

COMPLEX POWERS OF THE LAPLACIAN ON AFFINE NESTED FRACTALS AS CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. We give the first natural examples of Calderón-Zygmund operators in the theory of analysis on post-critically finite self-similar fractals. This is achieved by showing that the purely imaginary Riesz and Bessel potentials on nested fractals with 3 or more boundary points are of this type. It follows that these operators are bounded on L^p , $1 < p < \infty$ and satisfy weak 1-1 bounds. The analysis may be extended to infinite blow-ups of these fractals, and to product spaces based on the fractal or its blow-up.

1. INTRODUCTION

Complex powers of the Laplacian on Euclidean spaces and manifolds and their connection to pseudodifferential operators have been studied intensely (see, for example, [16, 17, 20, 29, 3] and the citations within). In this paper we define and study a class of operators built from the Laplace operator Δ on nested fractals [13], which are a type of post-critically finite self-similar fractal [12, 27]. The main focus is to show that the Riesz potentials $(-\Delta)^{i\alpha}$ and the Bessel potentials $(I - \Delta)^{i\alpha}$, $\alpha \in \mathbb{R}$, are Calderón-Zygmund operators in the sense of [21]. These operators are the first explicit examples of Calderón-Zygmund operators on a general class of self-similar fractals. The main result is as follows.

Theorem 1.1. *Let K be a nested fractal and X be either K or an infinite blow-up of K without boundary. Suppose T is a bounded operator on $L^2(\mu)$, where μ is the self-similar measure on X , and there is a kernel $K(x, y)$ such that*

$$T(f)(x) = \int_X K(x, y)f(y)d\mu(y)$$

for $f \in L^2(\mu)$ and almost all $x \notin \text{supp } f$. If $K(x, y)$ is a smooth function off the diagonal of $X \times X$ and satisfies

$$(1.1) \quad |K(x, y)| \lesssim R(x, y)^{-d}$$

$$(1.2) \quad |\Delta_2 K(x, y)| \lesssim R(x, y)^{-2d-1},$$

where $R(x, y)$ is the resistance metric on X and d is the dimension of X with respect to the resistance metric, then the operator T is a Calderón-Zygmund operator in the sense of [21, Section I.6.5].

This is proved as Theorem 3.2. In Sections 4 and 5 we show that the Riesz and Bessel potentials have kernels which satisfy conditions (1.1) and (1.2). We conclude that these potentials are Calderón-Zygmund operators and that

Date: March 1, 2010.

they are bounded on $L^p(X)$, $1 < p < \infty$. Furthermore, we study the general Bessel operators $(I - \Delta)^\alpha$, $\alpha \in \mathbb{C}$ and prove that, for $\operatorname{Re} \alpha < 0$, they are given by integration with respect to kernels which are smooth off the diagonal and they are bounded on $L^p(X)$. In Section 6 we extend our analysis to products of nested fractals and their infinite blowups.

Riesz and Bessel potentials for negative real powers in the context of metric measure spaces, including fractals, have been studied in [10] (see also [9]), however their results are not directly applicable in our setting. The main tool we use is estimates of kernels of the form (4.5) and powers of the Laplacian applied to these kernels. Our estimates are closely related to those in Section 4 of [10], where integrals obtained by replacing $m(t)dt$ in (4.5) by a non-negative measure $d\nu$ are treated. Our results are not contained in theirs, because we are primarily interested in complex-valued oscillatory functions $m(t)$ and need estimates of Laplacians of the kernel. Nor are their results contained in ours, because their work can apply to singular measures and measures for which the absolutely continuous part is not bounded.

Acknowledgement. The authors would like to thank Robert S. Strichartz for numerous and fruitful discussions during our work on this project.

2. BACKGROUND

In this paper K denotes a nested fractal in the sense of Lindström [13]. These are a subclass of the post-critically finite self-similar fractals, on which there is an analytic theory due to Kigami [12]. We will find it convenient to use the notation and constructions of Kigami to describe the analytic structure on K , rather than the equivalent probabilistic construction made by Lindström, and we only include enough information here to provide notation for our later results. Further details and proofs are in [12].

Nested fractals: Energy, Laplacian, smoothness, resistance metric. An iterated function system (i.f.s.) is a collection $\{F_1, \dots, F_N\}$ of contractions on \mathbb{R}^d . For such an i.f.s. there exists a unique invariant set K satisfying (see [11])

$$K = F_1(K) \cup \dots \cup F_N(K).$$

K is called the self-similar set associated to the i.f.s. For $\omega_1, \dots, \omega_n \in \{1, \dots, N\}$, $\omega = \omega_1 \dots \omega_n$ is a word of length n over the alphabet $\{1, \dots, N\}$. Then $K_\omega = F_\omega(K) := F_{\omega_1} \circ \dots \circ F_{\omega_n}(K)$ is called a cell of level n . The set of all finite words over $\{1, \dots, N\}$ is denoted by W_* . Each map F_i of the i.f.s. defining K has a unique fixed point x_i . Then K is a *post-critically finite (PCF) self-similar set* if there is a subset $V_0 \subseteq \{x_1, \dots, x_N\}$ satisfying

$$F_\omega(K) \cap F_{\omega'}(K) \subseteq F_\omega(V_0) \cap F_{\omega'}(V_0)$$

for any $\omega \neq \omega'$ having the same length. The set V_0 is called the *boundary* of K and the boundary of a cell K_ω is $F_\omega(V_0)$. We define $V_1 = \bigcup_i F_i(V_0)$, and, inductively, $V_n = \bigcup_i F_i(V_{n-1})$ for $n \geq 2$.

It is well known (see [11]) that for weights $\{\mu_1, \dots, \mu_N\}$ such that $0 < \mu_i < 1$ there is a unique self-similar measure

$$\mu(A) = \sum_{i=1}^N \mu_i \mu(F_i^{-1}(A)).$$

A nested fractal is of the above type, but in addition has a large symmetry group. For K to be a nested fractal one requires that for every pair of points $p, q \in V_0$ the reflection in the Euclidean hyperplane equidistant from p and q maps n cells to n cells. It is also required that any n cell that intersects the hyperplane at a non-boundary point (of the n cell) is mapped to itself by the reflection. For full details see [13].

A key feature of nested fractals is the existence of a regular self-similar Dirichlet energy form E on K with weights $0 < r_i < 1$, $i = 1, \dots, N$ such that

$$E(u) = \sum_{i=1}^N r_i^{-1} E(u \circ F_i).$$

The existence of such forms is non-trivial. In the case that all r_i are the same it is due to Lindström [13] via probabilistic methods. Kigami's approach to constructing these as limits of resistance forms may be found in [12, 27]. When the r_i are not all equal there is no known general solution.

Let $u \in \text{dom } E$ and f be continuous on X . We say $u \in \text{dom } \Delta$ with (Neumann) Laplacian $\Delta u = f$ if

$$E(u, v) = - \int_X f v d\mu$$

for all $v \in \text{dom } E$. We say that u is *smooth* if $\Delta^n u$ is continuous for all $n \geq 1$. The operator $-\Delta$ is non-negative definite and self-adjoint, with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \dots$ accumulating only at ∞ . We fix an orthonormal basis $\{\varphi_n\}$ for $L^2(\mu)$ where φ_n has eigenvalue λ_n and eigenvalues may be repeated and let D be the set of finite linear combinations of φ_n .

The *effective resistance metric* $R(x, y)$ on K is defined by

$$R(x, y)^{-1} = \min\{E(u) : u(x) = 0 \text{ and } u(y) = 1\}.$$

It is known that the resistance metric is topologically equivalent, but not metrically equivalent to the Euclidean metric [12, 27]).

Examples. An important example of a PCF self-similar set is the unit interval $I = [0, 1]$. In this case $V_0 = \{0, 1\}$. While I is not a fractal, Kigami's construction applies and one recovers the usual energy on the interval

$$E(u, v) = \int_0^1 u'(x)v'(x)dx,$$

and the usual Laplacian $\Delta u = u''$.

The simplest example of a fractal to which the theory applies is the Sierpinski gasket, which has been studied intensively (see, for example, [12, 27, 6, 2, 14, 24, 30]). To describe the Sierpinski gasket, consider a triangle in \mathbb{R}^2 with vertices $\{q_0, q_1, q_2\}$ and consider a set of three mappings $F_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i = 1, 2, 3$, defined by

$$F_i(x) = \frac{1}{2}(x - q_i) + q_i.$$

The invariant set of this iterated function system is the Sierpinski gasket and $V_0 = \{q_0, q_1, q_2\}$.

Blow-ups. In [23, 24] Strichartz defined fractal blow-ups of K . This construction generalizes the relationship between the unit interval and the real line to arbitrary PCF self-similar sets.

Let $w \in \{1, \dots, N\}^\infty$ be an infinite word. Then

$$F_{w_1}^{-1} \dots F_{w_m}^{-1} K \subseteq F_{w_1}^{-1} \dots F_{w_m}^{-1} F_{w_{m+1}}^{-1} K.$$

The *fractal blow-up* K_∞ is

$$K_\infty = \bigcup_{m=1}^{\infty} F_{w_1}^{-1} \dots F_{w_m}^{-1} K.$$

If C is an n cell in K , then $F_{w_1}^{-1} \dots F_{w_m}^{-1} C$ is called an $(n - m)$ cell. The blow-up depends on the choice of the infinite word w . In general there are an uncountably infinite number of blow-ups which are not homeomorphic. In this paper we assume that the infinite blow-up K_∞ has no boundary. This happens unless all but a finite number of letters in w are the same. One can extend the definition of the energy E and measure μ to K_∞ . The measure μ will be σ -finite rather than finite. As before, Δ is defined by $u \in \text{dom } \Delta$ with $\Delta u = f$ if $u \in \text{dom } E$, f is continuous, and

$$E(u, v) = - \int_{K_\infty} f v d\mu$$

for every $v \in \text{dom } E$.

It will be important in what follows that if K is a nested fractal and V_0 contains 3 or more points, then the Laplacian on an infinite blow-up without boundary has pure point spectrum [30, 15] and the eigenfunctions have compact support. We write $\{\lambda_n\}_{n \in \mathbb{Z}}$ for the eigenvalues of $-\Delta$, which are non-negative and accumulate only at 0 and ∞ . As for K , we take a basis $\{\varphi_n\}_{n \in \mathbb{Z}}$ of $L^2(\mu)$ consisting of eigenfunctions of $-\Delta$ with eigenvalues λ_n that may be repeated, and let D be the set of finite linear combinations of φ_n .

Notation for estimates. We write $A(y) \lesssim B(y)$ if there is a constant C independent of y , but which might depend on the fractal K , such that $A(y) \leq CB(y)$ for all y . We write $A(y) \sim B(y)$ if $A(y) \lesssim B(y)$ and $B(y) \lesssim A(y)$. If $f(x, y)$ is a function on $X \times X$, then we write $\Delta_1 f$ to denote the Laplacian of f with respect to the first variable and $\Delta_2 f$ to denote the Laplacian of f with respect to the second variable; repeated subscripts indicate composition, for example $\Delta_{21} = \Delta_2 \circ \Delta_1$.

Heat kernel on nested fractals. Let K be an affine nested fractal with Dirichlet form as above. Let μ be the unique self-similar probability measure for which $\mu_i = r_i^d$, so that K is of Hausdorff dimension d in the resistance metric, and let X be an infinite blow-up of K without boundary. A fundamental result we require is an estimate for the heat kernel corresponding to the Laplacian. Specifically, the semi-group $e^{t\Delta}$ is given by integration with respect to a positive heat kernel $h_t(x, y)$ which satisfies

$$(2.1) \quad t^{-\beta} \exp \left(-c_1 \left(\frac{R(x, y)^{d+1}}{t} \right)^\gamma \right) \lesssim h_t(x, y) \lesssim t^{-\beta} \exp \left(-c \left(\frac{R(x, y)^{d+1}}{t} \right)^\gamma \right),$$

where $\beta = d/(d+1)$, $R(x, y)$ is the effective resistance metric on X , and $\gamma = \frac{\gamma'}{d+1-\gamma'}$, where $0 < \gamma' < d+1$, is the chemical exponent, a constant depending on the fractal. The same estimates are valid for $t^k \frac{\partial^k}{\partial t^k} h_t(x, y)$. These estimates are originally due to Barlow and Perkins [1] for the case of the Sierpinski gasket and have been generalized beyond what is needed here. In particular, (2.1) is a special case of [4, Theorem 1.1(2) and Remark 3.7(2)], but see also [7] and [8].

The following lemma implies that the heat kernel is integrable with respect to x and y , and will be important later. The estimate is presumably well-known but we include it for the convenience of the reader.

Lemma 2.1. *If $y \in X$ we have that*

$$\int_X e^{-c\left(\frac{R(x,y)^{d+1}}{t}\right)^\gamma} d\mu(x) \sim t^{\frac{d}{d+1}}$$

for all $t > 0$. Similar estimates hold if we integrate with respect to y .

Proof. Fix $y \in X$ and for $n \geq 1$ let X_n denote the annulus centered at y of inner radius $(2^n t)^{\frac{1}{d+1}}$ and outer radius $(2^{n+1} t)^{\frac{1}{d+1}}$ in the resistance metric. Let X_0 denote the ball radius $t^{\frac{1}{d+1}}$ centered at y . Then $\mu(B(y, 2^{\frac{n}{d+1}} t^{\frac{1}{d+1}})) \sim (2^n t)^{\frac{d}{d+1}}$, and on X_n the integrand satisfies

$$e^{-c2^{(n+1)\gamma}} \leq e^{-c\left(\frac{R(x,y)^{d+1}}{t}\right)^\gamma} \leq e^{-c2^{n\gamma}}.$$

Thus the integral is bounded below by the contribution from X_0 and above by the series of terms from each X_n , which are exponentially decaying and therefore have sum bounded by a constant multiple of the largest term, which is readily computed to have size comparable to $t^{\frac{d}{d+1}}$.

$$\begin{aligned} t^{\frac{d}{d+1}} &\leq \sum_{n=0}^{\infty} (2^n t)^{\frac{d}{d+1}} e^{-c2^{(n+1)\gamma}} \lesssim \sum_{n=0}^{\infty} \int_{X_n} e^{-c\left(\frac{R(x,y)^{d+1}}{t}\right)^\gamma} d\mu(x) \\ &\lesssim \sum_{n=0}^{\infty} (2^n t)^{\frac{d}{d+1}} e^{-c2^{n\gamma}} \lesssim t^{\frac{d}{d+1}}. \quad \square \end{aligned}$$

3. SINGULAR INTEGRAL AND CALDERÓN-ZYGMUND OPERATORS ON FRACTALS

In this section we define singular integral and Calderón-Zygmund operators on fractals and infinite blow-ups of fractals without boundary. For this we do not need to assume that K is nested, but only that it is a PCF fractal supporting a Laplacian in the sense of Kigami [12]. As usual, X is either K or an infinite blow-up of K without boundary. The following definition can be made for any dense subspace of L^2 , but we consider only the subspace D of finite linear combinations of eigenfunctions.

Definition 3.1 ([21, Section I.6.5]). An operator T bounded on $L^2(\mu)$ is called a *Calderón-Zygmund operator* if T is given by integration with respect to a kernel $K(x, y)$, that is

$$Tu(x) = \int_X K(x, y)u(y)d\mu(y)$$

for $u \in D$ and almost all $x \notin \text{supp } u$, such that $K(x, y)$ is a function off the diagonal which satisfies the following conditions

$$(3.1) \quad |K(x, y)| \lesssim R(x, y)^{-d}$$

$$(3.2) \quad |K(x, y) - K(x, \bar{y})| \lesssim \eta \left(\frac{R(y, \bar{y})}{R(x, \bar{y})} \right) R(x, y)^{-d},$$

for some Dini modulus of continuity η and some $c > 1$. We say, in this case, that $K(x, y)$ is a *standard kernel*.

The operator T is a *singular integral operator* if the kernel $K(x, y)$ is singular at $x = y$.

The next theorem gives conditions which guarantee that (3.2) holds. In the succeeding sections we will show that the purely imaginary Riesz and Bessel potentials satisfy the hypothesis of this theorem. The proof is more involved in our case than the proofs of similar results in the real case due to the lack of a mean value theorem.

Theorem 3.2. *Let K be a PCF fractal with regular self-similar Dirichlet form. Suppose that X is equal to K or an infinite blow-up of K without boundary. If $\Delta_2 K(x, y)$ is continuous off the diagonal of $X \times X$ and*

$$\begin{aligned} |K(x, y)| &\lesssim R(x, y)^{-d} \\ |\Delta_2 K(x, y)| &\lesssim R(x, y)^{-2d-1} \end{aligned}$$

then

$$|K(x, y) - K(x, \bar{y})| \lesssim \left(\frac{R(y, \bar{y})}{R(x, \bar{y})} \right)^{1/2} R(x, y)^{-d},$$

for all $x, y, \bar{y} \in X$ such that $R(x, y) \geq cR(y, \bar{y})$, for some $c > 1$ depending only on the scaling values r_j of the Dirichlet form.

Suppose that X is an infinite blow-up such that

$$X = \bigcup_{n=1}^{\infty} F_{w_1}^{-1} \dots F_{w_n}^{-1} K,$$

where $w = (w_n)$ is an infinite word. For $n \geq 0$ we write $\omega|n$ for the finite word $w_1 \dots w_n$ and $r_{\omega|n} := r_{w_1} \dots r_{w_n}$. We say that a cell C has size $R > 0$ if C is an m cell such that $c_1 r_{\omega|n}^{-1} \leq R \leq c_1 r_{\omega|n}$ if $m < 0$ and $c_1 r_{\omega|n} \leq R \leq c_1 r_{\omega|n-1}$ if $m > 0$, where c_1 is the constant from the estimates in [27, page 110] (see also [27, Lemma 1.6.1 a)). Then, for a cell of size R we have that $\mu(C) \lesssim R^d$.

Lemma 3.3. *Suppose that C is a cell of size $R > 0$. Assume that f is a smooth function on C such that*

$$(3.3) \quad |f(x)| \lesssim R^{-\beta}$$

$$(3.4) \quad |\Delta f(x)| \lesssim R^{-\beta-d-1}$$

for all $x \in C$, where $\beta > 0$ is a constant. Then, for all y and \bar{y} in the interior of C we have

$$(3.5) \quad |f(y) - f(\bar{y})| \lesssim \left(\frac{R(y, \bar{y})}{R} \right)^{\frac{1}{2}} R^{-\beta}.$$

Proof. Assume that C is an m cell with $m < 0$ (if $m \geq 0$ then the proof is similar). Thus $C = F_{w_1}^{-1} \cdots F_{w_{-m}}^{-1}(K_0)$, where K_0 is a set similar to K and

$$r_{\omega|_{-m-1}}^{-1} \leq \frac{1}{c_1} R \leq r_{\omega|_{-m}}^{-1}.$$

Then

$$f(z) = h^C(z) - \int_C G^C(z, s) \Delta f(s) d\mu(s) \text{ for all } z \in C,$$

where h^C is the harmonic function on C such that $h^C|_{\partial C} = f|_{\partial C}$, and G^C is the (Dirichlet) Green function on C . Therefore

$$|f(y) - f(\bar{y})| \leq |h^C(y) - h^C(\bar{y})| + \left| \int_C (G^C(y, z) - G^C(\bar{y}, z)) \Delta u(z) d\mu(z) \right|.$$

Since h^C is harmonic on C it follows that $h^C \circ F_{w_1}^{-1} \circ \cdots \circ F_{w_{-m}}^{-1}$ is harmonic on K_0 . Let $y', \bar{y}' \in K_0$ such that $F_{w_{-m}} \circ \cdots \circ F_{w_1}(y) = y'$ and $F_{w_{-m}} \circ \cdots \circ F_{w_1}(\bar{y}) = \bar{y}'$. Then $R(y', \bar{y}') \lesssim r_{\omega|_{-m}} R(y, \bar{y}) \lesssim R(y, \bar{y})/R$. Moreover by the definition of the resistance metric

(3.6)

$$|h^C(y) - h^C(\bar{y})| \leq E(h^C \circ F_{w_1}^{-1} \circ \cdots \circ F_{w_{-m}}^{-1})^{1/2} R(y', \bar{y}')^{1/2} \lesssim R^{-\beta} \left(\frac{R(y, \bar{y})}{R} \right)^{1/2},$$

since $E(h^C \circ F_{w_1}^{-1} \circ \cdots \circ F_{w_{-m}}^{-1}) \lesssim \|h^C\|_\infty^2 \lesssim R^{-2\beta}$ by hypothesis.

Let $z' = F_{w_{-m}} \circ \cdots \circ F_{w_1}(z)$ so $G^C(y, z) = r_{\omega|_{-m}}^{-1} G(y', z')$, where G is the Green function on K_0 . Recall that the Green function is given by the following formula [12, Proposition 3.5.5] or [27, (2.6.15)]

$$G(y', z') = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r_\alpha \Psi(F_\alpha^{-1} y', F_\alpha^{-1} z'),$$

where

$$\Psi(u, v) = \sum_{s, s' \in V_1 \setminus V_0} g(s, s') \psi_s^{(1)}(u) \psi_{s'}^{(1)}(v),$$

and $\sum_{s, s' \in V_1 \setminus V_0} g(s, s')$ is a constant depending on the fractal K . By the self-similarity of the energy E we have that

$$|\psi_s^{(1)} \circ F_\alpha^{-1}(y') - \psi_s^{(1)} \circ F_\alpha^{-1}(\bar{y}')| \leq (r_\alpha^{-1} E(\psi_s^{(1)}))^{1/2} R(y', \bar{y}')^{1/2}.$$

Since $E(\psi_s^{(1)}) \lesssim 1$ and $|\psi_{s'}^{(1)}(z')| \leq 1$ for all $z' \in K_0$, we have that

$$\begin{aligned} |G(y', z') - G(\bar{y}', z')| &\lesssim \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r_\alpha r_\alpha^{-1/2} R(y', \bar{y}')^{1/2} \\ &\lesssim R(y', \bar{y}')^{1/2} \lesssim R(y, \bar{y})^{1/2} R^{-1/2} \end{aligned}$$

Then

$$\begin{aligned}
 \left| \int_C (G^C(y, z) - G^C(\bar{y}, z)) \Delta u_x(z) d\mu(x) \right| &\lesssim r_{\omega|^{-m}}^{-1} \left(\frac{R(y, \bar{y})}{R} \right)^{1/2} \int_C |\Delta f(z)| d\mu(z) \\
 &\lesssim \left(\frac{R(y, \bar{y})}{R} \right)^{1/2} r_{\omega|^{-m}}^{-1} R^{-1-\beta} \int_C R^{-d} d\mu(z) \\
 (3.7) \qquad \qquad \qquad &\lesssim \left(\frac{R(y, \bar{y})}{R} \right)^{1/2} R^{-\beta}.
 \end{aligned}$$

because $r_{\omega|^{-m}}^{-1} \simeq R$. Inequalities (3.6) and (3.7) imply the conclusion. \square

Proof of Theorem 3.2. Let $r = \max_{i=1, \dots, n} r_i$ and let $c > r^{-3-k_0}$, where k_0 is such that $r^{k_0} < 1/3$, and fix $x, y, \bar{y} \in X$ such that $R(x, y) \geq cR(y, \bar{y})$. Then $R(x, y) \sim R(x, \bar{y})$. Let $\{C_n\}$ be a partition of cells of X such that each C_n is a cell of size $r^{k_0+1}R(x, \bar{y})$, or, equivalently, of size $r^{k_0+1}R(x, y)$. Then there is a cell C of order some m in this family that contains both y and \bar{y} . We claim that x and y , and x and \bar{y} , respectively, do not belong to the same or adjacent $m-1$ cells. To see this, assume that $m < 0$, the proof for $m > 0$ being similar. Suppose that x and y belong to the same or adjacent $m-1$ cells. By the estimates on [27, page 110] (see also [27, Lemma 1.6.1] [28, Theorem 2.1]) we have that $R(x, y) \leq c_1 r_{\omega|^{-m-1}}^{-1} \leq r^{k_0} R(x, y) < R(x, y)/3$, which is a contradiction.

Let $f_x(z) = K(x, z)$ for all $z \in C$. By the hypotheses, $f_x(\cdot)$ has continuous Laplacian on C and satisfies (3.3) and (3.4) with $R = R(x, y)$ and $\beta = d$, so Lemma 3.3 implies the conclusion. \square

4. PURELY IMAGINARY RIESZ POTENTIALS

Let X be a nested fractal K or an infinite blow-up based on this fractal. We define the class of operators $(-\Delta)^{i\alpha}$, with $\alpha \in \mathbb{R}$. Recall that for $\lambda > 0$ and $\alpha \in \mathbb{R}$ we have

$$(4.1) \qquad \qquad \qquad \lambda^{i\alpha} = C_\alpha \lambda \int_0^\infty e^{-\lambda t} t^{-i\alpha} dt,$$

where $C_\alpha = 1/\Gamma(1-i\alpha)$ and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ if $\operatorname{Re} z > 0$ is the Gamma function.

Definition 4.1. Let $\alpha \in \mathbb{R}$. Recall that D is the set of finite linear combinations of eigenfunctions. For φ an eigenfunction with eigenvalue λ we define

$$(-\Delta)^{i\alpha} \varphi = \lambda^{i\alpha} \varphi$$

and thus, for $u \in D$,

$$(-\Delta)^{i\alpha} u = C_\alpha (-\Delta) \left(\int_0^\infty e^{t\Delta} u t^{-i\alpha} dt \right),$$

where C_α is the constant from (4.1).

We show that these operators are Calderón-Zygmund operators by proving that their kernels satisfy estimates of the form (3.1) and (3.2).

Before doing this we need the following lemma which says that in order to show the kernel coincides with a smooth function off the diagonal in the sense of Theorem 3.2, it suffices to differentiate inside the integral.

Lemma 4.2. *If $m \in L^\infty([0, \infty))$ and u is a smooth function with compact support on X not intersecting $\{x\}$ then*

$$\int_X \Delta^k u(y) \int_0^\infty m(t) h_t(x, y) dt d\mu(y) = \int_X u(y) \int_0^\infty m(t) \Delta_2^k h_t(x, y) dt.$$

Proof. Using the Green-Gauss formula (see, for example, [27, Theorem 2.4.1]) we have that

$$\begin{aligned} \int_X \Delta^k u(y) \int_0^\infty m(t) h_t(x, y) dt d\mu(y) &= \int_X \int_0^\infty \Delta^k u(y) m(t) h_t(x, y) dt d\mu(y) \\ &= \int_X \int_0^\infty u(y) m(t) \Delta^k h_t(x, y) dt d\mu \\ &= \int_X u(y) \int_0^\infty m(t) \Delta^k h_t(x, y) dt d\mu. \quad \square \end{aligned}$$

Proposition 4.3. *For $\alpha \in \mathbb{R}$ define*

$$K_{i\alpha}(x, y) = C_\alpha \int_0^\infty (-\Delta_1) h_t(x, y) t^{-i\alpha} dt.$$

Then $K_{i\alpha}$ is the kernel of $(-\Delta)^{i\alpha}$, in the sense that

$$(4.2) \quad (-\Delta)^{i\alpha} u(x) = \int_X K_{i\alpha}(x, y) u(y) d\mu(y)$$

for all $u \in D$ such that $x \notin \text{supp } u$. Moreover, the kernel $K_{i\alpha}(x, y)$ is smooth off the diagonal and satisfies the following estimates

$$(4.3) \quad |K_{i\alpha}(x, y)| \lesssim R(x, y)^{-d}$$

$$(4.4) \quad |\Delta_2 K_{i\alpha}(x, y)| \lesssim R(x, y)^{-2d-1}.$$

Proof. The proof of (4.2) is clear because $h_t(x, y)$ is the kernel of the heat operator and u is a linear combination of eigenfunctions. Both the smoothness and the desired estimates rely on the following computation using the estimate (2.1) and with $l = j + k$

$$\begin{aligned} \left| \int_0^\infty (-\Delta_2)^k (-\Delta_1)^{j+1} h_t(x, y) e^{-t} t^{-i\alpha} dt \right| &= \left| \int_0^\infty \frac{\partial^{l+1}}{\partial t^{l+1}} h_t(x, y) t^{-i\alpha} dt \right| \\ &\lesssim \int_0^\infty t^{-\frac{d}{d+1}-l} e^{-c\left(\frac{R(x,y)}{t}\right)^{d+1}} \frac{dt}{t} \\ &= R(x, y)^{-d-l(d+1)} \int_0^\infty u^{\frac{d}{d+1}+l} e^{-cu^\gamma} \frac{du}{u} \\ &\lesssim C(j+k) R(x, y)^{-d-(j+k)(d+1)}, \end{aligned}$$

where $C(m)$ denotes a constant depending only on k . Since the functions in the integrand are continuous on $X \times X$ and the integral converges uniformly on compact sets away from $R(x, y) = 0$ we conclude that $(-\Delta_2)^k (-\Delta_1)^j K_{i\alpha}(x, y)$ is continuous off the diagonal for each $j, k \geq 0$. \square

Corollary 4.4. *The operators $(-\Delta)^{i\alpha}$ are Calderón-Zygmund operators.*

Proof. Observe that $(-\Delta)^{i\alpha}$ extends from D to $L^2(\mu)$ by the spectral theorem. By Proposition 4.3, $(-\Delta)^{i\alpha}$ is given by integration against a kernel $K_{i\alpha}$ that is smooth off the diagonal and satisfies estimates (4.3) and (4.4). Theorem 3.2 implies that $(-\Delta)^{i\alpha}$ is a Calderón-Zygmund operator. \square

We believe that the Riesz potentials are singular integral operators, that is, the kernel $K_{i\alpha}(x, y)$ is singular on the diagonal for all $\alpha \in \mathbb{R} \setminus \{0\}$, but have not succeeded in proving this.

Theorem 4.5. *For $\alpha \in \mathbb{R}$, the operator $(-\Delta)^{i\alpha}$ defined originally on D extends to a bounded operator on $L^p(\mu)$ for $1 < p < \infty$, and satisfies weak 1-1 estimates.*

Proof. Theorem 3 of [21, page 19] implies that $(-\Delta)^{i\alpha}$ extends to a bounded operator on $L^p(\mu)$ for all $1 < p \leq 2$ and satisfies weak 1-1 estimates. A duality argument (see the proof of Theorem 1 from [19, page 29]) implies that $(-\Delta)^{i\alpha}$ extends to a bounded operator on $L^p(\mu)$ for all $2 < p < \infty$. Thus $(-\Delta)^{i\alpha}$ extends to a bounded operator on $L^p(\mu)$, for all $1 < p < \infty$, and satisfies weak 1-1 estimates. \square

Remark 4.6. The boundedness of $(-\Delta)^{i\alpha}$ on $L^p(\mu)$ for $1 < p < \infty$ can also be obtained using the general spectral multiplier theorem of [31] (see [25, Proposition 3.2]).

Remark 4.7 (Laplace type transforms). The only property of the function $t \mapsto t^{-i\alpha}$ used in the proof of Proposition 4.3 was its uniform boundedness. Therefore all the above results remain valid for a more general class of operators, namely the operators of Laplace transform type. Recall that a function $p : [0, \infty) \rightarrow \mathbb{R}$ is said to be of *Laplace transform type* if

$$p(\lambda) = \lambda \int_0^\infty m(t) e^{-t\lambda} dt,$$

where m is uniformly bounded.

Corollary 4.8. *Let p be of Laplace transform type. Then we can define an operator*

$$p(-\Delta)u = (-\Delta) \int_0^\infty m(t) e^{t\Delta} u dt$$

for $u \in D$ with a kernel

$$(4.5) \quad K_p(x, y) = \int_0^\infty (-\Delta_1) h_t(x, y) m(t) dt.$$

The kernel K_p is smooth off the diagonal and it satisfies the estimates

$$(4.6) \quad |K_p(x, y)| \lesssim R(x, y)^{-d}$$

$$(4.7) \quad |\Delta_2 K_p(x, y)| \lesssim R(x, y)^{-2d-1}.$$

Therefore, for a function p of Laplace transform type the operator $p(-\Delta)$ is a Calderón-Zygmund operator and it extends to a bounded operator on $L^q(\mu)$, $1 < q < \infty$.

We end this section by describing the dependence of the kernel $K_{i\alpha}$ on α .

Proposition 4.9. *If $x \neq y$, the map $\alpha \mapsto K_{i\alpha}(x, y)$ is differentiable.*

Proof. Let $x, y \in X$ such that $x \neq y$ and $f(t, \alpha) := \Delta_1 h_t(x, y)t^{-i\alpha}$. We know that $f(\cdot, \alpha) \in L^1(0, \infty)$ for all $\alpha \in \mathbb{R}$. Since

$$\frac{\partial}{\partial \alpha} f(t, \alpha) = \Delta_1 h_t(x, y)(-i)t^{-i\alpha} \ln t,$$

it suffices to show that $g(t) = t^{-d/(d+1)-1} \exp(-c \left(\frac{R(x, y)^{d+1}}{t}\right)^\gamma) |\ln t|$ is integrable and apply a standard theorem (for example, [5, Theorem 2.27]).

As $g(t)$ is continuous on $[0, 1]$ we look only at the integral over $[1, \infty)$. Using that $\ln t \leq \delta^{-1}t^\delta$ for any $\delta > 0$ we have

$$g(t) \leq \delta^{-1}t^{\delta - \frac{d}{d+1} - 1} \exp\left(-c \left(\frac{R(x, y)^{d+1}}{t}\right)^\gamma\right)$$

which is integrable on $[1, \infty)$ provided $\delta < \frac{d}{d+1}$. \square

5. BESSEL POTENTIALS

We next study the Bessel potentials on X , where X is a nested fractal K or an infinite blowup, without boundary, of K . Our analysis follows, in large, [19, Chapter 5.3] (see also [25]). In this spirit we consider the strictly positive operator $A = 1 - \Delta$. Then u is an eigenfunction of A if and only if it is an eigenfunction of $-\Delta$ and $A\varphi_n = (1 + \lambda_n)\varphi_n$. Recall that D is the set of finite linear combinations of the eigenfunctions φ_n .

To define the Bessel potentials on X we recall that, for $\lambda > 0$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha < 0$, we have that

$$(5.1) \quad \lambda^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-\lambda t} t^{-\alpha-1} dt,$$

where Γ is the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ if $\operatorname{Re} z > 0$.

Definition 5.1 (Bessel Potentials). Let $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha < 0$. For an eigenfunction φ with eigenvalue λ we want that $(I - \Delta)^\alpha \varphi = A^\alpha \varphi = (1 + \lambda)^\alpha \varphi$. This motivates us to define, for $u \in D$,

$$(I - \Delta)^\alpha u = A^\alpha u = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-t} e^{t\Delta} u dt.$$

Proposition 5.2. Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha < 0$.

(1) If $u = \sum a_k \varphi_k \in D$ then

$$A^\alpha u = \sum a_k (1 + \lambda_k)^\alpha \varphi_k.$$

(2) If $\beta \in \mathbb{C}$ such that $\operatorname{Re} \beta < 0$ then $A^\alpha A^\beta u = A^{\alpha+\beta} u$ for all $u \in D$.

Proof. For the first assertion, recall that by the spectral theorem, if $u = \sum a_k \varphi_k \in D$ is a finite sum then

$$e^{t\Delta} \left(\sum a_k \varphi_k \right) = \sum a_k e^{-t\lambda_k} \varphi_k.$$

Therefore we can exchange the sum and the integral in

$$\begin{aligned} A^\alpha \left(\sum a_k \varphi_k \right) &= \sum a_k \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(1+\lambda_k)t} dt \varphi_k \\ &= \sum a_k (1 + \lambda_k)^\alpha \varphi_k. \end{aligned}$$

The second assertion is an immediate consequence of the first. \square

Based on the above proposition we can extend the definition of A^α on D to arbitrary $\alpha \in \mathbb{C}$ via $A^\alpha = A^k A^{\alpha-k}$, where k is an integer such that $-1 \leq \operatorname{Re} \alpha - k < 0$, if $\operatorname{Re} \alpha \geq 0$. Then $\{A^\alpha\}_{\alpha \in \mathbb{C}}$ is a group so that $A^1 = A$.

We show next that the operators A^α , $\operatorname{Re} \alpha < 0$, defined originally on D , extend to bounded operators on $L^p(\mu)$ for all $1 \leq p \leq \infty$. We accomplish this by studying the kernels of the operators. The main tools we use are the heat kernel estimates together with Lemma 2.1. Since estimates of this type will be needed several times, we give the argument for the most general kernel we will encounter.

Proposition 5.3. *For $s \in \mathbb{R}$, $m \in L^\infty([0, \infty))$, and $x \neq y$ define*

$$(5.2) \quad L_{s,m}(x, y) = \int_0^\infty m(t) t^{\frac{s}{d+1}} h_t(x, y) e^{-t} \frac{dt}{t}.$$

Then

$$(5.3) \quad |L_{s,m}(x, y)| \lesssim \begin{cases} \frac{\Gamma(s)}{d-s} \left(R(x, y)^{s-d} + R(x, y)^{\frac{\gamma(s-d)}{\gamma+1}} \right) e^{-R(x, y)^{\frac{\gamma(d+1)}{\gamma+1}}} & \text{if } s < d, \\ (1 - \log R(x, y)) e^{-R(x, y)^{\frac{\gamma(d+1)}{\gamma+1}}} & \text{if } s = d, \\ \frac{1}{s-d} R(x, y)^{\frac{\gamma(s-d)}{\gamma+1}} e^{-R(x, y)^{\frac{\gamma(d+1)}{\gamma+1}}} & \text{if } s > d. \end{cases}$$

In particular, for $s > 0$, $L_{s,m}(x, \cdot) \in L^1(\mu)$ for all $x \in X$ and $\|L_{s,m}(x, \cdot)\|_1 \leq C$, with C a constant independent of x . Similarly $s > \frac{d}{2}$ implies $L_{s,m}(x, \cdot) \in L^2(\mu)$ for all $x \in X$ with a uniform bound on the L^2 norm. Moreover if $s > d$ then $L_{s,m}(x, y)$ is uniformly bounded.

Proof. For $s \in \mathbb{R}$ make the substitution $t = uR(x, y)^{d+1}$, from which

$$\begin{aligned} |L_{s,m}(x, y)| &\lesssim \int_0^\infty t^{\frac{s-d}{d+1}-1} e^{-t} e^{-c \left(\frac{R(x, y)^{d+1}}{t} \right)^\gamma} dt \\ &= R(x, y)^{s-d} \int_0^\infty t^{\frac{s-d}{d+1}-1} e^{-tR(x, y)^{d+1}} e^{-ct^{-\gamma}} dt \\ &= R(x, y)^{s-d} \int_0^\delta t^{\frac{s-d}{d+1}-1} e^{-tR(x, y)^{d+1}} e^{-ct^{-\gamma}} dt \\ &\quad + R(x, y)^{s-d} \int_\delta^\infty t^{\frac{s-d}{d+1}-1} e^{-tR(x, y)^{d+1}} e^{-ct^{-\gamma}} dt \\ &=: R(x, y)^{s-d} (I_1 + I_2), \end{aligned}$$

where $\delta = R(x, y)^{-\frac{d+1}{\gamma+1}}$. The intervals of validity of the estimates (5.3) arise naturally in estimating I_1 and I_2 .

To bound I_1 we use that $e^{-tR(x, y)^{d+1}} \leq 1$ on the interval, and make the change of variable $t = u^{-\gamma}$ to find

$$(5.4) \quad I_1 \leq \int_{\delta^{-\gamma}}^\infty u^{-\frac{s-d}{\gamma(d+1)}} e^{-cu} \frac{du}{u}.$$

The exponential decay in the integrand implies this is bounded by a constant multiple $c(s)$ of the integral over the unit length interval $[\delta^{-\gamma}, \delta^{-\gamma} + 1]$. It is easy to see $c(s) \leq 1$ for $s \geq d$ and $c(s) \leq \Gamma(s)$ otherwise. We bound the exponential term by $e^{-c\delta^{-\gamma}}$, and integrate the polynomial term to obtain $\frac{\gamma(d+1)}{d-s} u^{-\frac{s-d}{\gamma(d+1)}} \Big|_{\delta^{-\gamma}}^{1+\delta^{-\gamma}}$

unless $s = d$ where it is $\log u \Big|_{\delta^{-\gamma}}^{1+\delta^{-\gamma}}$. If $s < d$ we bound by the value at the upper endpoint, obtaining $(1 + \delta^{-\gamma})^{\frac{d-s}{d+1}} \leq 1 + R(x, y)^{\frac{d-s}{\gamma+1}}$. If $s = d$ it is easy to see the bound is by $1 + \gamma \log \delta \lesssim 1 - \log R(x, y)$. And if $s > d$ we bound by the value at the lower endpoint, which is $\delta^{-\gamma \frac{d-s}{d+1}} = R(x, y)^{\frac{d-s}{\gamma+1}}$. Combining these we have found

$$I_1 \lesssim \begin{cases} \frac{1}{d-s} (1 + R(x, y)^{-\frac{s-d}{\gamma+1}}) e^{-cR(x, y)^{\frac{\gamma(d+1)}{\gamma+1}}} & \text{if } s < d, \\ (1 - \log R(x, y)) e^{-cR(x, y)^{\frac{\gamma(d+1)}{\gamma+1}}} & \text{if } s = d, \\ \frac{1}{s-d} R(x, y)^{-\frac{s-d}{\gamma+1}} e^{-cR(x, y)^{\frac{\gamma(d+1)}{\gamma+1}}} & \text{if } s > d. \end{cases}$$

For the estimate of I_2 we use that $e^{-ct^{-\gamma}} \leq 1$ on the interval, so that with $u = tR(x, y)^{d+1}$ we obtain

$$I_2 \leq \int_{\delta}^{\infty} t^{\frac{s-d}{d+1}} e^{-tR(x, y)^{d+1}} \frac{dt}{t} = R(x, y)^{d-s} \int_{\delta R(x, y)^{d+1}}^{\infty} u^{\frac{s-d}{d+1}} e^{-u} \frac{du}{u}$$

This integral is the same as in (5.4), except that the power in the integrand is $\frac{s-d}{d+1}$ not $\frac{s-d}{\gamma(d+1)}$. Notice that the lower endpoint is $\delta R(x, y)^{d+1} = R(x, y)^{\frac{\gamma(d+1)}{\gamma+1}} = \delta^{-\gamma}$. We must therefore have the same estimates for the integral that we did for I_1 , but with the power $R(x, y)^{-\frac{s-d}{\gamma+1}}$ replaced by $R(x, y)^{-\frac{\gamma(s-d)}{\gamma+1}}$ throughout. Multiplying through by the leading $R(x, y)^{d-s}$ factor makes these estimates the same or smaller than the corresponding ones for I_1 , which completes the proof of (5.3).

Observe that the upper bound for $s > d$ is itself uniformly bounded, so $L_{s,m}(x, y)$ is uniformly bounded in this case. Also the bound is integrable for large $R(x, y)$ because it has exponential decay, and the singularity in the bound (which occurs when $s \leq d$) is integrable provided $s > 0$. Hence $\|L_{s,m}(x, \cdot)\|_1 \leq C$ with C independent of x . Finally we note that the singularity is in L^2 if $s = d$, or if $s < d$ and $2(s-d) + d < 0$, meaning $s > \frac{d}{2}$. \square

Corollary 5.4. For $\operatorname{Re} \alpha < 0$ define

$$K_{\alpha}(x, y) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} h_t(x, y) t^{-\alpha-1} e^{-t} dt.$$

Then $K_{\alpha}(\cdot, y)$ is integrable for all $y \in X$ with $\|K_{\alpha}(\cdot, y)\|_1 \leq C$ with C independent of y . The same statements are true for $K_{\alpha}(x, \cdot)$.

Proof. Let $\alpha = a + ib$, with $a < 0$. This is an immediate consequence of the L^1 estimate in Proposition 5.3 with $s = -a(d+1)$ and $m(t) = t^{ib}$. \square

As a consequence of the above result we obtain that the operators A^{α} are bounded operators on $L^p(\mu)$ for $1 \leq p \leq \infty$ if $\operatorname{Re} \alpha < 0$. We prove this statement in the following.

Theorem 5.5. Let α be such that $\operatorname{Re} \alpha < 0$. Then A^{α} is given by integration with respect to K_{α} , that is

$$(5.5) \quad A^{\alpha} f(x) = \int_X K_{\alpha}(x, y) f(y) d\mu(y),$$

for all $f \in D$. In particular, the operator A^{α} defined originally on D extends to a bounded operator on $L^p(\mu)$ for all $1 \leq p \leq \infty$.

Proof. The proof of (5.5) is clear. The second part follows immediately from the estimates of Corollary 5.4 by an argument analogous to the classical proof of the Young's inequality via the generalized Minkowski inequality. \square

Remark. The boundedness of the operators A^α on $L^p(\mu)$, for $1 < p < \infty$, can also be obtained using the spectral theorems of [31] (see [25, Proposition 3.2]).

Proposition 5.6. *On the compact set K the operator A^α is Hilbert-Schmidt when $\operatorname{Re} \alpha < -\frac{d}{2(d+1)}$.*

Proof. Let $\alpha = a + ib$, set $s = -a(d+1)$ and $m(t) = t^{ib}$. Then $s > \frac{d}{2}$ so we can apply Proposition 5.3 to see $\|K_\alpha(x, \cdot)\|_{L^2}$ is uniformly bounded. Integrating with respect to x produces a factor of $\mu(K) < \infty$, so $K_\alpha(x, y)$ is in L^2 of the product space. \square

Corollary 5.7. *Assume that X is compact and $\operatorname{Re} \alpha < -\frac{d}{2(d+1)}$. Then*

$$K_\alpha(x, y) = \sum_n (1 + \lambda_n)^\alpha \varphi_n(x) \varphi_n(y),$$

where the infinite sum converges in $L^2(\mu \times \mu)$.

Proposition 5.8. *If $\operatorname{Re} \alpha < 0$ then K_α is smooth off the diagonal. If in addition $\operatorname{Re} \alpha \leq -\frac{d}{d+1}$ then K_α is continuous and uniformly bounded. Also, the map $\alpha \mapsto K_\alpha(x, y)$ is analytic on $\{\operatorname{Re} \alpha < 0\}$, for all $x, y \in X$ with $x \neq y$.*

Proof. Recall $K_\alpha(x, y) = \int_0^\infty h_t(x, y) t^{-\alpha-1} e^{-t} dt$ with $\alpha = a + ib$. To obtain smoothness off the diagonal, it suffices by Lemma 4.2 that we differentiate inside the integral. Since h_t is the heat kernel, applying the Laplacian is the same as differentiating with respect to t , and we know $t^k \frac{\partial^k}{\partial t^k} h_t$ satisfies the same bounds as h_t itself. Then applying Proposition 5.3 for $m(t) = t^{ib}$ and $s = -(a+j+k)(d+1)$ one can see that $\Delta_1^k \Delta_2^j K_\alpha(x, y)$ is continuous off the diagonal for all $j, k \geq 1$. If $\operatorname{Re} \alpha \leq -\frac{d}{d+1}$ then a second application of the Proposition shows $K_\alpha(x, y)$ is uniformly bounded.

The second part follows by a standard argument (such as [22, Theorem 5.4]), since the map $F(\alpha, t) = h_t(x, y) t^{-\alpha-1} e^{-t}$ is analytic on α for each $t > 0$, and continuous in α and t . \square

5.1. Purely imaginary Bessel potentials. We turn our attention now to the study of the kernel of purely imaginary Bessel potentials, that is, operators of the form $(I - \Delta)^{i\alpha}$, with $\alpha \in \mathbb{R}$. We use formula (4.1) as the starting point and, for $\alpha \in \mathbb{R}$ and $u \in D$, we define

$$(I - \Delta)^{i\alpha} u = C_\alpha (I - \Delta) \int_0^\infty e^{t\Delta} u e^{-t} t^{i\alpha} dt.$$

For $\alpha \in \mathbb{R}$, we define the kernel of $(I - \Delta)^{i\alpha}$ via

$$(5.6) \quad G_{i\alpha}(x, y) = i\alpha C_\alpha \int_0^\infty h_t(x, y) e^{-t} t^{i\alpha-1} dt.$$

Theorem 5.9. *For $\alpha \in \mathbb{R}$, $G_{i\alpha}(x, y)$ defined in (5.6) is the kernel of $(I - \Delta)^{i\alpha}$, in the sense that*

$$(5.7) \quad (I - \Delta)^{i\alpha} u(x) = \int_X G_{i\alpha}(x, y) u(y) d\mu(y)$$

for all $u \in D$ such that $x \notin \text{supp } u$. Moreover, $G_{i\alpha}(x, y)$ is smooth off the diagonal.

Proof. As $u \in D$ is a finite sum, using integration by parts we have that

$$\begin{aligned} (I - \Delta)^{i\alpha} u &= C_\alpha \int_X \left(\int_0^\infty h_t(x, y) e^{-t^{i\alpha}} dt + \int_0^\infty \left(\frac{\partial}{\partial t} h_t(x, y) \right) e^{-t^{i\alpha}} dt \right) u(y) d\mu(y) \\ &= i\alpha C_\alpha \int_X \int_0^\infty h_t(x, y) e^{-t^{i\alpha-1}} dt u(y) d\mu(y). \end{aligned}$$

Thus (5.7) holds. From Lemma 4.2 we may apply powers of the Laplacian inside the integral in (5.6) to establish smoothness off the diagonal. Moreover the application of $\Delta_1^j \Delta_2^k$ is equivalent to replacing $h_t(x, y)$ with $\frac{\partial^{(j+k)}}{\partial t^{(j+k)}} h_t$, which satisfies the same estimates as $t^{-(j+k)} h_t$. Applying Proposition 5.3 with $s = -(j+k)(d+1)$, we see that

$$|\Delta_1^j \Delta_2^k G_{i\alpha}(x, y)| \lesssim R(x, y)^{-(j+k)(d+1)-d} e^{-R(x, y)^{\frac{\gamma(d+1)}{\gamma+1}}},$$

for all $j, k \geq 0$, so in particular $G_{i\alpha}(x, y)$ is smooth off the diagonal. \square

Corollary 5.10. *For $\alpha \in \mathbb{R}$, the operator $(I - \Delta)^{i\alpha}$ is a Calderón-Zygmund operator.*

Proof. The operator $(I - \Delta)^{i\alpha}$ extends to a bounded operator on $L^2(\mu)$ by the spectral theorem. The proof of Theorem 5.9 implies that $G_{i\alpha}$ satisfies the estimates (1.1) and (1.2). Thus $(I - \Delta)^{i\alpha}$ is a Calderón-Zygmund operator. \square

Corollary 5.11. *If $\alpha \in \mathbb{R}$, then the operator $(I - \Delta)^{i\alpha}$ defined originally on D extends to a bounded operator on $L^p(\mu)$ for $1 < p < \infty$ and satisfies weak 1-1 estimates.*

Remark 5.12. The boundedness of $(I - \Delta)^{i\alpha}$ on $L^p(\mu)$ for $1 < p < \infty$ can also be obtained using the general spectral multiplier theorem of [31] (see [25, Proposition 3.2]).

We also note that one can easily modify the proof of Proposition 4.9 to obtain the following result.

Proposition 5.13. *Let $x, y \in X$ with $x \neq y$. Then the map $\alpha \mapsto G_{i\alpha}(x, y)$ is differentiable.*

6. COMPLEX POWERS ON PRODUCTS OF FRACTALS AND BLOWUPS

In this section we extend our analysis of Calderón-Zygmund operators and the Riesz and Bessel potentials to finite products X^N , where X is either a nested fractal K or an infinite blow-up without boundary of K . The study of the energy, the Laplace operator, and the heat kernel estimates on products of PCF fractals was initiated by Strichartz in [26] (see, also, [27, 28]). We begin by reviewing the basic steps in his construction. Consider the product space X^N with the product measure μ^N . Notice that X^N is not, in general, a PCF fractal. Recall from [26] that a measurable function u on X^2 has *minimal regularity* if and only if for almost $x_2 \in X$, $u(\cdot, x_2) \in \text{dom } E$, and for almost every $x_1 \in X$,

$u(x_1, \cdot) \in \text{dom } E$. Such a function belongs to the domain of the energy on X^2 , $\text{dom } E^2$, if and only if

$$E^2(u) = \int_X E(u(\cdot, x_2)) d\mu(x_2) + \int_X E(u(x_1, \cdot)) d\mu(x_1)$$

exists and is finite [26, 27]. This definition can be easily generalized to X^N . Then we may define a Laplacian by the weak formulation

$$E^N(u, v) = - \int_{X^N} (\Delta u) v d\mu^N.$$

To avoid confusion, we will henceforth write Δ' for the Laplacian on X . Recall that we fixed orthonormal basis $\{\varphi_n\}_n$ for $L^2(\mu)$ such that each φ_n is an eigenfunction of Δ' . Then if $\underline{n} = (n_1, n_2, \dots, n_N)$ the functions

$$\varphi_{\underline{n}}(x) = \varphi_{n_1}(x_1) \varphi_{n_2}(x_2) \dots \varphi_{n_N}(x_N),$$

where $x = (x_1, x_2, \dots, x_N) \in X^N$, form an orthonormal basis for $L^2(\mu^N)$. Let D^N be the set of finite linear combinations of $\varphi_{\underline{n}}$.

The heat kernel on X^N is the product

$$(6.1) \quad h_t^N(x, y) = h_t(x_1, y_1) h_t(x_2, y_2) \dots h_t(x_N, y_N),$$

for $x, y \in X^N$ ([28, Theorem 6.1]). We extend the metric to X^N by

$$R^N(x, y) = \left(R(x_1, y_1)^{(d+1)\gamma} + R(x_2, y_2)^{(d+1)\gamma} + \dots + R(x_N, y_N)^{(d+1)\gamma} \right)^{1/((d+1)\gamma)}.$$

Then, if K is an affine nested fractal, the heat kernel estimates become [28, Theorem 2.2]

$$(6.2) \quad h_t^N(x, y) \sim t^{-\frac{Nd}{d+1}} \exp\left(-c \frac{R^N(x, y)^{d+1}}{t}\right)^\gamma$$

where the symbol \sim indicates that there are upper and lower bounds by constant multiples, possibly with distinct values of c . The same estimates are satisfied by $t^k \frac{\partial^k}{\partial t^k} h_t^N(x, y)$ for all $t > 0$.

6.1. Singular integral and Calderón-Zygmund operators on products.

We say that an operator T bounded on $L^2(\mu^N)$ is a *Calderón-Zygmund operator* on X^N if it is given by integration with respect to a kernel $K(x, y)$ which is a function off the diagonal and satisfies

$$(6.3) \quad |K(x, y)| \lesssim R^N(x, y)^{-Nd}$$

for all $x \neq y$ and

$$(6.4) \quad |K(x, y) - K(x, \bar{y})| \lesssim \eta \left(\frac{R^N(y, \bar{y})}{R^N(x, \bar{y})} \right) R^N(x, y)^{-Nd},$$

if $R^N(x, y) \geq cR^N(y, \bar{y})$, for some $c > 1$, where η is a Dini modulus of continuity. We say that T is a *singular integral operator* if $K(x, y)$ is singular at $x = y$. The next theorem, which is the main result of this section, extends Theorem 3.2 to the product setting.

Theorem 6.1. *Let K be a nested fractal and assume that X is either K or an infinite blow-up of K without boundary. Suppose that $T : L^2(\mu^N) \rightarrow L^2(\mu^N)$ is given by integration with respect to a kernel $K(x, y)$ which is smooth off the diagonal and satisfies the following estimates*

$$(6.5) \quad |K(x, y)| \lesssim R^N(x, y)^{-Nd}$$

$$(6.6) \quad |\Delta'_{y,i} K(x, y)| \lesssim R^N(x, y)^{-(N+1)d-1}, \quad i = 1, 2, \dots, N$$

where for $y = (y_1, y_2, \dots, y_N)$, $\Delta'_{y,i} K(x, y)$ is the Laplacian on X with respect to y_i . Then T is a Calderón-Zygmund operator. In particular, T extends to a bounded operator on $L^p(\mu^N)$ for all $1 < p < \infty$ and satisfies weak 1-1 estimates.

Proof. We prove the theorem for $N = 2$. The difference between this case and general N is merely notation. Let c' be the constant from Theorem 3.2 and let $c = 2c'$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $\bar{y} = (\bar{y}_1, \bar{y}_2)$ in X^2 such that $R^2(x, y) \geq cR^2(y, \bar{y})$. Then we can find $C = C_1 \times C_2$, where C_1 and C_2 are cells of size $r^{k_0+1}R^2(x, y)$, such that $y, \bar{y} \in C$ and $x \notin C$. Moreover, we have that $R(x_1, y_1) \geq c'R(y_1, \bar{y}_1)$ or $R(x_2, y_2) \geq c'R(y_2, \bar{y}_2)$. Assume that $R(x_1, y_1) \geq c'R(y_1, \bar{y}_1)$. Then

$$\begin{aligned} K(x, y) - K(x, \bar{y}) &= K(x, (y_1, y_2)) - K(x, (y_1, \bar{y}_2)) \\ &\quad + K(x, (y_1, \bar{y}_2)) - K(x, (\bar{y}_1, \bar{y}_2)). \end{aligned}$$

Let $u_{(x, y_1)}(z) = K(x, (y_1, z))$ for $z \in C_2$ and let $v_{(x, \bar{y}_2)}(z) = K(x, (z, \bar{y}_2))$ for $z \in C_1$. Since $R(x_1, y_1) \geq c'R(y_1, \bar{y}_1)$ we have that $u_{(x, y_1)}$ is smooth on C_2 and $v_{(x, \bar{y}_2)}$ is smooth on C_1 . Moreover, they satisfy the following estimates

$$\begin{aligned} |u_{(x, y_1)}(z)| &\lesssim R^2(x, y)^{-2d}, \\ |\Delta' u_{(x, y_1)}(z)| &\lesssim R^2(x, y)^{-3d-1}, \end{aligned}$$

for all $z \in C_1$, and

$$\begin{aligned} |v_{(x, \bar{y}_2)}(z)| &\lesssim R^2(x, y)^{-2d}, \\ |\Delta' v_{(x, \bar{y}_2)}(z)| &\lesssim R^2(x, y)^{-3d-1}, \end{aligned}$$

for all $z \in C_2$. Lemma 3.3 together with the fact that $R(y_i, \bar{y}_i) \leq R^2(y, \bar{y})$, $i = 1, 2$, implies that

$$|u_{(x, y_1)}(y_2) - u_{(x, y_1)}(\bar{y}_2)| \lesssim \left(\frac{R^2(y, \bar{y})}{R^2(x, y)} \right)^{\frac{1}{2}} R(x, y)^{-2d}$$

and

$$|v_{(x, \bar{y}_2)}(y_1) - v_{(x, \bar{y}_2)}(\bar{y}_1)| \lesssim \left(\frac{R^2(y, \bar{y})}{R^2(x, y)} \right)^{\frac{1}{2}} R(x, y)^{-2d}.$$

Thus (6.4) holds with $\eta(s) = s^{1/2}$. \square

6.2. Purely imaginary Riesz potentials on products. Having all the ingredients in place, we can define for $\alpha \in \mathbb{R}$ and $u \in D^N$

$$(-\Delta)^{i\alpha} u = C(-\Delta) \int_0^\infty e^{t\Delta} u t^{-i\alpha} dt.$$

The kernel of $(-\Delta)^{i\alpha}$ is given by the formula

$$K_{i\alpha}(x, y) = C \int_0^\infty (-\Delta_1) h_t^N(x, y) t^{-i\alpha} dt.$$

Using, basically, the same computations as in Section 4 we see that, for $\alpha \in \mathbb{R}$, $K_{i\alpha}(x, y)$ is smooth away from the diagonal and satisfies the following estimates:

$$(6.7) \quad |K_{i\alpha}(x, y)| \lesssim R^N(x, y)^{-Nd},$$

$$(6.8) \quad |\Delta'_{y,i} K_{i\alpha}(x, y)| \lesssim R^N(x, y)^{-(N+1)d-1}, \quad i = 1, 2, \dots, N,$$

so $(-\Delta)^{i\alpha}$ is a Calderón-Zygmund operator on X^N . We believe that they are singular integral operators but have not succeeded in proving this.

Corollary 6.2. *For $a \in \mathbb{R}$, the operator*

$$(-\Delta)^{i\alpha} u(x) = \int_{X^N} K_{i\alpha}(x, y) u(y) d\mu^N(y)$$

extends to a bounded operator on $L^p(\mu^N)$, for all $1 < p < \infty$, and satisfies weak 1-1-estimates.

Remark 6.3. The boundedness of $(-\Delta)^{i\alpha}$ on $L^p(\mu^N)$, $1 < p < \infty$, can also be deduced using the multivariable spectral results of [18].

The results about the dependence of the kernels on α extend easily to the product setting. Using essentially the proof of Proposition 4.9 we see that $\alpha \mapsto K_{i\alpha}(x, y)$ is differentiable in α for all $x, y \in X$ with $x \neq y$.

6.3. Bessel Potentials on products. Consider now the strictly positive operator $A = I - \Delta$. As before, for $\operatorname{Re} \alpha < 0$ and $u \in D^N$, we define

$$(6.9) \quad A^\alpha u = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-t} t^{-\alpha-1} e^{t\Delta} u dt.$$

with the kernel

$$K_\alpha(x, y) = \int_0^\infty h_t^N(x, y) e^{-t} t^{-\alpha-1} dt.$$

It is clear that the equivalent of Proposition 5.2 holds so we can define $A^\alpha = A^{\alpha-k} A^k$, where k is such that $-1 \leq \operatorname{Re} \alpha - 1 < 0$, if $\operatorname{Re} \alpha \geq 0$. Versions of all the statements established in Section 5 remain valid for this class of operators. The crucial ingredients in the proofs there were the heat kernel estimates (2.1) which we have now as (6.2).

In particular, we see from (6.2) that all of the integrals we encounter in the product setting differ from (5.2) only in that there is an extra factor of $t^{-\frac{(N-1)d}{d+1}}$, so Proposition 5.3 is valid if we replace $h_t(x, y)$ with $h_t^N(x, y)$ and replace each occurrence of $s - d$ in (5.3) with $s - Nd$, making the regions for the estimates $s < Nd$, $s = Nd$ and $s > Nd$.

The above quickly gives an analogue of Theorem 5.5. Notice that the singularity occurring for $s \leq Nd$ is in $L^1(d\mu^N)$ if $s > 0$, so that $\|K_\alpha(\cdot, y)\|_1$ is bounded by a constant independent of y and similarly for $\|K_\alpha(x, \cdot)\|_1$. This implies that for $\operatorname{Re} \alpha < 0$, A^α extends to be a bounded operator on $L^p(\mu^N)$ for all $1 \leq p \leq \infty$.

We also obtain product versions of Proposition 5.6 and Corollary 5.7. The singularity $R(x, y)^{s-Nd}$ is in $L^2(d\mu^N)$ if $2(s - Nd) + Nd > 0$, so if $s > \frac{Nd}{2}$. As $s = -\operatorname{Re}(\alpha)(d+1)$ we conclude that $\|K_\alpha(x, \cdot)\|_{L^2}$ is uniformly bounded for $\operatorname{Re}(\alpha) <$

$-\frac{Nd}{2(d+1)}$. Thus on the compact fractal K the kernel is in L^2 , the operator A^α is Hilbert-Schmidt, and we have the L^2 expansion

$$K_\alpha(x, y) = \sum_{\underline{n}} (1 + \lambda_{n_1} + \lambda_{n_2} + \cdots + \lambda_{n_N})^\alpha \varphi_{\underline{n}}(x) \varphi_{\underline{n}}(y).$$

Our estimates show that the kernel is always smooth away from the diagonal. If $\operatorname{Re}(\alpha) < -\frac{Nd}{d+1}$ then we have $s > Nd$, from which the kernel is also globally continuous and uniformly bounded. By the same argument as in Proposition 5.8 the map $\alpha \mapsto K_\alpha(x, y)$ is analytic on $\{\operatorname{Re} \alpha < 0\}$ for all $x \neq y \in X^N$.

For purely imaginary Bessel potentials $(I - \Delta)^{i\alpha}$ we have analogues of Theorem 5.9 and its corollaries. Specifically, we define

$$G_{i\alpha}(x, y) = i\alpha C_\alpha \int_0^\infty h_t^N(x, y) e^{-t^{i\alpha-1}} dt.$$

and verify that it represents $(I - \Delta)^{i\alpha}$ on those $u \in D^N$ with support away from x . By the previous reasoning about the analogue of Proposition 5.3 (with d replaced by Nd in the conclusions) we see that $G_{i\alpha}(x, y)$ is smooth off the diagonal and satisfies (6.7) and (6.8). Thus $(I - \Delta)^{i\alpha}$ is a Calderón-Zygmund operator and it extends to a bounded operator on $L^p(\mu^N)$ for all $1 < p < \infty$ and satisfies weak 1-1 estimates. The map $\alpha \mapsto G_{i\alpha}(x, y)$ is also differentiable for all $x \neq y$.

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