} r_x| = |\partial^{\gamma_j} \mu| \leq C$  independent of  $x$ . Otherwise we may apply Taylor's theorem to find  $|\partial^{\gamma_j} r_x(\xi)| \leq C|x - \xi|$  when  $|\gamma_j| = 1$  and  $|r_x(\xi)| \leq C|x - \xi|^2$ . Moreover we are only interested in these functions for  $\xi$  in the support of  $T_x g$ , on which they are uniformly bounded. Since the factors  $\partial^{\gamma_1}(T_x g)$  are also bounded we find that  $\Delta^l(e^{ir_x(\xi)} T_x g(\xi))$  is uniformly bounded, which gives the desired bound for  $B$ .

Combining the estimates for  $A$  and  $B$  we conclude that  $e^{i\mu} \in W(\mathcal{FL}^1, \ell^\infty)$ . By Lemma 1 the multiplier  $H_{e^{i\mu}}$  is then bounded on  $\mathcal{M}^{p,q}$  for  $1 \leq p, q \leq \infty$ .  $\square$

### 4.3. Proof of Theorem 1.

*Proof.* Let  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  be the test function defined in (7). We split the multiplier into two parts by writing  $\sigma(\xi) = e^{i|\xi|^\alpha} \chi(\xi) + e^{i|\xi|^\alpha} (1 - \chi(\xi)) = \sigma_{\text{sing}}(\xi) + \sigma_{\text{osc}}(\xi)$ . Then  $\sigma_{\text{sing}} \in \mathcal{FL}^1$  by Theorem 4 and  $H_{\sigma_{\text{sing}}}$  is bounded on all modulation spaces.

To deal with  $\sigma_{\text{osc}}$ , we set  $\tilde{\mu}(\xi) = |\xi|^\alpha(1 - \chi(2\xi))$ . Then  $\tilde{\mu}(\xi) = |\xi|^\alpha$  for  $|\xi| \geq 1$  and

$$\sigma_{\text{osc}}(\xi) = e^{i\tilde{\mu}(\xi)} (1 - \chi(\xi)).$$

By this construction we have removed the singularity of  $|\xi|^\alpha$  at the origin. Since  $\alpha \leq 2$ , all derivatives  $\partial^\beta \tilde{\mu}$  are bounded for  $|\beta| \geq 2$ . The multiplier  $e^{i\tilde{\mu}}$  is therefore bounded on all modulation spaces by Theorem 5. Clearly, the multiplier  $1 - \chi \in \mathcal{M}^{\infty,1}$  is bounded on all modulation spaces, so the fact that the bounded multipliers on  $\mathcal{M}^{p,q}$  form an algebra implies that the same is true of  $\sigma_{\text{osc}}$ . This completes the proof.  $\square$

It is not hard to see that we have in fact proven a more general result than Theorem 1. Inspecting the conditions needed in Theorems 4 and 5, we have shown the following.

**Corollary 2.** *Let  $d \geq 1$ ,  $\alpha \in [0, 2]$ , and define  $l = \lfloor d/2 \rfloor + 1$ . Assume that  $\mu \in \mathcal{C}^{2l}(\mathbb{R}^d \setminus \{0\})$  is homogeneous of order  $\alpha$  and all derivatives  $\partial^\beta \mu$  are bounded outside a neighborhood of 0 for  $2 \leq |\beta| \leq 2l$ . Then  $H_{e^{i\mu}}$  is bounded on all modulation spaces  $\mathcal{M}^{p,q}$  for  $1 \leq p, q \leq \infty$ .*

For  $1 < r < \infty$ , we let  $|\xi|_r = (\sum_{j=1}^d |\xi_j|^r)^{1/r}$  denote the  $r$ -norm on  $\mathbb{R}^d$ . With this notation,  $|\xi| = |\xi|_2$ . For  $1 \leq r < \infty$ , and  $\alpha \geq 0$ , let  $\mu(\xi) = e^{i|\xi|_{2r}^\alpha}$ . Then  $\mu(\xi) = |\xi|_{2r}^\alpha$  satisfies the conditions of Corollary 2 for all  $0 \leq \alpha \leq 2$ .

**Corollary 3.** *The multiplier  $e^{i|\xi|_{2r}^\alpha}$  is a bounded Fourier multiplier for all modulation spaces.*

**4.4. Improved Estimates for the Cases  $\alpha = 2$  and  $\alpha \in [0, 1]$ .** Boundedness of the Fourier multiplier operator  $H_{\sigma_2}$  on  $L^p(\mathbb{R}^d)$  was settled by Hörmander [17], who showed the more general result that when  $\phi$  is any quadratic polynomial the multiplier  $\sigma = e^{i\phi}$  is bounded only on  $L^2(\mathbb{R}^d)$ . We now give a different proof of part of Theorem 1, showing boundedness of  $H_{\sigma_2}$  by a time-frequency approach that is based on the so-called metaplectic invariance of the modulation spaces. This method gives a better bound for the operator norm.

**Theorem 6.** *Let  $d \geq 1$ , and let  $\sigma_2(\xi) = e^{i\pi t|\xi|^2}$ . Then:*

- (a)  $\sigma_2 \in (W(\mathcal{FL}^1, \ell^\infty) \cup \mathcal{M}^{1,\infty}) \setminus \mathcal{M}^{\infty,1}$ ,  
 (b)  $H_{\sigma_2}$  is bounded on all modulation spaces  $\mathcal{M}^{p,q}(\mathbb{R}^d)$ ,  $1 \leq p, q \leq \infty$ , and the operator norm satisfies the uniform estimate  $\|H_{\sigma_2}\|_{op} \leq c(d, p, q)(1 + t^2)^{d/4}$ .  
 (c) Let  $\sigma(\xi) = e^{-\pi\xi \cdot A\xi + 2\pi b \cdot \xi}$  be a generalized Gaussian so that  $A = B + iC$  for a positive-definite real-valued  $d \times d$ -matrix  $B$ , a symmetric real-valued matrix  $C$  and  $b \in \mathbb{C}^d$ . Then  $H_\sigma$  is bounded on all modulation spaces  $\mathcal{M}^{p,q}(\mathbb{R}^d)$ ,  $1 \leq p, q \leq \infty$ .

*Proof.* We use the Gaussian  $g(\xi) = e^{-\pi|\xi|^2}$  as a window for the short-time Fourier transform. Then the STFT  $V_g\sigma$  can be calculated explicitly by using Gaussian integrals.

$$\begin{aligned} V_g\sigma_2(x, \omega) &= \int_{\mathbb{R}^d} e^{i\pi t|\xi|^2} e^{-2\pi i\xi \cdot \omega} e^{-\pi|\xi-x|^2} d\xi \\ &= e^{-\pi|x|^2} \int_{\mathbb{R}^d} e^{-\pi(1-it)|\xi|^2} e^{2\pi\xi \cdot x} e^{-2\pi i\xi \cdot \omega} d\xi. \end{aligned}$$

The integral is the Fourier transform of a generalized Gaussian. By using a table or [13, Lemma 4.4.2], we obtain

$$V_g\sigma_2(x, \omega) = e^{-\pi|x|^2} (1-it)^{-d/2} e^{\pi(1-it)|x|^2} T_{tx} M_{-x} \left( e^{-\pi|\omega|^2/(1-it)} \right)$$

(where the square root  $(1-it)^{1/2}$  is taken with positive imaginary part). After taking absolute values and performing some cancellations we arrive at the expression

$$(15) \quad |V_g\sigma_2(x, \omega)| = (1+t^2)^{-d/4} e^{-\pi|\omega-tx|^2/(1+t^2)}.$$

Since  $\int_{\mathbb{R}^d} e^{-a|x|^2} dx = a^{-d/2}$ , the modulation space norms of  $\sigma_2$  are now easy to compute. It is trivial that

$$\int_{\mathbb{R}^d} \|V_g\sigma_2(\cdot, \omega)\|_{L^\infty} d\omega = \infty,$$

and therefore  $\sigma_2 \notin \mathcal{M}^{\infty,1}(\mathbb{R}^d)$ . On the other hand,

$$\|\sigma_2\|_{\mathcal{M}^{1,\infty}} = \sup_{\omega} \int_{\mathbb{R}^d} |V_g\sigma_2(x, \omega)| dx = (1+t^2)^{-d/4} \int_{\mathbb{R}^d} e^{-\frac{\pi t^2}{t^2+1}|x|^2} dx = (t^2+1)^{d/4} t^{-d},$$

and

$$\|\sigma_2\|_{W(\mathcal{FL}^1, \ell^\infty)} = \sup_x \int_{\mathbb{R}^d} |V_g\sigma_2(x, \omega)| d\omega = (1+t^2)^{-d/4} \int_{\mathbb{R}^d} e^{-\frac{\pi}{t^2+1}|\omega|^2} d\omega = (t^2+1)^{d/4}.$$

Consequently  $\sigma_2 \in \mathcal{M}^{1,\infty}$  and  $\sigma_2 \in W(\mathcal{FL}^1, \ell^\infty)$ , so Lemma 1 implies the boundedness of  $\sigma_2$  with an explicit form for the dependence on the parameter  $t$ :

$$\|H_{\sigma_2} f\|_{\mathcal{M}^{p,q}} \lesssim \|\sigma_2\|_{W(\mathcal{FL}^1, \ell^\infty)} \|f\|_{\mathcal{M}^{p,q}} \lesssim (1+t^2)^{d/4} \|f\|_{\mathcal{M}^{p,q}}.$$

The proof of (c) is similar, using the fact that after a change of coordinates, any quadratic function on  $\mathbb{R}^d$  can be written in the form  $\phi(\xi) = \langle \xi, C\xi \rangle$ , where  $C$  is a  $d \times d$  hermitian matrix.  $\square$

For the range  $\alpha \in [0, 1]$ , we now prove a stronger property of the multipliers  $e^{i|\xi|^\alpha}$ , which is also sufficient to show that  $H_{\sigma_\alpha}$  is bounded on all modulation spaces.

**Corollary 4.** *If  $\alpha \in [0, 1]$ , then  $\sigma_\alpha(\xi) = e^{i|\xi|^\alpha}$  belongs to  $\mathcal{M}^{\infty,1}(\mathbb{R}^d)$ .*

*Proof.* Let  $\chi$  be the smooth bump function defined by (7), and write

$$\sigma_\alpha(\xi) = \chi(\xi)\sigma_\alpha(\xi) + (1 - \chi(\xi))\sigma_\alpha(\xi) = \sigma_{\text{sing}}(\xi) + \sigma_{\text{osc}}(\xi).$$

Theorem 4 implies that  $\sigma_{\text{sing}} \in \mathcal{FL}^1 \subset \mathcal{M}^{\infty,1}$ . It is readily seen that the following estimate holds for all  $|\xi| \geq 1$  and  $|\beta| \geq 1$ :

$$|\partial^\beta \sigma_{\text{osc}}(\xi)| \leq C_\beta (1 + |\xi|)^{\alpha - |\beta|}.$$

Since  $\alpha \leq 1$ , all partial derivatives of  $\sigma_{\text{osc}}$  are bounded, and this fact implies that  $\sigma_{\text{osc}} \in \mathcal{C}^{d+1}(\mathbb{R}^d) \subset \mathcal{M}^{\infty,1}(\mathbb{R}^d)$ . For this embedding, see, e.g., [13, Thm. 14.5.3] or [21]. Thus  $\sigma_\alpha \in \mathcal{M}^{\infty,1}$  and the conclusion follows.  $\square$

**4.5. Further results.** Next we consider the related family of multipliers  $\sigma_{\alpha,\delta}$  defined by

$$\sigma_{\alpha,\delta}(\xi) = \mathfrak{F}(e^{i|\xi|^\alpha} |\xi|^{-\delta}) = \frac{\sin |\xi|^\alpha}{|\xi|^\delta}, \alpha, \delta > 0.$$

The following statement should be compared to results proved in [16, 22, 31].

**Theorem 7.** *Let  $d \geq 1$ , and let  $\alpha, \beta > 0$ .*

(a.) *If  $0 < \delta \leq \alpha \leq 1$ . Then  $\sigma_{\alpha,\delta} \in \mathcal{M}^{\infty,1}$ . Consequently,  $H_{\sigma_{\alpha,\delta}}$  is bounded on  $M^{p,q}$  for all  $1 \leq p, q \leq \infty$*

(b.) *If  $\alpha > 1$  and  $\delta \leq \alpha$ , then  $H_{\sigma_{\alpha,\delta}}$  is bounded on  $M^{p,q}$  for  $1 \leq p, q \leq \infty$  and  $|\frac{1}{p} - \frac{1}{2}| < \frac{\delta}{\alpha d}$ .*

*Proof.* (a) Using the smooth bump  $\chi$  defined by (7), we write

$$\sigma_{\alpha,\delta}(\xi) = \chi(\xi)\sigma_{\alpha,\delta}(\xi) + (1 - \chi(\xi))\sigma_{\alpha,\delta}(\xi) = \sigma_{\text{sing}}(\xi) + \sigma_{\text{osc}}(\xi).$$

We first show that  $\sigma_{\text{sing}} \in \mathcal{FL}^1$ . Using the same notation as in Theorem 4, we decompose the symbol as

$$\sigma_{\text{sing}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} |\xi|^{(2k+1)\alpha - \delta} \chi(\xi).$$

Set  $\psi_k(\xi) = |\xi|^{(2k+1)\alpha - \delta} \psi(\xi)$  and

$$\phi_k = |\xi|^{(2k+1)\alpha - \delta} \chi(\xi) = \sum_{j=1}^{\infty} 2^{-j(2k\alpha + \alpha - \delta)} \psi_k(2^j \xi).$$

Since  $(2k+1)\alpha - \delta > 0$ , the proof of Theorem 4 applies and we conclude that  $\sigma_{\text{sing}} \in \mathcal{FL}^1$  and hence  $H_{\sigma_{\text{sing}}}$  is bounded on all  $\mathcal{M}^{p,q}$ .

Now consider  $\sigma_{\text{osc}} = (\sin |\xi|^\alpha) |\xi|^{-\delta} (1 - \chi(\xi))$ . The multiplier  $\frac{1}{2i}(e^{i|\xi|^\alpha} - e^{-i|\xi|^\alpha})$  is bounded on all modulation spaces by Theorem 1. On the other hand, after removing the singularity at  $\xi = 0$ , the multiplier  $\kappa(\xi) = |\xi|^{-\delta} (1 - \chi(\xi))$  satisfies the conditions of the Hörmander-Mihlin multiplier theorem. In particular, all partial derivatives  $\partial^\beta \kappa$  are bounded. As before we conclude that  $|\xi|^{-\delta} (1 - \chi(\xi)) \in \mathcal{C}^{d+1}(\mathbb{R}^d) \subseteq \mathcal{M}^{\infty,1}$ , and thus  $H_\kappa$  is bounded on all modulation spaces  $\mathcal{M}^{p,q}$ . Consequently  $H_{\sigma_{\text{osc}}} = H_{\sin |\xi|^\alpha} H_\kappa$  is also bounded on  $\mathcal{M}^{p,q}$  for  $1 \leq p, q \leq \infty$ , and the theorem is proved.

(b) We still write  $\sigma_{\alpha,\delta}(\xi) = \sigma_{\text{sing}}(\xi) + \sigma_{\text{osc}}(\xi)$ , and the same argument as above shows that  $\sigma_{\text{sing}} \in \mathcal{FL}^1$  and hence  $H_{\sigma_{\text{sing}}}$  is bounded on all  $\mathcal{M}^{p,q}$  (in fact  $H_{\sigma_{\text{sing}}}$  is bounded on all  $L^p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ ). On the other hand it was proved in [16, 22, 31] that  $H_{\sigma_{\text{osc}}}$  is bounded on  $L^p(\mathbb{R}^d)$  whenever  $|\frac{1}{p} - \frac{1}{2}| < \frac{\delta}{\alpha d}$ . Consequently, using [10, Theorem 17] we conclude that  $H_{\sigma_{\text{osc}}}$  is bounded on  $\mathcal{M}^{p,q}$  whenever  $|\frac{1}{p} - \frac{1}{2}| < \frac{\delta}{\alpha d}$  and for all  $1 \leq q \leq \infty$ . This concludes the proof.  $\square$

*Remark 1.* In contrast to Theorem 7, the multipliers  $\tilde{\sigma}_{\alpha,\delta}(\xi) = e^{i\xi^{1\alpha}}|\xi|^{-\delta}$  for  $\alpha, \delta > 0$  and  $|\xi|^{-\delta} \sin|\xi|^\alpha$  for  $\delta > \alpha > 0$  are not bounded on  $L^p$  or on  $\mathcal{M}^{p,q}$ , because they are unbounded functions. Using arguments of this section, we can show that the Fourier multiplier with symbol

$$\tilde{\sigma}_{\alpha,\delta} - \sum_{k=0}^{\lfloor \delta/\alpha \rfloor} \frac{i^k \phi_{k,\alpha,\delta}}{k!}$$

is bounded on certain modulation spaces.

## 5. APPLICATIONS TO SOME CAUCHY PROBLEMS

**5.1. The Schrödinger equation.** Consider the linear free Schrödinger equation

$$(16) \quad \begin{cases} i \frac{\partial u}{\partial t}(x, t) &= \Delta_x u(x, t) \\ u(x, 0) &= f(x), x \in \mathbb{R}^d, t \geq 0, \end{cases}$$

where  $\Delta_x$  is the Laplacian. The formal solution to this equation is given by

$$(17) \quad u(x, t) = \int_{\mathbb{R}^d} e^{it|\xi|^2} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi = H_{\sigma_2^t} f(x),$$

where  $\sigma_2^t(\xi) = \sigma_2(\sqrt{t}\xi)e^{it|\xi|^2}$  is a bounded multiplier on modulation spaces by Theorem 1 and Theorem 6.

**Corollary 5.** *Let  $d \geq 1$ , and let  $u(x, t)$  be given by (17). Then, for any  $t \geq 0$ ,*

$$\|u(\cdot, t)\|_{\mathcal{M}^{p,q}} \leq C(t^2 + 4\pi^2)^{d/4} \|f\|_{\mathcal{M}^{p,q}}$$

for all  $1 \leq p, q \leq \infty$  and a constant  $C$  depending only on  $d, p$  and  $q$ .

*Remark 2.* This statement was also obtained with a different method in [1].

*Remark 3.* In particular, modulation space properties are preserved by the time evolution of the Schrödinger equation. This is in strong contrast to the standard  $L^p$ -theory where the  $L^p$ -property of the initial data is not preserved by the time evolution. see, for example, [29], where it was shown that  $L^2(\mathbb{R}^d) \ni f(x) \mapsto u(x, t) \in L^p(\mathbb{R}^{d+1})$  for  $p = 2(d+2)/d$ .

**5.2. The Wave Equation.** Consider now the following Cauchy problem for the wave equation

$$(18) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(x, t) &= \Delta_x u(x, t) \\ u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x). \end{cases}$$

Its formal solution is given by

$$(19) \quad u(x, t) = \int_{\mathbb{R}^d} \cos(t|\xi|) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi + \int_{\mathbb{R}^d} \frac{\sin t|\xi|}{|\xi|} \hat{g}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$

The time evolution requires an understanding of the continuity properties of the Fourier multipliers  $\sigma^t(\xi) = \cos t|\xi|$ , or equivalently  $\sigma_1^t(\xi) = e^{it|\xi|}$ , and  $m^t(\xi) = \frac{\sin t|\xi|}{|\xi|}$ . The first of these multipliers is known to be bounded on all  $L^p(\mathbb{R})$ , but only on  $L^2(\mathbb{R}^d)$  for all  $d \geq 1$ , [17, 19]. Theorems 1 and 7 yield the following result.

**Corollary 6.** *Let  $d \geq 1$ , and let  $u(x, t)$  be the solution of the wave equation as given by (19). Then, for any  $t \geq 0$ ,*

$$\|u(\cdot, t)\|_{\mathcal{M}^{p,q}} \leq C(t) (\|f\|_{\mathcal{M}^{p,q}} + \|g\|_{\mathcal{M}^{p,q}})$$

for all  $1 \leq p, q \leq \infty$ , where  $C(t) > 0$  depends on  $d, p$ , and  $q$ . Again, the solution to the Cauchy problem for the wave equation preserves the initial data in a modulation space.

*Remark 4.* Again, one should compare the space preserving estimate in the previous theorem to, for example, the following boundedness result  $\dot{L}_1^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \ni (f, g) \mapsto u(\cdot, t) \in L^q(\mathbb{R}^d)$ , for certain  $p \leq 2 \leq q$ , proved in [27]. Here,  $L_1^p$  denotes the appropriate homogeneous Sobolev space.

As previously mentioned, these results show that solutions to the Cauchy problems for the Schrödinger and the wave equation with initial data in a modulation space stay in the same space for all future time. This is a time-frequency version of the classical principle of conservation of energy [19, 27, 28, 29] for these Cauchy problems. The reader will recall that, in the context of Lebesgue spaces, this principle holds only on  $L^2$ .

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ÁRPÁD BÉNYI, DEPARTMENT OF MATHEMATICS, 516 HIGH STREET, WESTERN WASHINGTON UNIVERSITY, BELLINGHAM, WA 98225, USA

*E-mail address:* arpad.benyi@wwu.edu

KARLHEINZ GRÖCHENIG, FACULTY OF MATHEMATICS, UNIVERSITÄT WIEN, 1090 VIENNA, AUSTRIA

*E-mail address:* karlheinz.groechenig@univie.ac.at

KASSO A. OKOUDJOU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA

*E-mail address:* kasso@math.umd.edu

LUKE G. ROGERS, DEPARTMENT OF MATHEMATICS, MALOTT HALL, CORNELL UNIVERSITY, ITHACA, NY 14853, USA

*E-mail address:* luke@math.cornell.edu